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GRAPHIC APPLICATIONS OF SOME INTERPOLATING WEIGHTED MEAN FUNCTIONS

VITTORIA DEMICHELIS

ABSTRACT. Our aim in the present paper is to prove that some results about interpolating weighted mean functions are a useful tool for the design of curves.

In [**1**], a method is considered to construct weighted mean functions, with interpolation property. A few examples are given; they include the Shepard formula.

In the present paper we introduce the weights of interpolating means in a well-known Walsh theorem. In this way we can impose a finite number of interpolation constraints to an approximant (for example, a Bernstein-Bezier curve) with a preassigned error.

In [**5**] a class of piecewise weighted mean functions is introduced for the interpolation of a finite set of real values $f_i = f(x_i)$, $i = 1, \dots, n$, given at the points $x_1 < \dots < x_n$. At any point $x \in [x_i, x_{i+1}]$, $i = 1, \dots, n-1$, the interpolant is a weighted mean of the values f_i and f_{i+1}. These piecewise weighted mean functions are at least C^1 in $[x_1, x_n]$, satisfy a variation diminishing property and preserve positivity and monotonicity of the sequence $f_1, \dots, f_n$. In the present paper we use them to solve a histopolation problem.

0. Introduction. In the present paper we consider some graphic applications of interpolating weighted mean functions [**1, 5**], which interpolate a set of real values $f_i = f(x_i)$, $i = 1, \dots, n$ at distinct points $x_1, \dots, x_n \in I \subset R$.

In the first section we introduce weights of interpolating means in a Walsh theorem, in order to get a hybrid scheme for simultaneous approximation and interpolation with a preassigned error.

In the second section we recall some properties of piecewise weighted mean functions, which include functional precision, regularity class in $[x_1, x_n]$, variation diminishing. A further property is that positivity and monotonicity of the sequence $f_1, \dots, f_n$ are preserved. In the third

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section we use a piecewise interpolating mean to solve a histopolation problem.

1. Approximating curves with interpolation constraints. A wide class of weighted mean functions, with interpolation property, is defined in [**1**]. For a given set of real values $f_i = f(x_i)$, $i = 1, \dots, n$, and distinct nodes x_i, $i = 1, \dots, n$, arbitrarily distributed in $I \subset R$, the interpolating mean is given by

$$u(f; x, n) = \sum_{i=1}^{n} f_i p_i(x; n) \tag{1.1}$$

where the weight functions $p_i(x; n)$, $i = 1, \dots, n$ satisfy the conditions:

$$\begin{gathered} p_i(x, n) \geq 0; \qquad \sum_{i=1}^{n} p_i(x; n) = 1; \\ p_i(x_j; n) = \delta_{ij}, \qquad i, j = 1, \dots, n. \end{gathered} \tag{1.2}$$

We consider in particular the weight functions $p_i(x; n)$, which can be represented by the general formula:

$$p_i(x; n) = \frac{|\prod_{\substack{k=1\\k\neq i}}^{n} [\varphi(x) - \varphi(x_k)]|^{\alpha}}{\sum_{j=1}^{n} |\prod_{\substack{k=1\\k\neq j}}^{n} [\varphi(x) - \varphi(x_k)]|^{\alpha}}, \qquad i = 1, \dots, n; \tag{1.3}$$

where $\alpha \in R^+$ and $\varphi(x)$ can be particularized as follows [**1**]:

$$\varphi(x) = x, \qquad x \in I \subset R;$$

in this case the (1.1) becomes the well-known Shepard interpolation formula [**2**];

$$\varphi(x) = \cos x, \qquad x \in [0, \pi);$$
$$\varphi(x) = e^x, \qquad x \in I \subset R;$$

moreover, in general, $\varphi(x)$ can be any function strictly monotone and at least C^1 in I. Let $g(x)$ be a curve which approximates $f(x)$ in $I \subset R$ with a certain approximation degree. In order to get a hybrid scheme $g^*(x)$, which approximates $f(x)$ in I with the same degree of $g(x)$ and simultaneously interpolates $f(x)$ at the distinct points $x_1, \dots, x_n \in I$, we can use the Boolean sum operator [**3**] or, alternatively, a constructive Walsh theorem [**4**, pages 121, 122]. It provides a method for uniform approximation in the complex domain, under a finite number of interpolation conditions, by polynomials. This theorem, for a real function $f(x)$, can be formulated in the following way [**6**]:

