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ON THE SOLUTIONS OF FOURTH ORDER DIFFERENCE EQUATIONS

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1. Introduction. In this note we will study fourth order difference equations of the form

$$\Delta^4 y_n = f(n, y_{n+2}), \qquad n \in \mathbf{N}. \tag{E}$$

We denote by $\mathbf{N}$ the set of positive integers, by $\mathbf{R}$ the set of real numbers. For a function $x : \mathbf{N} \to \mathbf{R}$, the forward difference operators are defined as follows: $\Delta x_n = x_{n+1} - x_n$, $n \in \mathbf{N}$ and $\Delta^k x_n = \Delta(\Delta^{k-1} x_n)$ for $k > 1$.

By a solution of (E) we mean any sequence $y = \{y_n\}_{n=1}^{\infty}$ which satisfies (E) for all $n \in \mathbf{N}$. We call the solution y the zero (or trivial) solution if it is identically zero or if there exists $\nu \in \mathbf{N}$ such that $y_n = 0$ for all $n \geq \nu$.

A nonzero solution is oscillatory if, for every $m \in \mathbf{N}$ there exists $n \geq m$ such that $y_n y_{n+1} \leq 0$. Therefore, a nonoscillatory solution is such a sequence, which is eventually positive or eventually negative. We suppose that the function $f : \mathbf{N} \times \mathbf{R} \to \mathbf{R}$ satisfies condition $(*)$ if

$$x f(n, x) < 0 \quad \text{for all } n \in \mathbf{N},\ x \in \mathbf{R} \backslash \{0\}. \tag{$*$}$$

In his paper [**2**], W. Taylor considered two types of solutions of the fourth order linear difference equations (see also [**1**]). Relations between these types of solutions for the equation (E), and their oscillatory behavior are the main purposes of this note. Some of our theorems generalize results in the work of Taylor. This refers to Theorems 1, 2, 3 and 4 proved below and Theorems 1.3, 2.3, 2.4 and 2.5 of [**2**], respectively.

Following Taylor, we define operator F as

$$F(x_n) = x_{n+1} \Delta^3 x_n - \Delta x_n \Delta^2 x_n, \qquad n \in \mathbf{N}.$$

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We call a solution y for which $F(y_n) \geq 0$ for all $n \in \mathbf{N}$ an F_+-solution. If $F(y_n) < 0$ for some n, then y is called an F_--solution.

We use this Taylor operator to classify nonoscillatory solutions of the equation (E).

In fact, the operator F divided the set of solutions into two disjoint subsets, F_+ and F_- solutions. In Section 2 we shall prove that every nonoscillatory F_+-solution is monotonic and with square summable second difference and, moreover, these relations hold in both directions. We also prove that every F_--solution is unbounded with unbounded first difference. In Section 3 we will study equation (E) with some additional assumptions which allow us to determine the character (in relation to the operator F) and thereby properties of all or all nonoscillatory solutions. Furthermore, we formulate assumptions under which every F_+-solution is oscillatory.

We use the convention that void sum is equal to zero.

2. Properties of F-solutions.

Lemma. *Let the function f satisfy condition $(*)$, and let y be any solution of* (E). *Then $F(y_n)$ is a nonincreasing function on* $\mathbf{N}$.

Proof. By differencing $F(y_n)$, and applying equality (E), we get

$$\begin{aligned}\Delta F(y_n) &= [y_{n+2}\Delta^4 y_n + (\Delta y_{n+1})\Delta^3 y_n] \\ &\quad - [(\Delta y_{n+1})\Delta^3 y_n + (\Delta^2 y_n)^2] \\ &= y_{n+2} f(n, y_{n+2}) - (\Delta^2 y_n)^2 \leq 0.\end{aligned}$$

Hence, the monotonicity property of F follows. □

For the zero solution, $F(y_n) = 0$ for all sufficiently large n.

Theorem 1. *Let the function f satisfy condition $(*)$, and let y be a nontrivial F_+-solution of the equation* (E). *Then*

(i) $\sum_{n=1}^{\infty} [\Delta^{2+j} y_n]^2 < \infty$,

(ii) $\lim_{n\to\infty} \Delta^{2+j} y_n = 0$

for $j = 0, 1, 2, \dots$.

