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C^*-ALGEBRAS GENERATED BY COMMUTING ISOMETRIES

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ABSTRACT. C^*-algebras generated by commuting isometries are analyzed. It is shown that if a C^*-algebra is generated by a semigroup of commuting isometries whose range projections commute, then the C^*-algebra is nuclear. Not all C^*-algebras generated by commuting isometries are nuclear—the universal C^*-algebra generated by a commuting pair of isometries is shown to be nonnuclear.

1. Introduction. If G is a group, then—as is well known—its unitary representations correspond to the representations of the (full) group C^*-algebra $C^*(G)$. Thus, the algebras $C^*(G)$ can be used to reduce the representation theory of groups to that of a special case of the representation theory of C^*-algebras. Of course, the algebras $C^*(G)$ are important in their own right also, since they—and the corresponding reduced group C^*-algebras $C^*_{\mathrm{red}}(G)$—provide interesting examples in the theory of C^*-algebras. Indeed, the study of group C^*-algebras has played a significant role in the development of the general theory of C^*-algebras.

In analogy with the group case, one can associate with each cancellative semigroup M a C^*-algebra $C^*(M)$ that reflects the isometric representation theory of M (that is, the representations of M by isometries on Hilbert spaces). An early study of the algebras $C^*(M)$ was undertaken by R.G. Douglas in the special case that M is the positive cone of a subgroup of the additive group $\mathbf{R}$, see [**4**]. The more general analysis of C^*-algebras generated by commuting isometries undertaken by C.A. Berger, L.A. Coburn and A. Lebow in [**1**] is particularly relevant to the considerations of this paper. More recently, the author has made a detailed study of semigroup algebras for the case in which the semigroup M is the positive cone of an ordered group [**9, 10, 11**]. In this case $C^*(M)$ is primitive and nuclear and has many other nice properties, some of which are discussed below.

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A motivation for considering semigroup C^*-algebras and, more generally, isometric representations of semigroups, is that they arise naturally in the study of Toeplitz operators on generalized Hardy spaces and provide important tools for the analysis of this class of operators, see [**9, 11, 12, 13, 14**]. Semigroup C^*-algebras are particularly relevant to the index theory of Toeplitz operators [**11, 14**]. The connection with Toeplitz operators is discussed in more detail below, as it is needed for our consideration in this paper.

Semigroup C^*-algebras can be set in a much more general context than that considered here. They are special cases of crossed products introduced and studied by the author in [**10**] and [**15**]. These two papers involve a detailed analysis of the properties of crossed products of C^*-algebras by semigroups of their automorphisms. Some of the results of the present paper answer a question of nuclearity of semigroup C^*-algebras raised in [**15**].

Aside from having applications in operator theory and connections with generalized crossed products, the semigroup C^*-algebras $C^*(M)$ form an interesting class of C^*-algebras in their own right and have some rather surprising properties. For an abelian group G, the C^*-algebra $C^*(G)$ is, of course, commutative, but for an abelian semigroup that is not a group, the corresponding C^*-algebra is *not* commutative and can have a very complicated structure, as will be evident from examples discussed below.

The analysis we undertake in this paper involves the study not only of universal semigroup C^*-algebras $C^*(M)$, but of all C^*-algebras generated by abelian semigroups of isometries. In other words, we analyze general isometric representations of abelian semigroups. This analysis is initiated in Section 3, where a tensor algebra construction is discussed that provides a powerful tool in the study of isometric representations. A detailed study of this construction is undertaken in Section 4, and the technical analysis that results provides the basis for some of our principal results.

The tensor construction is used to show that the C^*-algebra generated by an isometric representation, $x \mapsto W_x$, is nuclear if the range projections $W_x W_x^*$ of the isometries W_x commute.

In another application of the tensor construction, it is shown that the universal C^*-algebra generated by a commuting pair of isometries is not nuclear, nor even sub-nuclear. The universal C^*-algebra generated by a commuting pair of isometries is the semigroup C^*-algebra $C^*(\mathbf{N}^2)$. The non-nuclearity result for it should be compared with the corresponding result for the universal C^*-algebra generated by two unitaries. This is the C^*-algebra $C^*(\mathbf{F}_2)$, where $\mathbf{F}_2$ is the free group on two generators. It is well-known that $C^*(\mathbf{F}_2)$ is not nuclear. The "bad" behavior of $C^*(\mathbf{F}_2)$ reflects the fact that $\mathbf{F}_2$ is not very well behaved (specifically, $\mathbf{F}_2$ is not amenable) and it is, perhaps, not surprising that $C^*(\mathbf{F}_2)$ should be a complicated object. However, the semigroup $\mathbf{N}^2$ is not complicated nor is it badly behaved, yet its algebra $C^*(\mathbf{N}^2)$ is quite complicated.

The principal results of this paper center around the tensor algebra representation mentioned above. It appears likely that the analysis made of it here will form a basis for further progress in the study of isometric representations of semigroups. In particular, the tensor representation may be useful in determining necessary and sufficient conditions on a semigroup to ensure that its C^*-algebra is primitive—this is a question that the author hopes to pursue elsewhere.

We indicate now how the paper is organized. Section 2 is a preliminary one that discusses some useful background material from the theory of semigroups. In Section 3 the tensor construction is developed and in Section 4 it is further analyzed in detail. One of the principal results of the paper, Theorem 4.8, is obtained as an application of this analysis (this theorem is the nuclearity result for isometric representations with commuting range projections that has already been mentioned). In Section 5 some properties of semigroup C^*-algebras are discussed; in particular, conditions are given that ensure the tensor representation is faithful. Finally, in Section 6, it is shown that $C^*(\mathbf{N}^2)$ is not nuclear.

2. Semigroups and groups with an order structure. In pursuing our objective of analyzing the isometric representations of a semigroup, the tensor algebra construction that we obtain in the next section plays a fundamental role. The construction appears to be possible only on condition that a certain (very mild) restriction is placed on the semigroup, namely, that it should admit an order unit. In this preliminary section we discuss this concept and some other material concerning semigroups that we shall need, including the well-known correspondence between semigroups and groups with an order structure.

We begin by making a convention on terminology:

Henceforth we shall use the term semigroup *to mean a cancellative abelian semigroup having a zero element.*

We shall usually write semigroups additively. A *subsemigroup* of a semigroup is a subset closed under the operation and containing the zero element.

Recall that a *pre-order* is a reflexive, transitive relation and that a *pre-ordered group* is a pair $(G, \leq)$ consisting of a discrete abelian group G and a translation-invariant pre-order $\leq$ on G. We shall also assume that $G = G^+ - G^+$, where G^+ is the *positive cone* of G, that is, the set of all elements x in G such that $x \geq 0$.

If the relation $\leq$ is also antisymmetric, in which case $\leq$ is a partial order, the pair $(G, \leq)$ is called a *partially ordered group*.

If G is a pre-ordered group, then G^+ is a semigroup. In the reverse direction, if M is a semigroup, then M is the positive cone of a pre-ordered group G. To see this, take G to be the Grothendieck enveloping group of M and define a pre-order on G by setting $x \leq y$ if $y - x \in M$. We shall need to refer frequently to this pre-order.

Note that G is partially ordered if and only if the only element of M having an additive inverse is the zero element.

If a partially ordered group G has the property that all of its elements are comparable, that is, for every pair of elements x, y we have $x \leq y$ or $y \leq x$, then G is called simply an *ordered group*.

Of course, all subgroups of $\mathbf{R}$ are ordered groups. More general examples can be obtained by taking finite products of such groups with the lexicographic order. An ordered group is automatically torsion-free, and every torsion-free abelian group admits an order relation making it an ordered group [**8**]. Note also that, in general, an abelian group may admit quite different order structures.

It is clear from the preceding observations that partially ordered groups and ordered groups exist in vast abundance. In consequence, their positive cones provide us with a correspondingly large supply of semigroups.

We consider now a special class of semigroups discussed by Berger, Coburn and Lebow in [**1**]—these authors were interested in this class in connection with the theory of isometric representations.

If M is a semigroup, it is said to be *boundedly generated* if it is generated by a set of elements, all of which are majorized by some fixed element z of M. In this case we shall call z a *generator bound.*

If M is equal to the additive semigroup $\mathbf{R}^+$, then M is boundedly generated by the set of elements in the open interval $(0, 1)$ and 1 is a generator bound. If $M = \mathbf{N}^n$, then M is boundedly generated by the usual basis elements $e_i = (\delta_{ij})_{j=1}^n$, $i = 1, \ldots, n$, and $e = (1, \ldots, 1)$ is a generator bound. Similarly, any finitely-generated semigroup is boundedly generated.

Not every semigroup is boundedly generated. A counterexample is given by $M = \mathbf{N}^{(\infty)}$, the direct sum of countably-infinitely many copies of $\mathbf{N}$. It is easily checked that no generating set for this semigroup is bounded.