Theorem 1.1. *Let the function $f(x)$ be given on the interval $I \subset R$; let $g(x) \in \mathbf{G}$ be a function which approximates $f(x)$ on I. Let $g_i^*(x) \in \mathbf{G}$, $i = 1, 2, \dots, n$, be such that $g_i^*(x_j) = \delta_{ij}$, where $\{x_1, x_2, \dots, x_n\}$ is a given set of distinct points on I.*

Then

$$g^*(x) = g(x) + \sum_{i=1}^{n} [f(x_i) - g(x_i)] g_i^*(x) \tag{1.4}$$

satisfies the conditions

1) $g^*(x_i) = f(x_i)$, $i = 1, 2, \dots, n$.
2) $g^*(x) \in \mathbf{G}$.
3) *If $|f(x) - g(x)| \le \varepsilon$, uniformly with respect to x on I, then*

$$|f(x) - g^*(x)| \le |f(x) - g(x)| + \varepsilon \sum_{i=1}^{n} |g_i^*(x)| \le \varepsilon(1 + M), \tag{1.5}$$

where

$$M = \max_{x \in I} \sum_{i=1}^{n} |g_i^*(x)|.$$

In the original Walsh theorem $g_i^*(x)$ is the i-th fundamental Lagrange polynomial of degree $(n-1)$ and so M depends upon I and $x_1, x_2, \dots, x_n$. In graphic applications it's better if the approximation error in (1.5) does not depend upon I and $x_1, \dots, x_n$.

Setting in Theorem 1.1 $g_i^* = p_i(x; n)$, $i = 1, \dots, n$, conditions 1) and 2) are satisfied and condition 3) holds, with $M = 1$ in (1.5). In this way, we get an approximation error of the hybrid scheme $g^*(x)$ which depends only upon the approximation error of $g(x)$.

To illustrate the behavior of the proposed method, we set in (1.3) $\varphi(x) = x$, $\alpha = 2$ and we particularize $g(x)$ by choosing a Bernstein

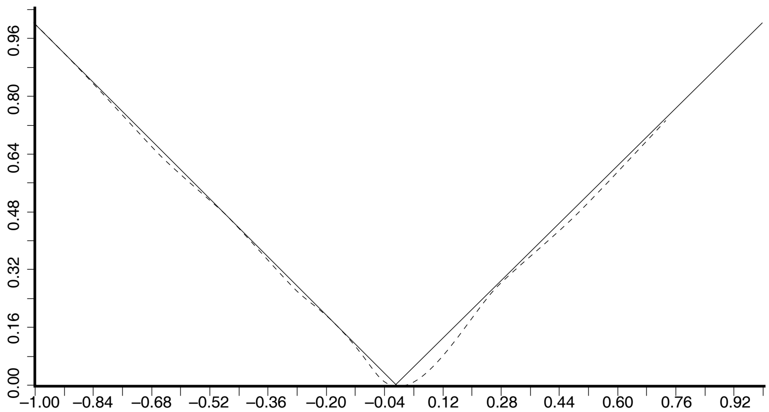

FIGURE 1.

polynomial of degree 50. Figure 1 shows $g^*(x)$, drawn by a dashed line, which approximates $f(x) = |x|$, $x \in [-1, 1]$ and interpolates $f(x)$ at seven distinct points in $[-1, 1]$.