Proof. Let y be a nontrivial F_+-solution of (E). To prove (i) for $j = 0$, we examine the sum $\sum_{k=1}^{n-1}(\Delta^2 y_k)^2$ and prove its boundedness independently of the upper limit of summation. Differencing $F(y_k)$ and substituting $f(k, y_{k+2})$ in the place of $\Delta^4 y_k$ we get

$$\Delta F(y_k) = y_{k+2} f(k, y_{k+2}) - (\Delta^2 y_k)^2.$$

Now summing from $k = 1$ to $n - 1$, we obtain

$$F(y_n) = F(y_1) + \sum_{k=1}^{n-1} y_{k+2} f(k, y_{k+2}) - \sum_{k=1}^{n-1} (\Delta^2 y_k)^2. \tag{1}$$

Since $F(y_n) > 0$ and f satisfies $(*)$, therefore

$$0 < F(y_1) + \sum_{k=1}^{n-1} y_{k+2} f(k, y_{k+2}) - \sum_{k=1}^{n-1} (\Delta^2 y_k)^2,$$

and consequently,

$$\sum_{k=1}^{n-1} (\Delta^2 y_k)^2 < F(y_1), \qquad n \geq 1. \tag{2}$$

Inequality (2) remains true for all $n \in \mathbf{N}$. Hence,

$$\sum_{k=1}^{\infty} (\Delta^2 y_k)^2 \leq F(y_1) < \infty.$$

We have proved condition (i) for $j = 0$.

To get (i) for $j > 0$ we use inductive argument. For this, we need to express $(\Delta^{2+j} y_k)^2$ in terms of the lower order differences. Since for arbitrary reals a, b we have $-2ab \leq a^2 + b^2$, therefore

$$\begin{aligned}(\Delta^{2+j} y_k)^2 &= [\Delta^{2+j-1} y_{k+1} - \Delta^{2+j-1} y_k]^2 \\ &\leq 2(\Delta^{2+j-1} y_{k+1})^2 + 2(\Delta^{2+j-1} y_k)^2.\end{aligned}$$

So

$$\sum_{k=1}^{\infty}(\Delta^{2+j}y_k)^2 \leq 2\sum_{k=1}^{\infty}(\Delta^{2+j-1}y_{k+1})^2 + 2\sum_{k=1}^{\infty}(\Delta^{2+j-1}y_k)^2$$
$$\leq 4\sum_{k=1}^{\infty}(\Delta^{2+j-1}y_k)^2.$$

Therefore, $\sum_{k=1}^{\infty}(\Delta^{2+j}y_k)^2 < \infty$ provided that $\sum_{k=1}^{\infty}(\Delta^{2+j-1}y_k)^2 < \infty$. We have proved $\sum_{k=1}^{\infty}(\Delta^2 y_k)^2 < \infty$, hence (i) holds for all $j \geq 0$. Condition (ii) follows directly from (i). □

Corollary 1. *If, in addition to the assumptions of Theorem* 1, *the function f satisfies the condition*

$$|f(n,x)| \geq \varepsilon, \quad \textit{for } (n,x) \in \mathbf{N} \times \{\mathbf{R}\backslash\{0\}\} \tag{3}$$

then equation (E) *does not possess a nontrivial F_+-solution.*

Proof. Suppose that there exists an F_+-solution $\{y_n\}_{n=1}^{\infty}$. By Theorem 1, we have $\lim_{n\to\infty}|\Delta^4 y_n| = 0$. On the other hand, from (3), $\limsup_{n\to\infty}|f(n,y_{n+2})| \geq \varepsilon > 0$. This contradiction proves the assertion. □

Remark 1. The following properties (see [**1**]) of nonoscillatory solutions of equation (E) are known. Every nonoscillatory solution $\{y_n\}_{n=1}^{\infty}$ can be one of the types:

(A4+) $\quad y_n > 0,\ \Delta y_n > 0,\ \Delta^2 y_n > 0,\ \Delta^3 y_n > 0,\ \Delta^4 y_n < 0,$

(A4−) $\quad y_n < 0,\ \Delta y_n < 0,\ \Delta^2 y_n < 0,\ \Delta^3 y_n < 0,\ \Delta^4 y_n > 0,$

(A2+) $\quad y_n > 0,\ \Delta y_n > 0,\ \Delta^2 y_n < 0,\ \Delta^3 y_n > 0,\ \Delta^4 y_n < 0,$

(A2−) $\quad y_n < 0,\ \Delta y_n < 0,\ \Delta^2 y_n > 0,\ \Delta^3 y_n < 0,\ \Delta^4 y_n > 0,$

for n sufficiently large, say $n \geq \nu \geq 1$.