We shall see (in the next section) that the tensor representation introduced by Berger, Coburn and Lebow for boundedly-generated semigroups exists in a more general situation. Instead of using a generator bound as they do in [**1**], we use the (perhaps more natural) idea of an order unit. Not only is the theory thereby obtained more general, it also leads to an approach that has great technical advantages—for instance, it facilitates the analysis of Section 4 that is of fundamental importance for a number of our applications of the tensor representation (the results of Section 4 have no analogue in the analysis undertaken in [**1**]).

If M is a semigroup, we say that the element z of M is an *order unit* if for each element x of M there exists a positive integer n such that $x \leq nz$. If G is the enveloping group of M, then z is an order unit for M if and only if z is an order unit for G in the usual sense used in the theory of groups with an order structured. (Order units play a useful role in the theory of dimension groups, in connection with the K-theoretic classification of AF algebras, see [**2**] and [**5**].)

It is clear that if z is a generator bound for M, it is an order unit. However, the converse is false, as we shall see presently. Nevertheless, the converse does hold in great generality as the following easy proposition shows.

Before proceeding to the proposition we need a definition: We say that M has the *Riesz interpolation property* if for any elements x_i, y_j of M with $x_i \leq y_j$, $i, j = 1, 2$, there exists an element $t \in M$ such that $x_i \leq t \leq y_j$, $i, j = 1, 2$. If G is the enveloping group of M, then M has the interpolation property if and only if G has the interpolation property (in the usual sense) as a pre-ordered group. It follows that if G is an ordered group or a dimension group, then M has the interpolation property [**5**, p. 16].

Proposition 2.1. *Let M be a semigroup having the Riesz interpolation property. Then an element z of M is a generator bound if and only if it is an order unit.*

Proof. It suffices to show that for any elements x, z of M, if $x \leq nz$, then x is a sum of elements $y_i \in M$ such that $y_i \leq z$. First, consider the case where $n = 2$, that is, suppose that $x \leq 2z$. Then $0, x - z \leq x, z$, so, using the interpolation property in the enveloping group G, there exists an element y of G such that $0, x - z \leq y \leq x, z$. Hence, $y, x - y \in M$ and $x = y + x - y$ with $y, x - y \leq z$. The general case follows from a similar argument and induction on n. $\square$

The simple argument given in the preceding proof is standard in the theory of interpolation groups. We have included it for the sake of completeness.

Now we give an example of a semigroup that is not boundedly generated but nevertheless possesses an order unit.

Example 2.2. Let z be the sequence defined by $z_n = 2n$ for $n \in \mathbf{N}$, and let M be the set of all sequences x of nonnegative integers having the following properties:

(1) There exists an integer N (dependent on x) such that $x_n \leq N z_n$ for all n;

(2) If $x_n \leq 2n^2$, then x_n is even.

Clearly, M is a semigroup under term-wise addition. Moreover, z is an order unit for M. For, if $x \in M$, then there exists a positive integer N such that $x_n \leq N z_n$ for all $n \in \mathbf{N}$; define $y \in M$ by setting

$y_n = 2(Nz_n - x_n)$ and observe that $x + (x + y) = 2Nz$, so $x \leq 2Nz$ for the pre-order relation $\leq$ of M. (We cannot simply set $y = Nz - x$ in this argument, as $Nz - x$ may not belong to M.)

We show now that M is not boundedly generated. If $x \in M$ and $x \leq nz$, then by Condition 2, x_n is even. Hence, for any element x of the subsemigroup M_n generated by the elements majorized by nz, we likewise have x_n is even. If $n > 0$, define the element y of M by setting $y_k = 0$ if $k \neq n$ and $y_n = 2n^2 + 1$. Since y_n is odd, y cannot belong to M_n, so $M_n \neq M$. Hence, nz is not a generator bound for M. It now follows easily that M can have no generator bound: For if z' were a generator bound, then $z' \leq nz$ for some positive integer n, and this implies that nz is a generator bound, contradicting what we have just proved.

Incidentally, not all semigroups admit an order unit—it is is easily seen that $\mathbf{N}^{(\infty)}$ has no order unit.

3. Isometric representations of semigroups. Let M be a semigroup. An *isometric representation* of M is a pair (H, W), where H is a Hilbert space and $W : x \mapsto W_x$ is a map from M to $B(H)$ such that each operator W_x is an isometry and $W_{x+y} = W_x W_y$ for all $x, y \in M$. This concept is, of course, the analogue for a semigroup of a unitary representation for a group.

It is always the case that a nontrivial isometric representation exists. Indeed, there exists a C^*-algebra $C^*(M)$—the *semigroup C^*-algebra* of M—and an injective isometric representation V of M, with $V_x \in C^*(M)$ for all $x \in M$, having the following universal property: If (H, W) is an isometric representation of M, then there exists a unique $*$-homomorphism φ from $C^*(M)$ to $B(H)$ such that $\varphi(V_x) = W_x$ for all $x \in M$. For details of the construction of $C^*(M)$ see [**10**]. The algebras $C^*(M)$ will be the principal objects of our considerations in Sections 5 and 6.

Suppose that M is a semigroup in which zero is the only additively-invertible element, and let G be the enveloping group of M. Regard G as a discrete topological group and denote its (compact) Pontryagin dual group by $\hat{G}$; also, let m be the normalized Haar measure of $\hat{G}$. If $x \in G$, let ε_x be the character on $\hat{G}$ defined by evaluation at x; thus, $\varepsilon_x(\gamma) = \gamma(x)$. It is well known that the family of elements $(\varepsilon_x)_{x \in G}$

forms an orthonormal basis for the Hilbert space $L^2(\hat{G}, m)$. Denote by $H^2(M)$ the Hilbert subspace having $(\varepsilon_x)_{x \in M}$ as an orthonormal basis. This space behaves like a generalized Hardy space, especially when the order $\leq$ on G is total; in this case much of the function theory of the circle can be carried over to this setting, see [**16**, Chapter 8].

If $f \in L^\infty(\hat{G}, m)$, the *Toeplitz* operator $T_f \in B(H^2(M))$ is defined by $T_f(g) = P(fg)$, for all $g \in H^2(M)$, where P is the projection of $L^2(\hat{G}, m)$ onto $H^2(M)$. As with function theory, much of the Toeplitz operator theory on the H^2 space of the circle extends to this setting, see [**9, 11, 12, 13, 14**]. However, in this paper we are interested in the C^*-algebra $T^*(M)$ generated by the Toeplitz operators $T_f (f \in C(\hat{G}))$ rather than in these operators themselves. It is shown in [**9**] that $T^*(M)$ acts irreducibly on $H^2(M)$.

The algebra $T^*(M)$ is generated by an isometric representation of M and is, therefore, a quotient algebra of $C^*(M)$. To see this, let $W_x = T_{\varepsilon_x}$ for all $x \in M$. Then the elements W_x generate $T^*(M)$ and the map, $W : x \mapsto W_x$, is an isometric representation of M on $H^2(M)$ [**9**]. Hence, W induces a surjective $*$-homomorphism $\varphi : C^*(M) \to T^*(M)$.

We shall return to the algebra $T^*(M)$ and the isometric represenation W again in Sections 4 and 5. In particular, we shall show in Section 4 that $T^*(M)$ is a nuclear C^*-algebra.

Note incidentally that if $M = \mathbf{N}$, then $H^2(M)$ is the usual Hardy space on the unit circle $\mathbf{T}$ and our generalized theory of Toeplitz operators reduces in this case to the usual classical theory.

Our objective now is to obtain a tensor algebra representation for C^*-algebras generated by isometric representations of semigroups with order unit. As mentioned in the Introduction, the construction we undertake will play a fundamental role in the sequel.

Let (H, W) be an isometric representation of a semigroup M admitting an order unit z. Let C be the C^*-algebra generated by W, that is, the C^*-subalgebra of $B(H)$ generated by the isometries W_x, $x \in M$. We apply the Wold-von Neumann decomposition to W_z. Thus, set

$$H_1 = \bigcap_{n=0}^{\infty} W_z^n(H) \quad \text{and} \quad H_2 = H \ominus H_1.$$

Then H_1 and H_2 are invariant spaces for W_z such that its restriction to H_1 is a unitary and its restriction to H_2 is a direct sum of a number of copies of the unilateral shift of multiplicity one. If $x \in M$, then, since W_x and W_z commute, H_1 is invariant for W_x. Moreover, the restriction of W_x to H_1 is a unitary. For there exists an element $y \in M$ and a positive integer n such that $x + y = nz$, so $W_xW_y = W_yW_x = W_z^n$. Hence, the corresponding equations hold for the restrictions of these operators to H_1 and, therefore, since the restriction of W_z to H_1 is invertible, the same is true of the restriction of W_x. Hence, $W_x(H_1) = H_1$. Therefore, H_1 reduces all of the operators W_x and, consequently, H_1 and H_2 are invariant spaces for C. Hence, the identity representation of C is a direct sum of the subrepresentations φ_1 and φ_2 on H_1 and H_2, respectively, obtained by restriction.

Clearly, $\varphi_1(C)$ is commutative, as it is generated by the commuting unitaries $\varphi_1(W_x)$, $x \in M$. We shall have little further interest in φ_1 and we now restrict attention to φ_2.