2. Piecewise interpolating mean functions. In [**5**] we introduce the "piecewise mean functions"; to do this, we suppose that the nodes are in increasing order, namely, $x_1 < x_2 < \cdots < x_n$, and we apply a formula of type (1.1) to the pairs of nodes x_i, x_{i+1}, $i = 1, \dots, (n-1)$; the resulting interpolation scheme is

$$u_2(f; x, n) = \sum_{j=i}^{i+1} f_j p_j(x; 2), \qquad i = 1, \dots, (n-1) \tag{2.1}$$

where

$$p_i(x; 2) = \frac{|\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}}{|\varphi(x) - \varphi(x_i)|^{\alpha_i} + |\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}} \tag{2.2}$$

$$p_{i+1}(x; 2) = \frac{|\varphi(x) - \varphi(x_i)|^{\alpha_i}}{|\varphi(x) - \varphi(x_i)|^{\alpha_i} + |\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}} \tag{2.3}$$

with $\alpha_i \in R^+$; we note that the conditions (1.2) are satisfied by the pair of weight functions defined by (2.2) and (2.3) and so $u_2(f; x, n)$

interpolates the values $f_1, \dots, f_n$ and, in each subinterval $[x_i, x_{i+1}]$, is a weighted mean of the values $f_i,\ f_{i+1}$. We can immediately derive that the following properties hold:

$$\min(f_i, f_{i+1}) \le u_2(f; x, n) \le \max(f_i, f_{i+1}), \qquad x \in [x_i, x_{i+1}]; \tag{2.4}$$

and if $f_i = f_{i+1} = c$, then

$$u_2(f; x, n) = c[p_i(x; 2) + p_{i+1}(x; 2)] = c, \qquad x \in [x_i, x_{i+1}]. \tag{2.5}$$

Property (2.4) insures that $u_2(f; x, n)$ preserves the positivity of the sequence $f_1, \dots, f_n$. Property (2.5) insures that $u_2(f; x, n)$ reproduces exactly the constant function in $[x_1, x_n]$.

In [**5**], we prove that for $\alpha > 1, u_2(f; x, n)$ is at least C^1 in $[x_1, x_n]$. Setting in (2.2) and (2.3) $\varphi(x) = x$ and $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$, the interpolating mean becomes the piecewise Shepard formula

$$S_2(f; x, n) = \frac{f_i|x - x_{i+1}|^\alpha + f_{i+1}|x - x_i|^\alpha}{|x - x_i|^\alpha + |x - x_{i+1}|^\alpha}, \qquad x \in [x_i, x_{i+1}]. \tag{2.6}$$

The continuity class of $S_2(f; x, n)$ in $[x_1, x_n]$ can be derived from the following Theorem [**2**] for the univariate Shepard's weights $p_k(x; 2)$, $k = i, i + 1$.

Theorem 2.1. *Let* $0 \le p < \alpha$, *then*

$$D^p p_k(x_j; 2) = \begin{cases} \delta_{k,j}, & p = 0 \\ 0, & 0 < p < \alpha \end{cases}$$

with $k, j = i, i + 1$.

Theorem 2.1 insures that, if $\alpha > 1$, then $S_2(f; x, n)$ has null derivative at the points $x_1, \dots, x_n$ up to order $\lfloor \alpha \rfloor$, where $\lfloor \alpha \rfloor$ is the largest integer less than α, and we can conclude that $S_2(f; x, n)$ is at least $C^{\lfloor \alpha \rfloor}$ in $[x_1, x_n]$;.

Following S. Karlin [**8**] we say that $u_2(f; x, n)$ is variation diminishing in $[x_1, x_n]$ if

$$S^-_{[x_1, x_n]}[u_2(f; x, n)] \le S^-(f_1, f_2, \dots, f_n)$$
$$S^-_{[x_1, x_n]}[u_2(f; x, n)] = \sup S^-[u_2(f; t_1, n), \dots, u_2(f; t_m, n)];$$

where the supremum is extended over all sets $t_1 < t_2 < \cdots < t_m (t_j \in [x_1, x_n])$, m is arbitrary but finite and $S^-(y_1, y_2, \dots, y_n)$ is the number of sign changes of the indicated sequence, zero terms being discarded.

We state the following theorem.