Suppose that the function f satisfies condition

$$f(n,-x) = -f(n,x) \quad \text{for } (n,x) \in \mathbf{N} \times \{\mathbf{R}\backslash\{0\}\}.$$

For such a function, one can check that if $\{y_n\}_{n=1}^{\infty}$ is an (A4+)-solution of (E), then the sequence $\{-y_n\}_{n=1}^{\infty}$ is an (A4−)-solution of (E) and conversely. The same property holds if y is an (A2+)-solution. So the set of nonoscillatory solutions of (E) possesses a kind of symmetry. This remark allows us to consider only (A4+) or (A2+) solutions, especially when the asymptotic behavior is studied. However, we do not suppose that this condition is fulfilled.

The following two theorems give necessary and sufficient conditions for a nonoscillatory solution of (E) to be an F_+-solution.

Theorem 2. *Let the function f satisfy condition* $(*)$, *and let y be a nonoscillatory solution of* (E). *Then y is an F_+-solution if and only if it is an* (A2)*-solution.*

Proof. We will prove the theorem for an eventually positive solution, for negative solutions the proof is similar.

Necessity. Let y be an eventually positive F_+-solution. Suppose to the contrary that it is an (A4)-solution. Then from $\Delta^3 y_n > 0$, $\Delta^2 y_\nu > 0$ it follows that $\Delta^2 y_n > \Delta^2 y_\nu > 0$ for $n > \nu$. But this contradicts condition (ii) of Theorem 1, so y is an (A2+)-solution.

Sufficiency. Let y be an (A2+)-solution. We will show positivity of the operator F on the whole sequence y. Take some m, $m \geq \nu$, where ν is taken from the definition of the (A2+)-solution. Then by (A2+) we get $F(y_m) > 0$. By the lemma, the sequence $\{F(y_n)\}$ is nonincreasing; therefore, $F(y_j) \geq F(y_m) > 0$ for all $j < m$. Since m was taken arbitrary, so $F(y_n) > 0$ for all $n \in \mathbf{N}$, that is, $\{y_n\}_{n=1}^{\infty}$ is an F_+-solution. □

Theorem 3. *Let the function f satisfy condition* $(*)$, *and let y be a nonoscillatory solution of* (E). *Then y is an F_+-solution if and only if*

$$\sum_{j=1}^{\infty} (\Delta^2 y_j)^2 < \infty. \tag{4}$$

Proof. Necessity follows directly from Theorem 1. To prove sufficiency, let y be a nonoscillatory solution of (E) for which condition (4) is fulfilled. Suppose that y is an F_--solution. Then for some $m \in \mathbf{N}$ we have $F(y_m) < 0$. Hence, by the lemma

$$F(y_n) \le F(y_m) < 0, \quad \text{for all } n,\ n \ge m. \tag{5}$$

Since y is nonoscillatory then applying Remark 1, y can be an (A2) or an (A4)-solution. We exclude both of the cases. For an (A4)-solution, we get the contradiction $\sum_{j=1}^{\infty}(\Delta^2 y_j)^2 = \infty$. If y is an (A2)-solution, then there exists a $\nu \in \mathbf{N}$ such that $F(y_n) > 0$ for all $n \ge \nu$. This time we obtain contradiction with (5). Thus, y is an F_+-solution. □

Remark 2. Theorem 2 shows that the operator F divides the set of nonoscillatory solutions of (E) into two disjoint subsets: F_+-solutions which are the same as (A2)-solutions and F_--solutions which are the same as (A4)-solutions. Theorem 3 describes another property of elements of these sets. Namely, for every element y of the first set $\sum_{j=1}^{\infty}(\Delta^2 y_j)^2 < \infty$, while for the second $\sum_{j=1}^{\infty}(\Delta^2 y_j)^2 = \infty$.

The next theorem characterizes F_--solutions of (E).

Theorem 4. *Let the function f satisfy condition $(*)$, and let y be an F_--solution of* (E). *Then y has an unbounded first difference.*

Proof. Let y be an F_--solution. We will consider separately two possible cases of y:

(a) y is oscillatory,

(b) y is nonoscillatory.