To avoid trivialities, we suppose that C is noncommutative, which implies that $H_2 \neq 0$.

As we observed above, the operator $\varphi_2(W_z)$ is a direct sum of copies of the unilateral shift, so there exists a Hilbert space L and a unitary U from H_2 onto the Hilbert space tensor product $H^2 \otimes L$ such that $U\varphi_2(W_z)U^* = S \otimes 1$. Here, and in the sequel, H^2 denotes the usual Hardy space on the unit circle $\mathbf{T}$ and S denotes the unilateral shift on the standard orthonormal basis $(\varepsilon_n)_{n\in\mathbf{N}}$ of H^2. We write ψ for the representation of C on $H^2 \otimes L$ given by $\psi(T) = U\varphi_2(T)U^*$ for all $T \in C$.

We shall refer to the pair $(H^2 \otimes L, \psi)$ as a *tensor representation* of C, or of W, *associated* to the order unit z (the use of the term "tensor" is justified by Theorem 3.1 below).

Recall that the *commutator ideal* of a C^*-algebra C is the smallest closed ideal K of C such that C/K is commutative. Equivalently, K is the closed ideal generated by the additive commutators $[T, T'] = TT' - T'T$, $T, T' \in C$.

We shall find it useful to make some more notational conventions. We shall always write E_1 for the rank-one projection $1 - SS^*$ and, more generally, E_n for the projection $1 - S^nS^{*n}$. Henceforth, $\mathbf{A}$ will denote the Toeplitz algebra on H^2, that is, the C^*-subalgebra of $B(H^2)$

generated by all Toeplitz operators T_f with continuous symbols f. Furthermore, we shall always denote the set of compact operators on H^2 by $\mathbf{K}$. As is well known, $\mathbf{K}$ is the commutator ideal of $\mathbf{A}$ [**3**, p. 181].

Theorem 3.1. *Let M be a semigroup and z an order unit. Suppose that (H, W) is an isometric representation of M and that the C^*-algebra C generated by W is noncommutative. Denote by K the commutator ideal of C. Let $(H^2 \otimes L, \psi)$ be a tensor representation of C associated to z. Then there exists a unital C^*-subalgebra B of $B(L)$ such that $\psi(C)$ is a C^*-subalgebra of the C^*-tensor product $\mathbf{A} \otimes B$ and the restriction of ψ to K is an isomorphism of K onto $\mathbf{K} \otimes B$.*

Proof. If an orthonormal basis $(\eta_i)_i$ is fixed in L, then each bounded operator T on $H^2 \otimes L$ has an operator matrix $(V_{ij})_{ij}$ with entries $V_{ij} \in B(H^2)$ given by

$$\langle V_{ij} f, g\rangle = \langle T(f \otimes \eta_j), g \otimes \eta_i\rangle, \qquad f, g \in H^2.$$

Clearly, T commutes with $S \otimes 1$ if and only if each matrix entry V_{ij} commutes with S. In this case, V_{ij} is an analytic Toeplitz operator, so $V_{ij} = T_{f_{ij}}$, for some function f_{ij} belonging to H^∞ [**6**, p. 79].

Let x be an element of M. Since z is an order unit, there exists an element y of M and a positive integer n such that $x + y = nz$. By the preceding paragraph, since W_x and W_y commute with W_z and $\psi(W_z) = S \otimes 1$, the operator matrices of $\psi(W_x)$ and $\psi(W_y)$ are of the form $(T_{f_{ij}})$ and $(T_{g_{ij}})$, where f_{ij} and g_{ij} belong to H^∞. Because

$$W_x W_y = W_y W_x = W_z^n \tag{3.1}$$

we have $\psi(W_x) = \psi(W_y^*)\psi(W_z)^n$ and therefore $f_{ij} = \bar{g}_{ji}\varepsilon_n$. Hence, f_{ij} belongs to the linear span of the vectors $\varepsilon_0, \dots, \varepsilon_n$, so

$$T_{f_{ij}} = \sum_{m=0}^{n} \lambda_{ij}(m) S^m, \tag{3.2}$$

for scalars $\lambda_{ij}(m)$ satisfying the equation $\lambda_{ij}(m)E_1 = E_1 S^{*m} T_{f_{ij}} E_1$. Since the operator matrix $(\lambda_{ij}(m)E_1)_{ij}$ is the matrix of the bounded operator $T = (E_1 S^{*m} \otimes 1)\psi(W_x)(E_1 \otimes 1)$ on $H^2 \otimes L$, the corresponding scalar matrix $(\lambda_{ij}(m))_{ij}$ is the matrix of a bounded operator, $A_m(x)$ say, on L. (The sesquilinear form

$$(\eta, \eta') \mapsto \langle T(\varepsilon_0 \otimes \eta), \varepsilon_0 \otimes \eta' \rangle$$

defines a bounded operator on L whose matrix is $(\lambda_{ij}(m))_{ij}$.) From (3.2) we now get

$$\psi(W_x) = \sum_{m=0}^{n} S^m \otimes A_m(x). \tag{3.3}$$

Let B be the (unital) C^*-subalgebra of $B(L)$ generated by the operators $A_m(x)$, for arbitrary m and x. Because the isometries W_x generate C, the image algebra $\psi(C)$ is a C^*-subalgebra of $\mathbf{A} \otimes B$. Since $S \otimes 1 = \psi(W_z)$, and S generates $\mathbf{A}$, the C^*-algebra $\mathbf{A} \otimes 1$ is contained in $\psi(C)$. For each x and m, we have $E_1 \otimes A_m(x) = (E_1 S^{*m} \otimes 1)\psi(W_x)(E_1 \otimes 1)$, so $E_1 \otimes A_m(x)$ belongs to $\psi(C)$. Hence, the C^*-algebra $E_1 \otimes B$ is contained in $\psi(C)$ and, therefore, as $E_1 \otimes 1 = \psi(1 - W_z W_z^*)$, and $1 - W_z W_z^* \in K$, we have $E_1 \otimes B \subseteq \psi(K)$.

Let $J = \{T \in \mathbf{A} \mid T \otimes B \subseteq \psi(K)\}$. It follows from the inclusion $\mathbf{A} \otimes 1 \subseteq \psi(C)$ that J is a closed ideal in $\mathbf{A}$. The inclusion $E_1 \otimes B \subseteq \psi(K)$ implies that $E_1 \in J$, so $J \cap \mathbf{K} \neq 0$. Hence, since $\mathbf{K}$ is simple, $\mathbf{K} \subseteq J$ and therefore $\mathbf{K} \otimes B \subseteq \psi(K)$.

We claim that $\mathbf{K} \otimes B = \psi(K)$, and to see the reverse inclusion we need only show that the projection $Q = 1 - W_z W_z^*$ generates K as a closed ideal in C (since $\psi(Q) = E_1 \otimes 1$, the containment $\psi(K) \subseteq \mathbf{K} \otimes B$ then follows). Suppose then that I is the closed ideal in C generated by Q. In the quotient algebra C/I the element $W_z + I$ is a unitary and therefore so is $W_x + I$ for all $x \in M$, by Equation (3.1). Hence, C/I is generated by commuting unitaries, implying that C/I is commutative. It follows that $K \subseteq I$. Since $Q \in K$, we get the reverse inclusion also, giving $K = I$, as required. Therefore, $\psi(K) = \mathbf{K} \otimes B$, as claimed.

To see that ψ is injective on K, let φ_1 and φ_2 be the representations of C on H_1 and H_2, as in the remarks preceding this theorem. Since the direct sum $\varphi_1 \oplus \varphi_2$ is the identity representation of C on H, we have $\ker(\varphi_1) \cap \ker(\varphi_2) = 0$. However, ψ is unitarily equivalent to φ_2, so $\ker(\psi) = \ker(\varphi_2)$. Suppose now that $a \in K$ and that $\psi(a) = 0$. Since

$\varphi_1(C)$ is commutative, $\varphi_1(K) = 0$, so a belongs to the kernels of φ_1 and ψ. Therefore, $a = 0$. Thus, ψ is an isomorphism of K onto $\mathbf{K} \otimes B$. $\square$

The tensor representation ψ of C was obtained by Berger, Coburn and Lebow in their very fine paper [**1**] in the case that z is a generator bound for M. The essential idea of the proof of the more general result obtained here, that of using the Wold-von Neumann decomposition to obtain the representation of $\psi(W_x)$ given in (3.3), is taken from [**1**]. That aside, the proof given here is quite different from that given in [**1**].

There is an important technical difference in our development of the tensor representation as compared to that undertaken in [**1**]. We obtained the representation in (3.3) for all elements x of M, whereas in [**1**] this representation is essentially considered explicitly only for elements $x \leq z$. The more general version of (3.3) complicates our proof a little, but it is essential for our analysis, as will be evident at many points in the sequel, especially in Section 4.

Without further development, the usefulness of the tensor representation as it stands appears to be somewhat restricted, although there are some interesting immediate consequences of Theorem 3.1 that we shall see presently. In [**1**] no detailed general analysis of B is undertaken and the description given involving projections and unitaries appears to be of limited usefulness (the priorities in [**1**] are quite different from ours). What is needed in order to get the applications of the tensor representation that we obtain below is an in-depth analysis of B. Such an analysis is undertaken in Section 4.