Theorem 2.2. *The piecewise interpolating mean function $u_2(f; x, n)$ is variation diminishing in $[x_1, x_n]$ and it holds*

$$S^-_{[x_1,x_n]}[u_2(f; x, n)] = S^-(f_1, \dots, f_n).$$

A proof of this theorem can be found in [**5**]. Finally, $u_2(f; x, n)$ preserves the monotonicity of the sequence $f_1, \dots, f_n$ by virtue of the following proposition, proved in [**5**].

Proposition 2.1. *If $f_1, \dots, f_n$ are monotonic, then $u_2(f; x, n)$ is monotonic in $[x_1, x_n]$.*

3. Histopolation by piecewise Shepard formula. To give a graphic application of the piecewise means, we use $S_2(f; x, n)$ to solve the following histopolation problem [**7**]. Let $F = \{F_1, \dots, F_N\}$ be a histogram, where F_i is the frequency for the uniform class interval

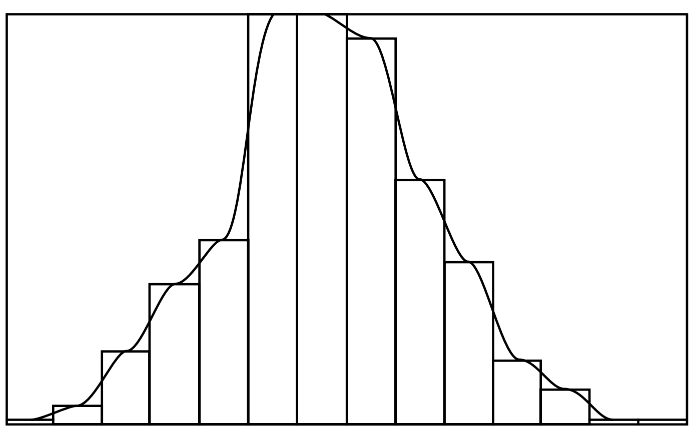

FIGURE 2. $S_2(F; x, 16)$, $\alpha = 1.5$.

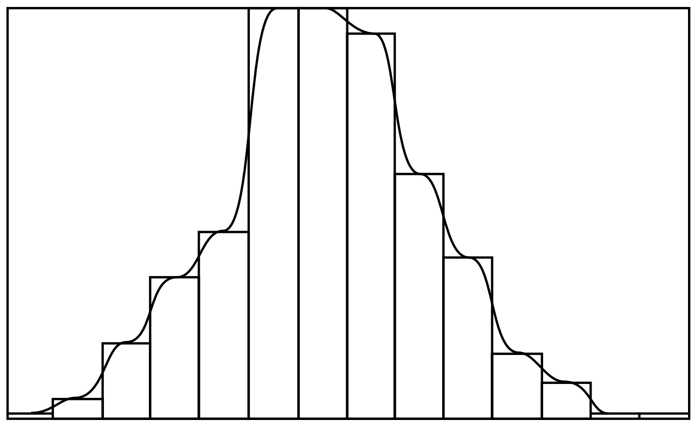

FIGURE 3. $S_2(F; x, 16)$, $\alpha = 2$.

$[X_{i-1}, X_i]$, with $X_i - X_{i-1} = h$, $i = 1, \dots, N$. In many practical applications one is interested in the construction of a function s, at least continuously differentiable in $[X_0, X_N]$, which satisfies the area matching condition

$$\int_{X_0}^{X_N} s(x)\, dx = h \sum_{i=1}^{N} F_i \tag{3.1}$$

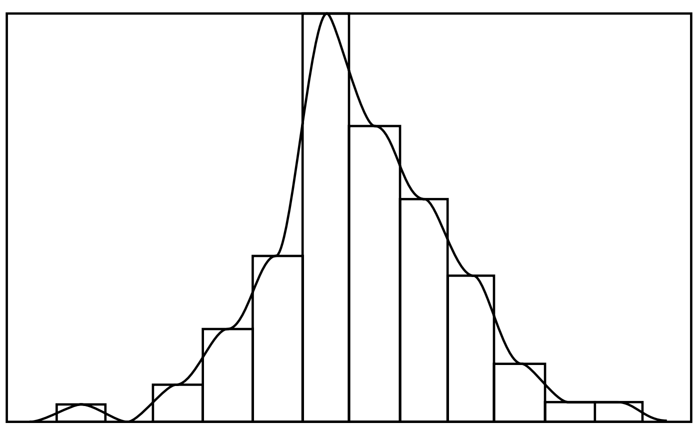

FIGURE 4. $S_2(\overline{F}; x, 16)$, $\alpha = 1.5$.

