ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

IRRATIONAL SUMS

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1. Introduction. In this note we give some sufficient conditions for the irrationality of the sum of the series $\sum_{n=1}^{\infty} 1/H(f(n))$, where $(H(k))_{k\geq 0}$ is a sequence of integers, positive from some point on, satisfying a homogeneous linear recurrence relation with integer coefficients, and f is a strictly increasing function from the set of positive integers to the set of nonnegative integers.

We will refer to such a sequence $(H(k))_{k\geq 0}$ simply as a "recurrent sequence," and the symbol f will always denote a strictly increasing function from the set of positive integers to the set of nonnegative integers.

Let us agree that the symbol $\sum 1/H(f(n))$ denotes the summation of all those terms $1/H(f(n))$ for which $H(f(n)) > 0$.

All of our results are based on the following theorem of C. Badea [**1**].

Theorem A (Badea [**1**]). *If $(a_k)_{k\geq 0}$ is a sequence of positive integers such that $a_{k+1} > a_k^2 - a_k + 1$ for all sufficiently large k, then $\sum 1/a_k$ is irrational.*

A simple example to show that the converges of Badea's Theorem A is false is the series $\sum 1/n! = e$. Another easy example to see that the converse of Badea's result is false is the following. Let $\{c_n\}$, $n \geq 1$, be a nonperiodic sequence of 2s and 5s, and let $a_n = 10^n/c_n$, $n \geq 1$. Then $\sum 1/a_n$ is irrational, and $a_{n+1} \leq a_n^2 - a_n + 1$, $n \geq 3$.

Thus our goal is to find simple conditions on $H(k)$ and $f(n)$ which ensure that $H(f(n+1)) > H(f(n))^2 - H(f(n)) + 1$ for all sufficiently large n.

To avoid complications, *from now on we will always assume that the characteristic polynomial of the recurrent sequence $H(k)$ has a unique*

Received by the editors on July 20, 1992, and in revised form on January 25, 1994.

(real) root $\beta > 1$ *of maximum modulus.*

It then follows from standard properties of recurrence relations (see, for example [**6**]) that there exist numbers $A > 0$ and $c \geq 0$ such that $\lim_{k\to\infty} H(k)/(k^c\beta^k) = A$. (If β is a root of multiplicity 1, then $c = 0$.)

2. Main results.

Theorem 1. *If* $f(n+1)-2f(n) \to \infty$ *as* $n \to \infty$ *and* $f(n+1) \geq f(n)^2$ *for all sufficiently large* n*, then* $\sum 1/H(f(n))$ *is irrational for every recurrent sequence* $H(k)$*.*

Proof. Assume that $H(k)/k^c\beta^k \to A$ as $k \to \infty$ (where $\beta > 1$, $A > 0$ and $c \geq 0$). To apply Badea's result, we need to show that $H(f(n+1))/(H(f(n))^2 - H(f(n)) + 1) > 1$ for sufficiently large n. We do this by dividing the numerator and denominator of the lefthand side of this inequality by $f(n+1)^c\beta^{f(n+1)}$.

Since $H(f(n+1))/f(n+1)^c\beta^{f(n+1)} \to A > 0$ as $n \to \infty$, then $H(f(n+1))/f(n+1)^c\beta^{f(n+1)} > (2/3)A$ for all sufficiently large n.

Next,

$$\begin{aligned}\frac{H(f(n))^2 - H(f(n)) + 1}{f(n+1)^c\beta^{f(n+1)}} \\ &= \frac{f(n)^{2c}}{f(n+1)^c}\frac{1}{\beta^q}\left(\frac{H(f(n))^2}{f(n)^{2c}\beta^{2f(n)}} - \frac{H(f(n))}{f(n)^{2c}\beta^{2f(n)}}\right) \\ &\quad + \frac{1}{f(n+1)^c\beta^{f(n+1)}},\end{aligned}$$

where $q = f(n+1) - 2f(n)$. Since the expression inside the large brackets converges to A^2 and the other term converges to 0, for sufficiently large n (using also $f(n)^{2c}/f(n+1)^c \leq 1$)

$$\frac{H(f(n))^2 - H(f(n)) + 1}{f(n+1)^c\beta^{f(n+1)}} < \beta^{-q}(A^2 + 1) + (1/3)A.$$

Finally,

$$\frac{H(f(n+1))}{H(f(n))^2 - H(f(n)) + 1} > \frac{(2/3)A}{\beta^{-q}(A^2+1) + (1/3)A} > 1,$$

as required. □

Corollary 1. *For every recurrent sequence $H(k)$, $\sum 1/H(2^{2^n})$ is irrational.*

For the next result, we weaken the condition on f and strengthen the condition on $H(k)$.