In a very weak sense, the algebra B can be "explicitly" identified immediately. It is isomorphic to the C^*-subalgebra QCQ of C, where $Q = 1 - W_z W_z^*$, since $B \cong E_1 \otimes B$ and $E_1 \otimes B = \psi(QCQ)$ (the restriction of ψ to QCQ is injective because $QCQ \subseteq K$). Of course, the problem with this identification is that in general we know no more about QCQ than we do about B. (Despite this, the identification of B with QCQ is occasionally of use.)

The following result addresses the question of precisely when the tensor representation is an isomorphism of C onto $\psi(C)$.

Theorem 3.2. *The tensor representation ψ of C is faithful if and only if K is an essential ideal of C.*

Proof. Since ψ is injective on K, we have $K \cap \ker(\psi) = 0$. Hence, if K is essential in C, the ideal $\ker(\psi)$ cannot be nonzero and so ψ is a faithful representation of C.

We now show that $\mathbf{K} \otimes B$ is an essential ideal of $\psi(C)$ and the theorem will clearly follow, as $\psi(K) = \mathbf{K} \otimes B$. Suppose that T is an element of $\psi(C)$ such that $T(\mathbf{K} \otimes B) = 0$, and let $(V_{ij})_{ij}$ be the operator matrix of T. If T' belongs to $\mathbf{K}$, then the equation $T(T' \otimes 1) = 0$ implies that $V_{ij}T' = 0$. Hence, $V_{ij}\mathbf{K} = 0$ and therefore $V_{ij} = 0$, as $\mathbf{K}$ is an essential ideal of $B(H^2)$. Consequently, $T = 0$ and so $\mathbf{K} \otimes B$ is essential in $\psi(C)$, as required. □

Henceforth, we shall write $\hat{E}_n$ for the projection $E_n \otimes 1$ in $\mathbf{K} \otimes B$. If $Q_n = 1 - W_{nz}W_{nz}^*$, then Q_n belongs to K and $\psi(Q_n) = \hat{E}_n$. Since E_{n+1} is the projection of H^2 onto the linear span of the vectors $\varepsilon_0, \dots, \varepsilon_n$, we have $\lim_{n\to\infty} E_n = 1$ in the strong topology and therefore the sequence $(E_n)_n$ is an approximate unit for $\mathbf{K}$. It follows that the sequence $(\hat{E}_n)_n$ is an approximate unit for $\mathbf{K} \otimes B$ and therefore, by Theorem 3.1, $(Q_n)_n$ is an approximate unit for K.

Let $\mathbf{E}$ be the closed linear span of the set $\{E_n \mid n \geq 1\}$. Since the E_n form an increasing sequence of projections, $\mathbf{E}$ is clearly a C^*-algebra, and since all the E_n are of finite rank, $\mathbf{E}$ is contained in $\mathbf{K}$. Of course, an operator T of $B(H^2)$ commutes with all the E_n if and only if it is diagonal with respect to the basis $(\varepsilon_n)_n$ of H^2 and if, in addition, T is compact, then the diagonal sequence converges to zero and therefore T belongs to $\mathbf{E}$. Hence, $\mathbf{E}$ is its own commutant in $\mathbf{K}$.

In the next theorem we relate the algebra B to the commutant in K of the projections Q_n. We given an immediate application of the theorem in Corollary 3.4; it will also be used in Section 4.

Before proceeding to the theorem, we recall a result from elementary linear algebra that will be used in the following proof and elsewhere.

If $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ belong to vector spaces X and Y, respectively, and if $\sum_{m=0}^{n} X_m \otimes Y_m = 0$, then linear independence of the X_m implies that all the Y_m are equal to zero; likewise, linear independence of the Y_m implies that the X_m are equal to zero.

Theorem 3.3. *The tensor representation ψ of C gives an isomorphism from the commutant in K of the projetions $Q_n = 1 - W_{nz}W_{nz}^*$, $n \geq 1$, onto the C^*-tensor product $\mathbf{E} \otimes B$.*

Proof. Since $\psi : K \to \mathbf{K} \otimes B$ is an isomorphism and $\psi(Q_n) = \hat{E}_n$, the statement of the theorem is equivalent to asserting that the commutant in $\mathbf{K} \otimes B$ of the $\hat{E}_n$ is $\mathbf{E} \otimes B$. Hence, we need only show that if T is an element of $\mathbf{K} \otimes B$ commuting with the $\hat{E}_n$, then T belongs to $\mathbf{E} \otimes B$. Since the projections $\hat{E}_n$ form an approximate unit for $\mathbf{K} \otimes B$, we have $T = \lim_{n\to\infty} T_n$, where $T_n = \hat{E}_n T \hat{E}_n$. Clearly, each term T_n commutes with all the projections $\hat{E}_m$, and T will be shown to belong to $\mathbf{E} \otimes B$ if we show that each T_n belongs to $\mathbf{E} \otimes B$. Thus, we may reduce to the case where $T = \hat{E}_N T \hat{E}_N$, for some integer N. It follows that T belongs to the algebraic tensor product $E_N \mathbf{K} E_N \otimes B$ (this is in fact the same as the C^*-tensor product, that is, it is complete, since $E_N \mathbf{K} E_N$ is a finite-dimensional C^*-algebra). We can therefore write T as a sum of elementary tensors

$$T = \sum_{m=0}^{r} A_m \otimes B_m,$$

where the A_m belong to $E_N \mathbf{K} E_N$ and the B_m belong to B. Moreover, we may suppose that the B_m are linearly independent. For each projection E_n,

$$\sum_{m=0}^{r} E_n A_m \otimes B_m = \sum_{m=0}^{r} A_m E_n \otimes B_m,$$

so, by linear independence of the B_m, we have $E_n A_m = A_m E_n$. Since this holds for arbitrary n, all the A_m belong to $\mathbf{E}$. Hence, $T \in \mathbf{E} \otimes B$. □

Corollary 3.4. *If the commutant in K of the projections Q_n is nuclear (in particular, if it is commutative), then C is nuclear.*

Proof. It follows from the theorem that if the hyothesis above holds, then $\mathbf{E} \otimes B$ is nuclear and therefore B is nuclear. Hence, $\mathbf{K} \otimes B$ is nuclear. Consequently, K too is nuclear, by Theorem 3.1. Since C is an extension by K of the commutative—and therefore nuclear—C^*-algebra C/K, it follows that C is nuclear. $\square$

Remark 3.5. One of the interesting consequences of the identification of the commutator ideal K given by the tensor representation ψ is that it shows that K is a stable C^*-algebra. It follows that the center $Z(K)$ of K is trivial. Equivalently, the center of $\mathbf{K} \otimes B$ is equal to zero. For, $Z(\mathbf{K} \otimes B) = Z(\mathbf{K}) \otimes Z(B) = 0$, as $Z(\mathbf{K}) = 0$.

We shall use this remark in Section 5.

A natural question arises from the tensor construction concerning the kind of algebra B is. For instance, is B of Type I? Not surprisingly, the answer is no in general. In fact, B is of Type I if and only if C is. For, if B is of Type I, so is $\mathbf{K} \otimes B$ and, since K is isomorphic to $\mathbf{K} \otimes B$ and since C is an extension by K of the commutative algebra C/K, it follows that C is of Type I. Conversely, since B is isomorphic to the C^*-subalgebra QCQ of C, where $Q = 1 - W_z W_z^*$, if C is of Type I, so is B.

There may be a sense in which it is true that the C^*-algebra generated by a nonunitary isometric representation is rarely of Type I. As support for this contention, consider the semigroup C^*-algebra $C^*(M)$ in the case that M is the positive cone of a finitely-generated ordered group G. It is shown in [**11**] that $C^*(M)$ is of Type I if and only if G is isomorphic (as an ordered group) to the ordered group $\mathbf{Z}^n$, for some n, where $\mathbf{Z}^n$ is endowed with the lexicographic order. Thus, at least in this case, the requirement that $C^*(M)$ be of Type I imposes a very restrictive condition on M.

It would be interesting to characterize, in the general situation, the semigroups whose universal C^*-algebras are of Type I.

An obvious question concerning the tensor representation is whether it is surjective, that is, whether $\psi(C) = \mathbf{A} \otimes B$. The answer is negative in general. More specifically, ψ is not surjective if B is not commutative. For, suppose that $\psi(C) = \mathbf{A} \otimes B$. If T is a nonzero commutator of B, then $1 \otimes T$ belongs to the commutator ideal $\mathbf{K} \otimes B$ of $\psi(C)$ and therefore $\lim_{n\to\infty} \hat{E}_n(1 \otimes T) = 1 \otimes T$; since $T \neq 0$, this implies that $\lim E_n = 1$, so $1 \in \mathbf{K}$, a contradiction. Hence, if $\psi(C) = \mathbf{A} \otimes B$, then

B is commutative.