Case (a). Suppose that the sequence $\{\Delta y_n\}_{n=1}^{\infty}$ is bounded, that is, for some constant C_1,

$$|\Delta y_n| < C_1, \quad \text{for all } n \in \mathbf{N}. \tag{6}$$

To shorten the notation, we introduce an operator H defined by

$$H(y_n) = y_n \Delta^2 y_n - (\Delta y_n)^2, \qquad n \in \mathbf{N}. \tag{7}$$

We will show that $H(y_n)$ tends to minus infinity, and following this together with (6), $\{y_n\}$ is an unbounded sequence. As a result of this and the oscillatory character of $\{y_n\}$, we obtain contradiction with (6). Differencing (7) yields

$$\begin{aligned}\Delta H(y_n) &= y_{n+1}\Delta^3 y_n + \Delta y_n \Delta^2 y_n \\ &\quad - (\Delta y_{n+1}\Delta^2 y_n + \Delta y_n \Delta^2 y_n) \\ &= y_{n+1}\Delta^3 y_n - \Delta y_{n+1}\Delta^2 y_n \\ &= F(y_n) - (\Delta^2 y_n)^2, \qquad n \in \mathbf{N}.\end{aligned}$$

Since y is an F_--solution, then for some μ, $F(y_\mu) < 0$; moreover, by the lemma,

$$F(y_n) \le F(y_\mu), \quad \text{for } n \ge \mu.$$

Hence,

$$\Delta H(y_k) \le F(y_\mu), \quad \text{for } k \ge \mu. \tag{8}$$

Summing inequality (8) from μ to $n-1$, we get

$$H(y_n) \le H(y_\mu) + \sum_{k=\mu}^{n-1} F(y_\mu) = H(y_\mu) + F(y_\mu)(n-\mu);$$

allowing n to tend to infinity we obtain

$$H(y_n) \to -\infty \quad \text{as } n \to \infty. \tag{9}$$

This property of the operator H together with (6) leads us by definition (7) to the conclusion

$$\lim_{n\to\infty} y_n \Delta^2 y_n = -\infty. \tag{10}$$

Using once more the estimate (6), we obtain

$$\begin{aligned}\sup_{n\in\mathbf{N}} |\Delta^2 y_n| &= \sup_{n\in\mathbf{N}} |\Delta y_{n+1} - \Delta y_n| \\ &\le \sup_{n\in\mathbf{N}} |\Delta y_{n+1}| + \sup_{n\in\mathbf{N}} |\Delta y_n| \le 2C_1.\end{aligned}$$

Therefore, (10) yields

$$\lim_{n\to\infty} |y_n| = \infty,$$

and so for any constant C there exists $n(C) \in \mathbf{N}$ such that $|y_n| \geq C$ for all $n \geq n(C)$. Since $\{y_n\}_{n=1}^{\infty}$ is oscillatory, then there exists an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $y_{n_k} y_{n_k+1} \leq 0$ and at least one of y_{n_k}, y_{n_k+1} is different from zero.

Let $\mathcal{H} = \inf\{k : n_k \geq n(C)\}$. Therefore, for $k \geq \mathcal{H}$ we have

$$y_{n_k} y_{n_k+1} < 0$$

and by the oscillatory behavior of y in the points n_k, we get $|\Delta y_{n_k}| = |y_{n_k}| + |y_{n_k+1}| \geq 2C$. The constant C was taken arbitrary, so we can choose an increasing sequence of integers $\{n_{k_t}\}_{t=1}^{\infty}$ such that $\lim_{t\to\infty} |\Delta y_{n_{k_t}}| = \infty$.

This is a contradiction with (6).

Case (b). By Theorem 2, the nonoscillatory F_--solution has to be an (A4)-solution. Then, for an eventually positive solution, we obtain

$$\Delta^2 y_j > \Delta^2 y_\nu > 0, \qquad j > \nu;$$

thus, after suitable summation,

$$\Delta y_n \geq \Delta y_\nu + (n-\nu)\Delta^2 y_\nu, \qquad n \geq \nu.$$

Hence, $\Delta y_n \to \infty$ as $n \to \infty$. $\square$

Remark 3. The above theorem shows that every F_--solution is unbounded. It is evident because, supposing that the solution y is bounded, that is, $|y_n| \leq C$ for some constant C and all $n \in \mathbf{N}$, we get $|\Delta y_n| \leq |y_{n+1}| + |y_n| \leq 2C$; that is, boundedness of the first difference, which is impossible by Theorem 4. Therefore, every bounded solution is an F_+-solution. But we did not prove that all unbounded solutions are F_--solutions.