Theorem 2. *If $f(n+1) - 2f(n) \to \infty$ as $n \to \infty$, then $\sum 1/H(f(n))$ is irrational for every recurrent sequence $H(k)$ for which β has multiplicity 1. (Recall that $\beta > 1$ is the unique root of maximum modulus of the characteristic polynomial of $H(k)$.)*

Proof. The proof of Theorem 1, with c set equal to 0 throughout, gives a proof of Theorem 2. □

Corollary 2. *Let $H(k)$ be a recurrent sequence for which β has multiplicity 1. Then for every $\varepsilon > 0$, $\sum 1/H([(2+\varepsilon)^n])$ is irrational. For every $0 < \varepsilon < 1$, $\sum 1/H(2^n - [(2-\varepsilon)^n])$ is irrational.*

Theorem 3. *Let $H(k)$ be a recurrent sequence for which β has multiplicity 1. Then there exists an integer P such that for every pair of fixed integers s, p with $s > 0$, $-\infty < p \le P$, $\sum 1/H(s2^n + p)$ is irrational.*

Proof. Assume that $H(k)/\beta^k \to A$ as $k \to \infty$, where $\beta > 1$ and $A > 0$. Let s, p be given with $s > 0$ and $p < -\log A/\log\beta$. Let $f(n) = s2^n + p$, $n \ge 1$. Since $f(n+1) - 2f(n) = -p$,

$$\frac{H(f(n))^2 - H(f(n)) + 1}{\beta^{f(n+1)}} = \frac{1}{\beta^{-p}}\left(\frac{H(f(n))^2}{\beta^{2f(n)}} - \frac{H(f(n))}{\beta^{2f(n)}}\right) + \frac{1}{\beta^{f(n+1)}} \to \beta^p A^2.$$

Thus, since $H(f(n+1))/\beta^{f(n+1)} \to A$,

$$\frac{H(f(n+1))}{H(f(n))^2 - H(f(n)) + 1} \to \frac{1}{\beta^p A}.$$

Since $\beta^p A < 1$ by the choice of p,

$$\frac{H(f(n+1))}{H(f(n))^2 - H(f(n)) + 1} > 1$$

for sufficiently large n, and therefore $\sum 1/H(f(n)) = \sum 1/H(s2^n + p)$ is irrational, by Badea's theorem. □

3. Remarks. For the Fibonacci sequence $F(k)$, where

$$F(0) = 0, \qquad F(1) = 1, \qquad F(k+2) = F(k+1) + F(k), \quad k \geq 0,$$
$$F(k) = (1/\sqrt{5})(((1+\sqrt{5})/2)^k - ((1-\sqrt{5})/2)^k),$$
$$\beta = (1+\sqrt{5})/2, \qquad A = 1/\sqrt{5},$$

$-\log A/\log\beta = 1.67\ldots$. Thus, according to the proof of Theorem 3, $\sum 1/F(s2^n + p)$ is irrational for every fixed pair of integers $s > 0$ and $p \leq 1$. This is a generalization of a result of C. Badea [**1**], who showed, answering a question of Erdös and Graham [**2**], that $\sum 1/F(2^n + 1)$ is irrational.

More generally, let $H(0) = 0$, $H(1) = 1$, $H(k+2) = aH(k+1) + bH(k)$, $k \geq 0$, where $a \geq 1$, $b \geq 1$. Then $H(k) = (1/\sqrt{a^2+4b})(((a + \sqrt{a^2+4b})/2)^k - ((a - \sqrt{a^2+4b})/2)^k)$, $\beta = (a + \sqrt{a^2+4b})/2$, $A = 1/\sqrt{a^2+4b}$, and $\beta^p A < 1$ for $p \leq 1$, so again $\sum 1/H(s2^n + p)$ is irrational for every fixed pair of integers $s > 0$ and $p \leq 1$. This extends a result of Kuipers [**4**], who showed this in the case $b = 1$ and $p = 0$. (One can relax the requirement $a \geq 1$, $b \geq 1$ to $a = 1$, $b \geq 1$ or $a \geq 2$, $a^2 + 4b > 0$. In these cases $A < 1 < \beta$, so that $\beta^p A < 1$ holds for $p \leq 0$ and $\sum 1/H(s2^n + p)$ is irrational for $s > 0$ and $p \leq 0$.)

If $a^2 + 4b < 0$, so that the characteristic polynomial $x^2 - ax - b$ of the sequence $H(k)$ no longer has a unique root of maximum modulus, it is easy to verify that the sequence $H(k)$ has infinitely many negative terms, for any nontrivial initial values $H(0), H(1)$. For such a sequence the present methods give no information about the irrationality of $\sum 1/H(f(n))$ for any function f.

Some examples of polynomials for which $\beta > 1$ and b has multiplicity 1 (β is the unique root of maximum modulus of the given polynomial) are discussed in Hua and Wang [**3**], including the polynomials $x^d - x^{d-1} - \cdots - x - 1$, $d \geq 2$, (which come from the generalized Fibonacci sequences $F(0) = F(1) = \cdots F(d-2) = 0$, $F(d-1) = 1$, $F(k+d) = F(k+d-1) + F(k+d-2) + \cdots + F(k+1) + F(k)$, $k \geq 0$), $x^d - Lx^{d-1} - 1$, $d \geq 2$, $L \geq 2$, and $x^t - t^2 r^{t-1} x^{t-1} + (-1)^{t-2} A_{t-2} r^{t-2} x^{t-2} + \cdots - A_1 r x - 1 = 0$, $t \geq 2$, where

$$A_1 = \binom{2t}{1}, \qquad A_k = \binom{2t}{k} - A_1 \binom{2t-2}{k-1} - \cdots - A_{k-1} \binom{2t-2k+2}{1},$$

$t - 2 \geq k > 1$, and the positive integer r satisfies $t^2 > 2/r^{t-1} + |A_1|/r^{t-2} + \cdots + |A_{t-2}|/r$.

Acknowledgment. We are grateful to the referee for several helpful remarks.

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