Some of the preceding remarks indicate that, in general, the algebra B is a complicated one. Moreover, it seems to occur only rarely that B is completely explicitly identifiable. Nevertheless, the tensor representation can be used to obtain many interesting results, some of which we have already seen and more of which we shall see in the following sections. The key to applying the tensor representation is the fact that B can be *partially* identified in many important cases and that such partial identification is often enough to enable C to be effectively analyzed.

We turn, in the following section, to a closer study of the algebra B.

4. Analysis of the tensor representation. In this section we undertake a detailed analysis of the tensor representation. We then give an application of this analysis to show that if all the range projections of the isometries in an isometric representation W of a semigroup commute, then the C^*-algebra generated by W is nuclear.

We begin by fixing some notation:

Throughout this section (*except in Theorems* 4.8 *and* 4.9), M *denotes a semigroup with an order unit* z *and* W *denotes an isometric representation of* M. *The* C^**-algebra generated by* W *is denoted by* C *and is assumed to be noncommutative. Its commutator ideal is denoted by* K. *We use the symbol* ψ *to signify a tensor represenation associated to* W *and* z.

Thus, ψ is a $*$-homomorphism from C into the C^*-tensor product $\mathbf{A} \otimes B$, for a certain unital C^*-algebra B, and the restriction of ψ is an isomorphism of K onto $\mathbf{K} \otimes B$.

We begin our analysis of B by constructing in it a unitary representation of the enveloping group G of M:

Denote by τ the $*$-homomorphism from $\mathbf{A}$ to B mapping S to 1. Then there exists a unital $*$-homomorphism π from $\mathbf{A} \otimes B$ onto B such that for all $T \in \mathbf{A}$ and $T' \in B$ we have $\pi(T \otimes T') = \tau(T)T'$.

Suppose now that $x \in M$ and write $U_x = \pi\psi(W_x)$. Since W_x is an isometry, so also is U_x, and since $x+y = nz$ and therefore $W_x W_y = W_{nz}$ for some $y \in M$ and some positive integer n, we get $U_x U_y = U_z^n = 1$, as $\pi\psi(W_z) = \pi(S \otimes 1) = 1$. Hence, each operator U_x is a unitary. The homomorphism, $x \mapsto U_x$, from M into the unitary group of B clearly extends to a unitary representation U of G in B.

Recall that for each $x \in M$ we may write $\psi(W_x) = \sum_{m=0}^{\infty} S^m \otimes A_m(x)$, where the elements $A_m(x)$ belong to B and all but a finite number are equal to zero. Set $P_m(x) = A_m(x)U_x^*$ and

$$T_x = \sum_{m=0}^{\infty} S^m \otimes P_m(x).$$

Clearly,

$$\psi(W_x) = T_x(1 \otimes U_x).$$

Moreover, $\sum_{m=0}^{\infty} P_m(x) = 1$, since $U_x = \pi\psi(W_x) = \pi(\sum_{m=0}^{\infty} S^m \otimes A_m(x)) = \sum_{m=0}^{\infty} \tau(S^m)A_m(x) = \sum_{m=0}^{\infty} A_m(x)$.

Denote by D the C^*-subalgebra of B generated by the elements $P_m(x)$, where x and m are arbitrary. Obviously, B is generated by $D \cup U_G$.

Theorem 4.1. *The algebra D is invariant under conjugation by the elements of U_G, that is, $U_x D U_x^* = D$ for all $x \in G$.*

Proof. Suppose $x, y \in M$. The equation $W_{x+y} = W_x W_y$ implies that $T_{x+y}\hat{U}_{x+y} = T_x \hat{U}_x T_y \hat{U}_y$, where for $t \in M$, we set $\hat{U}_t = 1 \otimes U_t$. Therefore, using the fact that T_x is an isometry, $T_x^* T_{x+y} = \hat{U}_x T_y \hat{U}_x^*$. Clearly,

$$T_x^* T_{x+y} = \sum_{m=0}^{\infty} S^m \otimes A_m + \sum_{m=1}^{\infty} S^{*m} \otimes B_m, \tag{4.1}$$

for some elements A_m and B_m belonging to D (with all but finitely many A_ms and B_ms equal to zero). Also,

$$\hat{U}_x T_y \hat{U}_x^* = \sum_{m=0}^{\infty} S^m \otimes U_x P_m(y) U_x^*. \tag{4.2}$$

Since the operators $1, S^1, S^{*1}, S^2, S^{*2}, \ldots$ are linearly independent, we may equate corresponding terms in Equations (4.1) and (4.2). Hence,

$U_x P_m(y) U_x^* = A_m$, so $U_x P_m(y) U_x^*$ belongs to D. It follows that $U_x D U_x^* \subseteq D$. The reverse inclusion also holds—it follows from the observation that $x \leq nz$ for some n, together with the equations $U_{nz-x} = U_z^n U_{-x} = U_{-x}$. Hence, $U_x D U_x^* = D$ for all $x \in M$ and therefore all $x \in G$. □

Theorem 4.2. *If D is nuclear, so is C.*

Proof. Conjugation by the unitaries U_x induces an action α of G on D, that is, a homomorphism $\alpha : G \to \operatorname{Aut} D$. Since $D \cup U_G$ generates B, it follows from the universal property of the crossed product that B is a quotient C^*-algebra of $D \times_\alpha G$. Since D is nuclear, therefore, by a well-known result in the theory of crossed products [**7**], $D \times_\alpha G$ is also nuclear (this uses the fact that G is abelian). It follows that B is nucelar. Therefore, C too is nucelar, by a similar argument to that given in the proof of Corollary 3.4. □

Remark 4.3. Let $x, y \in M$. We saw in the proof of Theorem 4.1 that $\hat{U}_x T_y \hat{U}_x^* = T_x^* T_{x+y}$, where $\hat{U}_x = 1 \otimes U_x$. Hence,

$$\hat{U}_x T_y \hat{U}_x^* = \sum_{m,n=0}^{\infty} S^{*m} S^n \otimes P_m(x)^* P_n(x+y).$$

It follows that for $r \geq 0$,

$$U_x P_r(y) U_x^* = \sum_{m,n}{}' P_m(x)^* P_n(x+y),$$

where the symbol $\sum'$ signifies that the summation is over all m, n for which $S^{*m} S^n = S^r$. Clearly, however, if m, n satisfy this condition, then $n = m + r$ and conversely. Hence,

$$U_x P_r(y) U_x^* = \sum_{m=0}^{\infty} P_m(x)^* P_{m+r}(x+y). \tag{4.3}$$

This expression makes explicit the action of G on D and will be used below.

Remark 4.4. For the proof of the next lemma we shall need linear independence of the operators $S^m S^{*n} (m, n \in \mathbf{N})$. To see that this holds, suppose that $\lambda_{m,n}$ are scalars for which

$$\sum_{m,n=0}^{N} \lambda_{m,n} S^m S^{*n} = 0. \tag{4.4}$$

Multiplying on the right by E_1, we get $\sum_{m=0}^{N} \lambda_{m,0} S^m E_1 = 0$ (as $S^* E_1 = 0$) and, similarly, multiplying on the left by E_1, we get $\sum_{n=0}^{N} \lambda_{0,n} E_1 S^{*n} = 0$. Hence, since $\sum_{m=0}^{N} \lambda_{m,0} S^m \varepsilon_0 = 0$, all the coefficients $\lambda_{m,0}$ vanish. Since $\sum_{n=0}^{N} \bar{\lambda}_{0,n} S^n E_1 = 0$, all the numbers $\lambda_{0,n}$ likewise vanish. Therefore, Equation (4.4) becomes $\sum_{m,n=1}^{N} \lambda_{m,n} S^m S^{*n} = 0$. After cancelling S on the left and S^* on the right, we get $\sum_{m,n=0}^{N-1} \lambda_{m+1,n+1} S^m S^{*n} = 0$, and the argument is now completed by induction.

As observed earlier, for each $x \in M$, we have

$$\psi(W_x) = \sum_{m=0}^{\infty} S^m \otimes P_m(x) U_x,$$

where the elements $P_m(x)$ belong to D and all but finitely many are equal to zero. So far, the only explicit restriction we have been able to impose on the $P_m(x)$ is that $\sum_{m=0}^{\infty} P_m(x) = 1$. We show now that in the case that the range projections of the isometries W_x commute, we can say a lot more.