By Theorem 2, we can prove a statement similar to Theorem 4 (unfortunately for nonoscillatory solutions). Namely:

Every nonoscillatory F_+-solution of (E) has a bounded first difference.

An open question is: does there exist an oscillatory F_+-solution with an unbounded first difference?

3. Existence results. In the next two theorems we give some additional assumptions on the function f, limiting the types of the equations which possess F_+ or nonoscillatory F_+-solutions.

Theorem 5. *Let the function f satisfy condition $(*)$. There exists a positive constant δ such that*

$$xf(n,x) \le -\delta/n, \quad \text{for every } n \in \mathbf{N},\ x \in \mathbf{R}\backslash\{0\}. \tag{11}$$

Then equation (E) *does not possess a nontrivial F_+-solution.*

Proof. Suppose, on the contrary, that there exists a nontrivial F_+-solution y of the equation (E). Then

$$0 \le F(y_m) = F(y_1) + \sum_{j=1}^{m-1} y_{j+2} f(j, y_{j+2}) - \sum_{j=1}^{m-1} (\Delta^2 y_j)^2.$$

Hence,

$$\begin{aligned} -\sum_{j=1}^{m-1} y_{j+2} f(j, y_{j+2}) &\le -\sum_{j=1}^{m-1} y_{j+2} f(j, y_{j+2}) + \sum_{j=1}^{m-1} (\Delta^2 y_j)^2 \\ &\le F(y_1) < \infty. \end{aligned}$$

Thus,

$$-\sum_{j=1}^{\infty} y_{j+2} f(j, y_{j+2}) \le F(y_1) < \infty.$$

On the other hand, by (11),

$$-y_{j+2} f(j, y_{j+2}) \ge \delta/j.$$

Therefore,

$$-\sum_{j=1}^{\infty} y_{j+2} f(j, y_{j+2}) \ge \delta \sum_{j=1}^{\infty} 1/j = \infty.$$

This contradiction completes the proof. □

Theorem 6. *Let the function f satisfy condition $(*)$, and let δ be a positive constant such that*

$$\operatorname{sgn}(x) f(n,x) < -\delta x^{-2} \quad \textit{for every } n \in \mathbf{N},\ x \in \mathbf{R}\backslash\{0\}. \tag{12}$$

Then every nonoscillatory solution of (E) *is an F_--solution.*

Proof. Let y be an eventually positive F_+-solution of (E). By Theorem 2, y is an (A2)-solution. Therefore, by definition of an (A2)-solution

$$\Delta y_n \leq \Delta y_\nu, \quad \text{for } n \geq \nu.$$

Hence,

$$y_n \leq y_\nu + \sum_{j=\nu}^{n-1} \Delta y_\nu = y_\nu + (n-\nu)\Delta y_\nu, \qquad n \geq \nu. \tag{13}$$

The same reasoning as in the proof of Theorem 5 leads us to the estimate

$$-\sum_{j=\nu}^{\infty} y_{j+2} f(j, y_{j+2}) \leq F(y_\nu) < \infty. \tag{14}$$

On the other hand, by (12),

$$f(j, y_{j+2}) < -\delta (y_{j+2})^{-2}.$$

Hence,

$$-\sum_{j=\nu}^{\infty} y_{j+2} f(j, y_{j+2}) \geq \sum_{j=\nu}^{\infty} \delta (y_{j+2})^{-1}. \tag{15}$$

Using (13) in (15), we conclude

$$-\sum_{j=\nu}^{\infty} y_{j+2} f(j, y_{j+2}) \geq \sum_{j=\nu}^{\infty} \delta [y_\nu + (j+2-\nu)\Delta y_\nu]^{-1} = \infty$$

in contradiction to (14). □

In the theorem below we state, under suitable assumptions, nonexistence of nonoscillatory F_+-solutions.