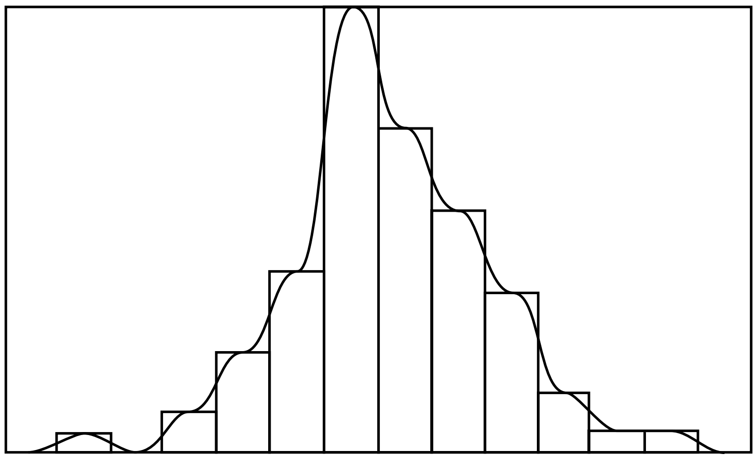

FIGURE 5. $S_2(\overline{F};x,16)$, $\alpha=2$.

and which, in addition, reflects the shape of the histogram. For example, monotonicity and positivity should be carried over from the given data F to the approximating function s.

Let $S_2(F;x,n)$, with $n=N+2$, be the piecewise Shepard formula for the set of data

$$\begin{aligned} x_1 &= X_0, \qquad f_1=F_1; \qquad x_2=\frac{X_0+X_1}{2}, \qquad f_2=F_1;\\ x_3 &= \frac{X_1+X_2}{2}, \qquad f_3=F_2;\cdots;\\ x_i &= \frac{X_{i-2}+X_{i-1}}{2}, \qquad f_i=F_{i-1};\ldots;\\ x_{N+1} &= \frac{X_{N-1}+X_N}{2}, \qquad f_{N+1}=F_N; \qquad x_{N+2}=X_N, \qquad f_{N+2}=F_N. \end{aligned}$$

In [**7**] we prove that the condition (3.1) is satisfied for $s(x) = S_2(F;x,n)$. The histopolation curve $S_2(F;x,n)$ with $\alpha>1$ is at least $C^{\lfloor\alpha\rfloor}$ in $[x_1,x_n]$, by virtue of Theorem 2.1, and carries over from the given data F positivity and monotonicity by virtue of (2.4) and Proposition 2.1, respectively.

TABLE 1.

i	X_i	F_i	$\overline{X}_i$	$\overline{F}_i$
0	10.5		62.5	
1	11.5	1	63.5	0
2	12.5	5	64.5	1
3	13.5	20	65.5	0
4	14.5	38	66.5	2
5	15.5	50	67.5	5
6	16.5	110	68.5	9
7	17.5	110	69.5	22
8	18.5	104	70.5	16
9	19.5	66	71.5	12
10	20.5	44	72.5	8
11	21.5	18	73.5	3
12	22.5	10	74.5	1
13	23.5	1	75.5	1
14	24.5	1	76.5	0

Table 1 gives the extreme points of class intervals and the relevant frequencies for the two histograms F and $\overline{F}$, which are taken from [**9**]. The graphs of Figures 2 and 3 show the histopolation curve $S_2(F; x, 16)$, respectively, with $\alpha = 1.5$ and $\alpha = 2$. The graphs of Figures 4 and 5 show $S_2(\overline{F}; x, 16)$, respectively, with $\alpha = 1.5$ and $\alpha = 2$.

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Dipartimento di Matematica, Universitá di Torino, Via Carlo Alberto 10, 10123 Torino, Italy