Lemma 4.5. *Let x, y be elements of M.*

(1) *The projection $W_x W_x^*$ commutes with all the projections $W_{nz} W_{nz}^*$, $n \in \mathbf{N}$, if and only if all the elements $P_m(x)$ are projections. In this case, the $P_m(x)$ are orthogonal, that is, $P_m(x) P_n(x) = 0$ if $m \neq n$.*

(2) *If $W_x W_x^*$, $W_y W_y^*$ and $W_{x+y} W_{x+y}^*$ commute with all the projections $W_{nz} W_{nz}^*$, then $P_n(x)$ and $P_n(y)$ commute with $P_m(x+y)$ for all $n, m \in \mathbf{N}$.*

Proof. Suppose that the commutator $[W_x W_x^*, W_{nz} W_{nz}^*] = 0$ for all n. If $T_x = \sum_{m=0}^{\infty} S^m \otimes P_m(x)$, then $T_x T_x^* = \psi(W_x W_x^*)$. Hence,

$[1 - T_xT_x^*, \hat{E}_n] = 0$, since $\hat{E}_n = \psi(1 - W_{nz}W_{nz}^*)$. Therefore, by the characterization given in Theorem 3.3 of the commutant in $\mathbf{K} \otimes B$ of the $\hat{E}_n$, the projection $1 - T_xT_x^*$ belongs to $\mathbf{E} \otimes B$. Choose a positive integer N such that $x \leq Nz$. Then $P_m(x) = 0$ for $m > N$ and we may write $Nz = x + t$ for some element $t \in M$, so $W_{Nz} = W_xW_t$ and therefore $W_{Nz}W_{Nz}^* = W_xW_tW_t^*W_x^* \leq W_xW_x^*$. Hence, $\psi(W_{Nz}W_{Nz}^*) \leq \psi(W_xW_x^*)$, that is, $1 - \hat{E}_N \leq T_xT_x^*$. Therefore, $1 - T_xT_x^* \leq \hat{E}_N$, so $1 - T_xT_x^*$ belongs to $E_N\mathbf{E} \otimes B$. Using the fact that $E_N\mathbf{E}$ is the linear span of the projections $E_1, \ldots, E_N$ and hence is contained in the linear span of the projections S^mS^{*m}, $m \leq N$, we may write two expressions for $T_xT_x^*$, namely,

$$T_xT_x^* = \sum_{m,n=0}^{N} S^mS^{*n} \otimes P_m(x)P_n(x)^*$$

and

$$T_xT_x^* = \sum_{m=0}^{N} S^mS^{*m} \otimes B_m,$$

where the elements B_m belong to B. Since the operators S^mS^{*n}, $m, n \in \mathbf{N}$, are linearly independent (Remark 4.4), corresponding terms in each of the two expressions for $T_xT_x^*$ are equal and therefore $P_m(x)P_n(x)^* = 0$ for $n \neq m$. Since $\sum_{n=0}^{N} P_n(x)^* = 1$, we have $P_m(x) = \sum_{n=0}^{N} P_m(x)P_n(x)^* = P_m(x)P_m(x)^*$. Therefore, the $P_m(x)$ are orthogonal projections.

Suppose conversely that the $P_m(x)$ are projections. Since their sum is equal to 1, they are orthogonal and therefore

$$\psi(W_xW_x^*) = T_xT_x^* = \sum_{m=0}^{\infty} S^mS^{*m} \otimes P_m(x).$$

Hence, $\psi(W_xW_x^*)$ commutes with each $\hat{E}_n$ and so $\psi(1 - W_xW_x^*)$ commutes with $\hat{E}_n = \psi(1 - W_{nz}W_{nz}^*)$. Since $1 - W_xW_x^*$ and $1 - W_{nz}W_{nz}^*$ belong to K and ψ is injective on K, therefore $1 - W_xW_x^*$ and $1 - W_{nz}W_{nz}^*$ commute. Hence, Condition (1) is proved.

Now suppose that x and y are elements of M for which $W_xW_x^*$, $W_yW_y^*$ and $W_{x+y}W_{x+y}^*$ commute with all the projections $W_{nz}W_{nz}^*$.

By Equation (4.3) we have, for each $r \geq 0$,

$$U_x P_r(y) U_x^* = \sum_{n=0}^{\infty} P_n(x) P_{n+r}(x+y).$$

Hence, if $Q = U_x P_r(y) U_x^*$, then, using the orthogonality of the sequence $(P_n(x))_n$ and of $(P_n(x+y))_n$, we have

$$\begin{aligned} Q = QQ^* &= \sum_{n,m=0}^{\infty} P_n(x) P_{n+r}(x+y) P_{m+r}(x+y) P_m(x) \\ &= \sum_{n=0}^{\infty} P_n(x) P_{n+r}(x+y) P_n(x) \end{aligned}$$

and therefore

$$P_n(x) Q = P_n(x) P_{n+r}(x+y) P_n(x) = Q P_n(x).$$

Consequently, $P_n(x)P_{n+r}(x+y)P_n(x)$ is a projection. However, if R and P are projections such that RPR is also a projetion, then R and P commute. (Set $A = RPR - PR$. Then $A^*A = RPR - (RPR)^2 = 0$, so $A = 0$, that is, $PR = RPR$ and therefore $PR = RP$.) Hence, $P_n(x)$ commutes with $P_{n+r}(x+y)$. Thus, the commutator $[P_n(x), P_m(x+y)]$ vanishes if $n \leq m$; that it also vanishes if $n > m$ is a consequence of the following lemma and the observation that $W_{x+y}W_{x+y}^* = W_x W_y W_y^* W_x^* \leq W_x W_x^*$. By symmetry, $[P_n(y), P_m(x+y)]$ vanishes also. This proves Condition (2). $\square$

Lemma 4.6. *Let x and y be elements of M such that $W_xW_x^*$ and $W_yW_y^*$ commute with all the projections $W_{nz}W_{nz}^*$. Then $W_yW_y^* \leq W_xW_x^*$ if and only if $P_m(y)P_n(x) = 0$ whenever $m < n$.*

Proof. By orthogonality, $T_xT_x^* = \sum_{n=0}^{\infty} S^n S^{*n} \otimes P_n(x)$ and $T_yT_y^* = \sum_{n=0}^{\infty} S^n S^{*n} \otimes P_n(y)$. If $W_yW_y^* \leq W_xW_x^*$, then $\psi(W_yW_y^*) \leq \psi(W_xW_x^*)$, that is, $T_yT_y^*T_xT_x^* = T_yT_y^*$. This can be rewritten as

$$(4.5) \qquad \sum_{r=0}^{\infty} S^r S^{*r} \otimes P_r(y) = \sum_{m,n=0}^{\infty} S^m S^{*m} S^n S^{*n} \otimes P_m(y) P_n(x).$$

Hence,

$$P_r(y) = \sum_{m,n}{}' P_m(y)P_n(x), \tag{4.6}$$

where we are using $\sum'$ to indicate that the summation is over all m, n for which $S^m S^{*m} S^n S^{*n} = S^r S^{*r}$, that is, for which $m, n \leq r$ and either $m = r$ or $n = r$. By orthogonality of the sequence $(P_n(x))_n$, if we multiply both sides of (4.6) on the right by $P_N(x)$ with $N > r$, the righthand side vanishes and therefore $P_r(y)P_N(x) = 0$. This proves the forward implication in the statement of the lemma.

Suppose conversely that $P_m(y)P_n(x) = 0$ whenever $m < n$. In this case,

$$\begin{aligned} T_y T_y^* T_x T_x^* &= \sum_{m,n=0}^{\infty} S^m S^{*m} S^n S^{*n} \otimes P_m(y)P_n(x) \\ &= \sum_{r=0}^{\infty} S^r S^{*r} \otimes \sum_{n=0}^{r} P_r(y)P_n(x) \\ &= \sum_{r=0}^{\infty} S^r S^{*r} \otimes P_r(y) \end{aligned}$$

(since $\sum_{n=0}^{\infty} P_n(x) = 1$)

$$= T_y T_y^*.$$

Hence, $\psi(W_y W_y^*) \leq \psi(W_x W_x^*)$. Therefore, $\psi(1 - W_x W_x^*) \leq \psi(1 - W_y W_y^*)$. By injetivity of ψ on K we therefore have $1 - W_x W_x^* \leq 1 - W_y W_y^*$, that is, $W_y W_y^* \leq W_x W_x^*$. This completes the proof of the lemma. □

Theorem 4.7. *If all the projections $W_x W_x^*$ commute, then D is commutative.*

Proof. Suppose that all the $W_x W_x^*$ commute. Then for any elements $x, y \in M$ and all $m, n \in \mathbf{N}$, the projections $P_n(x)$ and $P_m(x + y)$ commute, by Condition (2) of Lemma 4.5.

Suppose that N is a fixed nonnegative integer and $x \in M$. Since $W_{x+Nz} = W_x W_{Nz}$, we have $\psi(W_{x+Nz}) = \psi(W_x)\psi(W_{Nz})$, that is, $T_{x+Nz}(1 \otimes U_{x+Nz}) = T_x(1 \otimes U_x)T_{Nz}(1 \otimes U_{Nz})$. Hence, using the fact that $U_{x+Nz} = U_x$ and then cancelling out the factor $1 \otimes U_x$, we get $T_{x+Nz} = T_x(S^N \otimes 1)$. Thus,

$$\sum_{m=0}^{\infty} S^m \otimes P_m(x+Nz) = \sum_{m=0}^{\infty} S^{m+N} \otimes P_m(x).$$

Hence,

$$P_{m+N}(x+Nz) = P_m(x). \tag{4.7}$$

Suppose now that x and y are arbitrary elements of M. Since z is an order unit, there exists a positive integer N such that $y \leq Nz$. Hence, $y \leq x + Nz$, so $[P_{m+N}(x+Nz), P_n(y)] = 0$ for all $m, n \in \mathbf{N}$, by the observations of the first paragraph of this proof. Therefore, by (4.7), we have $[P_m(x), P_n(y)] = 0$. Thus, D is generated by commuting projections and therefore D is commutative. □

Up to this point our results of this section have assumed the existence of an order unit in M. However, the next two theorems do not—although they are derived from the preceding results.