Theorem 7. *Let the function f satisfy condition $(*)$. If, for arbitrary positive constant ε there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$|f(n,x)| > \delta n^{-2} \quad \textit{for all } n \in \mathbf{N} \textit{ and } |x| > \varepsilon, \tag{16}$$

then every F_+-solution of (E) *is oscillatory.*

Proof. Let y be a nonoscillatory F_+-solution of (E). Then

$$\lim_{n\to\infty} \Delta^2 y_n = \lim_{n\to\infty} \Delta^3 y_n = 0. \tag{17}$$

We show that condition (16) yields unboundedness of $\Delta^2 y_n$. For this, we deduce estimation of $\Delta^2 y_n$ in terms of $f(n, y_{n+2})$ in consequence of which and condition (16) we obtain a contradiction.

Suppose that $\{y_n\}_{n=1}^{\infty}$ is eventually positive, say for $n \geq \nu$ (proof in the case of an eventually negative solution is similar). Summing equation (E) we obtain

$$\Delta^3 y_n - \Delta^3 y_k = \sum_{j=k}^{n-1} f(j, y_{j+2}).$$

Hence,

$$-\Delta^3 y_k = \sum_{j=k}^{\infty} f(j, y_{j+2}). \tag{18}$$

Therefore the series $\sum_{j=k}^{\infty} f(j, y_{j+2})$ converges. From (18) we get

$$-\Delta^2 y_n + \Delta^2 y_m = \sum_{k=m}^{n-1} \sum_{j=k}^{\infty} f(j, y_{j+2}). \tag{19}$$

As we have noticed, there exists a finite limit of the sum on the lefthand side of (19) as $n \to \infty$, so the sum on the righthand side has a finite limit as $n \to \infty$. Hence,

$$\begin{aligned}\sum_{k=m}^{n-1}\sum_{j=k}^{\infty} f(j, y_{j+2}) &= \sum_{k=m}^{n-2}(k+1-m)f(k, y_{k+2}) \\ &\quad + (n-m)\sum_{k=n+1}^{\infty} f(k, y_{k+2}) \\ &\leq \sum_{k=m}^{n-2}(k+1-m)f(k, y_{k+2})\end{aligned}$$

for $m \geq \nu$. So we have from (19)

$$\Delta^2 y_n - \Delta^2 y_m \geq -\sum_{k=m}^{n-2}(k+1-m)f(k, y_{k+2})$$

and consequently by (16) and (17),

$$\begin{aligned}-\Delta^2 y_m &= \lim_{n\to\infty}[\Delta^2 y_n - \Delta^2 y_m] \\ &\geq \sum_{k=m}^{\infty}(k+1-m)[-f(k, y_{k+2})] \\ &\geq \sum_{k=m}^{\infty}(k+1-m)\delta k^{-2} = \infty.\end{aligned}$$

Because $-\Delta^2 y_m$ is finite, the obtained contradiction proves our theorem.

Theorem 8. *Let the function f satisfy condition $(*)$, and let f be nondecreasing on $(0, \infty)$ and nonincreasing on $(-\infty, 0)$. If*

$$\sum_{j=1}^{\infty} j^3 |f(j, C)| = \infty, \tag{20}$$

for every constant $C \neq 0$, then equation (E) *does not possess a nonoscillatory bounded solution.*

Proof. Let y be a nonoscillatory bounded solution of (E). Then, by Remark 3, y is an F_+-solution and consequently, by Theorem 2, y is an (A2)-solution (say (A2+)-solution). Using monotonicity of the function f, we prove convergence of the series $\sum_{j=\nu}^{\infty}(j+3)^{(3)}|f(j,C_1)|$ which in turn is at variance with (20). Let us denote

$$v_n = \sum_{k=0}^{3} \frac{(-1)^k}{(3-k)!}(n+2-k)^{(3-k)}\Delta^{3-k}y_n, \qquad n \in \mathbf{N}, \tag{21}$$

where $n^{(k)} = n(n-1)(n-2)\cdots(n-k+1)$.