The following theorem is one of the principal results of the paper.

Theorem 4.8. *Let M be a semigroup, and let W be an isometric representation of M for which all the range projections $W_x W_x^*$ commute. Then the C^*-algebra C generated by W is nuclear.*

Proof. If F is a finite subset of M, let M_F be the subsemigroup of M generated by F. Clearly, $M = \cup_F M_F$ and the family $(M_F)_F$ is upwards-directed, that is, for any two members M_{F_1} and M_{F_2}, there is a third M_F containing both of them. Let C_F be the C^*-subalgebra of C generated by the isometries W_x, $x \in M_F$. Then the family $(C_F)_F$ is also upwards-directed and the union $\cup_F C_F$ is dense in C. Therefore, by a well-known result in the theory of nuclear C^*-algebras, C is nuclear if all the subalgebras C_F are nuclear. Thus, to prove the theorem we may, and do now, assume that M is finitely generated. Obviously, we may also assume that C is not commutative. Hence, since M admits an order unit (because it is finitely generated), we can apply Theorem 4.7 to deduce that the algebra D derived from the tensor representation is commutative. Consequently, D is nucelar and, therefore, by Theorem 4.2, C is nuclear. $\square$

An alternative proof of Theorem 4.8 is possible using the work of H. Salas [**17**]. A C^*-algebra generated by a semigroup of isometries with commuting range projections can be realized as a groupoid C^*-algebra, as Salas showed. The groupoid is a transformation group groupoid determined by an abelian group action cut down to a closed subset. Such groupoids are amenable in Renault's sense and therefore yield nuclear C^*-algebras. The author is grateful to the referee for this observation.

Theorem 4.9. *Let M be a semigroup in which the only element that has an additive inverse is the zero element. Let $T^*(M)$ be the C^*-algebra generated by all Toeplitz operators with continuous symbols on the generalized Hardy space $H^2(M)$. Then $T^*(M)$ is nuclear.*

Proof. If W is the canonical isometric representation, $x \mapsto T_{\varepsilon_x}$, of M on $H^2(M)$, then $T^*(M)$ is the C^*-algebra generated by W, see Section 3. If $x, y \in M$, and $Q_x = W_x W_x^*$, then $Q_x(\varepsilon_y) = \varepsilon_y$ if $x \leq y$ and $Q_x(\varepsilon_y) = 0$ if $x \not\leq y$. Hence, Q_x is diagonal with respect to the standard orthonormal basis $(\varepsilon_y)_{y \in M}$. Consequently, $[Q_x, Q_y] = 0$ for all x and y and, therefore, by Theorem 4.8, $T^*(M)$ is nuclear. $\square$

5. The universal C^*-algebra of a semigroup. We begin the section by discussion some known results, mostly from [**9**], concerning the C^*-algebra $C^*(M)$, for M a semigroup, in order to set up the necessary background and because we shall need to use many of these results.

Firstly, the elements V_x, $x \in M$, generate $C^*(M)$, where V denotes the canonical isometric representation of M in $C^*(M)$. Of course, V_x is not a unitary in general. In fact, it is a unitary if and only if x is additively invertible in M. It is immediate from this that $C^*(M)$ is commutative if and only if M is a group. (In this case, $C^*(M)$ is just the usual group C^*-algebra.)

In general, $C^*(M)$ is highly noncommutative. For instance, if M is equal to the additive semigroup $\mathbf{N}$ of the natural numbers, then $C^*(M)$ is isomorphic to the Toeplitz C^*-algebra $\mathbf{A}$ and therefore $C^*(M)$ is primitive in this case.

As we saw in Section 3, a similar representation of the C^*-algebra $C^*(M)$ in terms of Toeplitz operators holds in great generality. We now consider this representation in a little more detail.

Suppose that M is a semigroup in which zero is the only additively invertible element, and let $\varphi : C^*(M) \to T^*(M)$ be the canonical representation. Let G be the enveloping group of M. There exists a $*$-homomorphism π_T from $T^*(M)$ onto $C(\hat{G})$ such that $\pi_T(T_f) = f$ for all $f \in C(\hat{G})$; the kernel of π_T is the commutator ideal of $T^*(M)$. Moreover, since the map, $x \mapsto \varepsilon_x$, is an isometric representation of M, we have a $*$-homomorphism $\pi_C : C^*(M) \to C(\hat{G})$ such that $\pi_C(V_x) = \varepsilon_x$ for all $x \in M$. The kernel of π_C is the commutator ideal of $C^*(M)$. Clearly, $\pi_C = \pi_T \varphi$.

The representation φ is not faithful in general, but it *is* faithful for a very important class of examples. If the order relation $\leq$ is total on G (equivalently, if for all $x, y \in M$, either $x = y + t$ or $y = x + t$ for some element t of M), then φ is faithful [**9**, Theorem 3.14]. In fact, in this case, $C^*(M)$ has a very strong property: If (H, W) is any isometric representation of M such that W_x is nonunitary for all nonzero x, then the induced representation $\varphi : C^*(M) \to B(H)$ is faithful (this is a kind of uniqueness property for the algebra $C^*(M)$). Not all semigroup C^*-algebras have this "uniqueness" property; see [**9**]. A consequence of faithfulness of the representation $\varphi : C^*(M) \to T^*(M)$ is that $C^*(M)$ is primitive, since (as we observed already in Section 3) $T^*(M)$ acts irreducibly on $H^2(M)$.

If a C^*-algebra is primitive, all its nonzero closed ideals are essential. I do not know whether $C^*(M)$ is necessarily primitive under the assumption that M has no nontrivial, additively-invertible elements; however, in this case, at least one proper ideal is essential, namely the commutator ideal, as the following theorem shows.

Theorem 5.1. *Let M be a nonzero semigroup in which the only element having an additive inverse is the zero element. Then the commutator ideal K of $C^*(M)$ is essential.*

Proof. Let G denote the Grothendieck enveloping group of M, and let $\varphi : C^*(M) \to T^*(M)$ be the canonical representation of $C^*(M)$. Let π_C and π_T be the $*$-homomorphisms from $C^*(M)$ and $T^*(M)$, respectively, onto $C(\hat{G})$ described in the remarks preceding this theorem. To show that K is essential in $C^*(M)$, let a be an element of $C^*(M)$ such that $aK = 0$. Then, denoting by K' the commutator ideal of $T^*(M)$, we have $\varphi(a)K' = 0$, since $\varphi(K) = K'$ (because $\varphi(C^*(M)) = T^*(M)$). However, as $T^*(M)$ is primitive, K' is an essential ideal, and therefore $\varphi(a) = 0$. Hence, $\pi_C(a) = \pi_T\varphi(a) = 0$, so $a \in K$, as $K = \ker(\pi_C)$. Therefore, $aK = 0 \Rightarrow aa^* = 0$, that is, $a = 0$. Thus, K is essential in $C^*(M)$, as required. □

It follows from the preceding theorem and from Theorem 3.2 that if M is nontrivial and admits an order unit and its only additively-invertible element is its zero element, then the tensor representation ψ of $C^*(M)$ associated to the canonical isometric representation V is faithful.

The following result uses the tensor representation.

Theorem 5.2. *Let M be a semigroup admitting an order unit, and let Z be the center of $C^*(M)$. Then $Z = \mathbf{C}1$ if and only if the only element of M that has an additive inverse is the zero element.*

Proof. If x is an additively-invertible element of M, then V_x is a unitary commuting with all the generating elements V_y, $y \in M$, so V_x belongs to Z. Moreover, V_x is nonscalar, if $x \neq 0$, see [**10**]. Hence, if $Z = \mathbf{C}1$, then zero is the only additively-invertible element of M.

Now suppose conversely that M does not admit nontrivial additively-invertible elements and let $a \in Z$. Let φ be the canonical $*$-homomorphism from $C^*(M)$ onto $T^*(M)$. Clearly, $\varphi(a)$ belongs to the commutant of $T^*(M)$ in $B(H^2(M))$, and by irreducibility of $T^*(M)$, this commutant is equal to the set of scalar operators, so $\varphi(a) = \lambda$ for some $\lambda \in \mathbf{C}$. Hence, $a - \lambda \in \ker(\varphi)$. However, since $\pi_C = \pi_T\varphi$, the commutator ideal $K = \ker(\pi_C)$ of $C^*(M)$ contains $\ker(\varphi)$, so $a-\lambda \in K$. Since, by the tensor representation and Remark 3.5, the center of K is trivial, $a = \lambda$. Hence, $Z = \mathbf{C}1$. This proves the theorem. $\square$

6. Nuclearity and semigroup C^*-algebras. If G is a discrete group (not assumed to be abelian), then $C^*(G)$ is nuclear if and only if G is amenable [**7**]. In particular, as we observed earlier, $C^*(\mathbf{F}_2)$ is not nuclear (since $\mathbf{F}_2$ is not amenable).

In analogy with the group situation, let us say that a semigroup M is *amenable* if $C^*(M)$ is nuclear.

Of course, if every element of M is additively invertible, that is, if M is a group, then M is amenable (as $C^*(M)$ is commutative in this case). In the semigroup case amenability of M is tied up with its natural order structure, as the following result illustrates (for more on the connection with the order structure, see [**10**], where, incidentally, essentially the following result was obtained, but by an entirely different method).

Theorem 6.1. *Let M be a semigroup such that for every pair of its elements x, y, there exists an element t such that $x = y+t$ or $y = x+t$. Then M is amenable.*

Proof. If $x = y + t$, then $V_xV_x^* = V_yV_tV_t^*V_y^* \leq V_yV_y^*$, so $V_xV_x^*$ and $V_yV_y^*$ commute. Hence, the hypothesis implies that all the range projections $V_xV_x^*$ commute, and therefore, by Theorem 4.8, the C^*-algebra generated by the isometric representation V, that is, the algebra $C^*(M)$, is nuclear. $\square$

The semigroup $\mathbf{N}^2$ is one of the simplest examples in which there exist elements not comparable with respect to the natural pre-order. We are now going to show that it is nonamenable. The proof involves an explicit identification of the commutator ideal of $C^*(\mathbf{N}^2)$ using the tensor algebra representation.

Theorem 6.2. *The semigroup $\mathbf{N}^2$ is not amenable. Indeed, $C^*(\mathbf{N}^2)$ is not subnuclear (that is, it is not a C^*-subalgebra of a nuclear C^*-algebra).*

Proof. Let B be the C^*-algebra free product of the C^*-algebras $C(\mathbf{T})$ and $\mathbf{C}^2$. We are going to show that the commutator ideal of $C^*(\mathbf{N}^2)$ is isomorphic to $\mathbf{K} \otimes B$. We do this by using the fact that B is the universal C^*-algebra generated by a unitary and a projection. More explicitly, let U be the canonical unitary generator of $C(\mathbf{T})$, that is, U is the inclusion function of $\mathbf{T}$ in $\mathbf{C}$, and let P be the projection $(1,0)$ of $\mathbf{C}^2$. If B' is another C^*-algebra generated by a unitary U' and a projection P', then there exists a $*$-homomorphism from $C(\mathbf{T})$ to B' mapping U onto U', and there exists a unital $*$-homomorphism from $\mathbf{C}^2$ to B' mapping P onto P'. It follows, therefore, from the universal property of a free product, that there is a $*$-homomorphism from B onto B' mapping U and P onto U' and P', respectively.

Now set $W_1 = S \otimes PU + 1 \otimes (1-P)U$ and set $W_2 = S \otimes U^*(1-P) + 1 \otimes U^*P$ and let C be the C^*-subalgebra of $\mathbf{A} \otimes B$ generated by W_1 and W_2. It is easily verified that W_1 and W_2 are commuting isometries and that $W_1W_2 = S \otimes 1$. Reasoning as in the proof of Theorem 3.1, the commutator ideal of C is seen to be $\mathbf{K} \otimes B$.

If V_1 and V_2 are the canonical isometries generating $C^*(\mathbf{N}^2)$, then, by the universal property of $C^*(\mathbf{N}^2)$, there is a $*$-homomorphism φ from $C^*(\mathbf{N}^2)$ onto C mapping V_1 and V_2 onto W_1 and W_2, respectively. We are going to show that φ is an isomorphism by constructing an inverse for it.

Then tensor representation associated to the generator bound $z = (1,1)$ and the canonical isometric representation of $\mathbf{N}^2$ in $C^*(\mathbf{N}^2)$ gives the existence of a C^*-algebra B' containing unitaries U_1 and U_2 and projections P_1 and P_2 such that we may identify V_1 with $S \otimes P_1U_1 + 1 \otimes (1-P_1)U_1$ and V_2 with $S \otimes P_2U_2 + 1 \otimes (1-P_2)U_2$ and identify the C^*-algebra $C^*(\mathbf{N}^2)$ with the C^*-subalgebra $A \otimes B'$ generated by these isometries. (We are using the fact that the tensor representation is faithful in this case; the reason it is faithful is that the commutator ideal is essential in $C^2(\mathbf{N})$, by Theorem 5.1, because the only additively-invertible element of $\mathbf{N}^2$ is the zero element.) Using the fact that $V_1V_2 = S \otimes 1$, an easy argument shows that $U_1U_2 = 1$ and $P_2 = U_1^*(1-P_1)U_1$. Now let ρ be the $*$-homomorphism from B into B' mapping U and P onto U_1 and P_1, respectively. Then the $*$-homomorphism $\mathrm{id} \otimes \rho$ mapping $\mathbf{A} \otimes B$ into $\mathbf{A} \otimes B'$ sends W_1 and W_2 onto V_1 and V_2, respectively. The restriction ψ of $\mathrm{id} \otimes \rho$ to C gives a $*$-homomorphism onto $C^*(\mathbf{N}^2)$; clearly, ψ is the inverse of φ.

Thus, $C^*(\mathbf{N}^2)$ is isomorphic to C and therefore the commutator ideal of $C^*(\mathbf{N}^2)$ is isomorphic to the commutator ideal of C, that is, to $\mathbf{K} \otimes B$. To prove the theorem, we need only show now that B is not subnuclear. To see this, let α be the automorphism of $C^*(\mathbf{F}_2)$ obtained by permuting the canonical unitary generators u_1 and u_2. Observe that the crossed product $C^*(\mathbf{F}_2) \times_\alpha \mathbf{Z}_2$ is generated by u_1 and u_2 and by the symmetry v implementing the automorphism α. Set $q = (v+1)/2$. Since $u_2 = \alpha(u_1) = vu_1v$, the algebra $C^*(\mathbf{F}_2) \times_\alpha \mathbf{Z}_2$ is generated by the unitary u_1 and the projection q. A routine verification using the universal property of the crossed product now shows that $C^*(\mathbf{F}_2) \times_\alpha \mathbf{Z}_2$ is the universal C^*-algebra generated by a unitary and a projection; therefore, this crossed product is isomorphic to B. Thus, we have an embedding of $C^*(\mathbf{F}_2)$ into B; since $C^*(\mathbf{F}_2)$ is not subnuclear [**7**], neither is B. $\square$

If a semigroup M is non-amenable, then any semigroup N admitting M as a quotient (that is, admitting a surjective homomorphism onto M) is also nonamenable. For, if there is a surjective homomorphism from N onto M, then the induced $*$-homomorphism from $C^*(N)$ to $C^*(M)$ is surjective and, since the quotient of a nuclear C^*-algebra is again nuclear, therefore $C^*(N)$ is not nuclear.

Consequently, any semigroup having $\mathbf{N}^2$ as a quotient is not amenable. In particular, $\mathbf{N}^n$ is nonamenable for all integers $n > 2$. Likewise, $\mathbf{N}^{(\infty)}$ is nonamenable. A consequence is that the semigroup $\mathbf{N}^\times$ of positive integers, with multiplication as operation, is not amenable. The reason is that $\mathbf{N}^\times$ is isomorphic to $\mathbf{N}^{(\infty)}$, by the prime factorization theorem.

Here are some easy ways of enlarging the supply of nonamenable semigroups: If M is an arbitrary semigroup, then the product $M \times \mathbf{N}^2$ is nonamenable. If M_1 and M_2 are semigroups having $\mathbf{N}$ as a quotient, then the product semigroup $M_1 \times M_2$ is nonamenable. Of course, in both cases the reason that the product semigroup is nonamenable is that it admits $\mathbf{N}^2$ as a quotient.

An obvious question now presents itself: Does every nonamenable semigroup have $\mathbf{N}^2$ as a quotient? The answer is no, as we shall now show.

Let $M = \{(m,n) \in \mathbf{N}^2 \mid m \neq 1, n \neq 1\}$. Clearly, M is a subsemigroup of $\mathbf{N}^2$. Moreover, if $x \in \mathbf{N}^2$, there exists $y, t \in M$ such that $x + t = y$. It follows that the $*$-homomorphism from $C^*(M)$ to $C^*(\mathbf{N}^2)$ induced by the inclusion homomorphism from M into $\mathbf{N}^2$ is surjective. Hence, $C^*(M)$ is nonnuclear and M is nonamenable. However, M does not have $\mathbf{N}^2$ as a quotient. To see this, we assume the contrary and obtain a contradiction: Let θ be a homomorphism from M onto $\mathbf{N}^2$. Clearly, θ extends to a homomorphism from $\mathbf{Z}^2$ onto itself. By elementary group theory, any such surjective homomorphism must be an isomorphism, so θ itself must be an isomorphism. However, it is easily checked that M cannot be generated by two elements, so M cannot be isomorphic to $\mathbf{N}^2$, a contradiction.

It follows from Theorem 6.1 that if every pair of elements of a semigroup M are comparable relative to its natural pre-order, M is amenable. A possibility that I am not able to rule out is that the converse holds.

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