Differencing (21) using formulae for differences of sum and products, we get $\Delta v_n = (1/6)(n+3)^{(3)}\Delta^4 y_n$, and hence

$$\Delta v_n - \frac{1}{6}(n+3)^{(3)}f(n, y_{n+2}) = 0.$$

Summing the above equality from ν to $n-1$, we obtain

$$v_n - C - \frac{1}{6}\sum_{j=\nu}^{n-1}(j+3)^{(3)}f(j, y_{j+2}) = 0,$$

where $C = v_\nu$. Therefore, we have the following equality

$$\sum_{k=0}^{3} \frac{(-1)^k}{(3-k)!}(n+2-k)^{(3-k)}\Delta^{3-k}y_n - \frac{1}{6}\sum_{j=\nu}^{n-1}(j+3)^{(3)}f(j, y_{j+2}) = C. \tag{22}$$

By definition of an (A2+)-solution, we get from (22),

$$-y_n - \frac{1}{6}\sum_{j=\nu}^{n-1}(j+3)^{(3)}f(j, y_{j+2}) < C. \tag{23}$$

Because y is increasing and bounded, then for some constant C_1 we have $y_n \le C_1$ and $f(j, y_{j+2}) \le f(j, C_1)$. Therefore, from (23),

$$-\sum_{j=\nu}^{n-1}(j+3)^{(3)}f(j, C_1) \le -\sum_{j=\nu}^{n-1}(j+3)^{(3)}f(j, y_{j+2}) < 6C_1 + 6C = C_2.$$

Hence,

$$\sum_{j=\nu}^{n-1}(j+3)^{(3)}|f(j,C_1)| \le C_2, \quad \text{for } n > \nu.$$

Tending with n to infinity, we obtain

$$\sum_{j=\nu}^{\infty}(j+3)^{(3)}|f(j,C_1)| \le C_2,$$

but $(j+3)^{(3)} = (j+3)(j+2)(j+1) \ge j^3$, so we obtain contradiction with (20). $\square$

4. Examples. Consider the equation

$$\text{(Ex)} \qquad \Delta^4 y_n = a_n y_{n+2}^3, \qquad n \in \mathbf{N},$$

where

$$a_n = \frac{-24(n+2)^3}{(n+4)^{(5)}(n+1)^3(n+3)^3}, \qquad n \in \mathbf{N}.$$

For this equation $f(n,x) = a_n x^3$, we see that condition $(*)$ is fulfilled. By Theorem 1, for any nontrivial F_+-solution u of (Ex), we have

$$\text{(24)} \qquad \sum_{n=1}^{\infty}[\Delta^{2+j}u_n]^2 < \infty \quad \text{and} \quad \lim_{n\to\infty}\Delta^{2+j}u_n = 0.$$

If, furthermore, u is nonoscillatory, then u should be an (A2)-solution. An easy calculation will show that u, defined by

$$\text{(25)} \qquad u_n = n - 1/n, \qquad n \in \mathbf{N}$$

is a nonoscillatory, F_+-solution of (Ex). Differences of sequence (25) are given by the formula

$$\Delta^i u_n = (-1)^{i+1}\frac{i!}{(n+i)^{(i+1)}}, \quad \text{for } i \ge 2,\ n \in \mathbf{N},$$

for which conditions (24) are satisfied. Moreover, since

$$\Delta u_n = 1 + \frac{1}{n(n+1)}, \qquad n \in \mathbf{N}$$

this sequence is an (A2+)-solution.

The condition $(*)$ is satisfied for the functions $f(n,x) = a_n x$ where $\{a_n\}$ is any sequence of negative numbers; therefore, our theorems hold for linear equations

$$\Delta^4 y_n = a_n y_{n+2}, \qquad n \in \mathbf{N},$$

with a negative sequence of coefficients.

Let us observe that the same sequence (25) (with the same properties) is an (F_+,A2+)-solution of the equation

$$\Delta^4 y_n = \frac{-24[(n+2)^4+1]}{(n+2)^2(n+4)^{(5)}[y_{n+2}^2+2]}, \qquad n \in \mathbf{N}$$

for which condition $(*)$ is not satisfied.

On the other hand, condition $(*)$ does not hold for the equation

$$\Delta^4 y_n = y_{n+2}, \qquad n \in \mathbf{N},$$

while this equation possesses an F_+-solution $u_n = \sin(n\pi/3)$, $n \in \mathbf{N}$, for which the second difference $\Delta^2 u_n = -\sin((n+1)\pi/3)$ is not square summable; that is, $\sum_{n=1}^{\infty} [\Delta^2 u_n]^2 = \infty$. Furthermore, $\lim_{n\to\infty} \Delta^2 u_n$ does not exist (compare Theorem 1).

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