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ON QUADRATIC SYSTEMS WITH A DEGENERATE CRITICAL POINT

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ABSTRACT. We study phase portraits of quadratic systems with exactly two critical points, one of them degenerate. This problem has already been considered in [**10**], where part of the results are obtained by computer. Here we deal with these systems in terms of semi-complete families of rotated vector fields. This new approach allows us to prove most of the bifurcation diagrams that we obtain.

1. Introduction. A quadratic system, QS, is a system of two real autonomous differential equations

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where the dot indicates derivative in respect to a real variable t, and P and Q are polynomials in two variables with degree at most two and not both with degree less than or equal to one. We denote by $X = (P,Q)$ the vector field associated to (1).

QS gives the simplest example of nonlinear differential equations and also presents most of the difficulties that nonlinear systems have. For instance, it was not proved that a given QS had a finite number of limit cycles until 1987 (see [**4**]). Nothing is known about the maximum number of limit cycles that a QS may have, except that it is greater than or equal to four. Hence, both the simplicity and the complexity that QS present, have been the reasons for which particular subfamilies of such systems have been extensively studied. In this way we can recall, for instance, the following subfamilies: homogeneous QS, QS with a start nodal point, bounded QS, QS without finite critical points, QS with an invariant straight line, QS with a degenerate critical point, QS with exactly one critical point, QS with a weak focus, etc., (see [**14**]).

In the above subfamilies two fundamental questions are studied: the phase portraits of the QS and the number of limit cycles that they may have.

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The purpose of this paper is the study of phase portraits of QS, on the Poincare sphere (see [**15**], for instance), with two finite critical points, where one of them, p, is a degenerate critical point with $\det DX(p) = \operatorname{tr} DX(p) = 0$.

It is obvious that two different possibilities arise, either $DX(p) \equiv 0$ or $DX(p) \not\equiv 0$. If we assume that $DX(p) \equiv 0$, the two subcases that we obtain are:

(i) p is an isolated critical point and hence X is a homogeneous QS.

(ii) p is not an isolated critical point.

As we mentioned above, subcase (i) has been extensively studied, see for instance [**1, 12**].

In the second subcase, either a conic or a straight line is a common factor of P and Q in (1). Consequently, in this subcase, the study of (1) follows easily from the study of linear systems.

Therefore, only the second case, that is, $DX(p) \not\equiv 0$ is considered in this paper.

We will follow the following notation:

QS with a critical point p for which $\det DX(p) = \operatorname{tr} DX(p) = 0$ and $DX(p) \not\equiv 0$ will be called QSD.

It is obvious that a QSD can always be written in the following canonical form

$$\dot{x} = y + P_2(x,y), \qquad \dot{y} = Q_2(x,y), \tag{2}$$

where $P_2(x,y) = lx^2 + mxy + ny^2$, and $Q_2(x,y) = ax^2 + bxy + cy^2$. It suffices to make a translation (to transport p to 0), a linear change of coordinates (to put $DX(p)$ in Jordan's canonical form) and finally a rescaling of t.

It is not difficult to confirm that a QSD has, at most, three critical points. QS with exactly one critical point are extensively studied in [**8**]. Here we study the QSD with exactly two critical points.

QSD have been already studied in two different papers [**6, 10**]. In the first one Coppel shows that this subfamily of QS has, at most, one limit cycle, and that when it exists it is hyperbolic. In the second one the problem of determining phase portraits of QSD is solved with the help of the computer.

The difference between our work and this latter one is that we deal, mainly, with semi-complete families of rotated vector fields (see [**9, 13**]). So we can follow the evolution of the separatrices and limit cycles, when they exist. This new approach allows us to prove most of the characteristics of the bifurcation diagrams that the systems considered present. Nevertheless, in two parts of the paper our analytic approach has not been sufficient and we also have needed the help of the computer. At this point the software [**3**] has been used.

The case of QSD with exactly three critical points also admits this approach but we do not study this case here.

The structure of this paper is as follows: in Section 2 we make a classification of QSD, and of QSD with exactly two critical points. Section 3 deals with QSD with exactly two critical points and without limit cycles. The study of QSD with exactly two critical points and which may have limit cycles is made in Section 4.

2. Classification of QS with a degenerate critical point. First we need a classification of the QSD.

Lemma 2.1. *A* QSD *is affine equivalent, scaling the variable t, if necessary, to one of the following*:

(a) $\dot{x} = y + x^2 + ny^2$, $\dot{y} = y^2$,

(b) $\dot{x} = y + lx^2 + mxy + ny^2$, $\dot{y} = xy$, with $l \neq 0$,

(c) $\dot{x} = y + lx^2 + mxy$, $\dot{y} = x^2 + bxy + cy^2$.

Proof. First we take a system of coordinates so that the QSD is written as in (2). If $a = b = 0$, then the change $x_1 = cx + (2l)^{-1}cmy$, $y_1 = l^{-1}c^2y$, $t_1 = c^{-1}lt$, converts it into (a). If $a = 0$ and $b \neq 0$, we can take $c = 0$ (making the change $x_1 = bx + cy$, $y_1 = by$), and (2) becomes (b). When $a \neq 0$, we can assume $n = 0$ (if necessary we can make the change of variables $x_1 = y - rx$, $y_1 = y$ where $r \neq 0$ satisfies $a + (b - a)r + (c - m)r^2 - nr^3 = 0$) and $a = 1$ (making the change $x_1 = ax$, $y_1 = ay$) and then (2) is written as in (c). $\square$

The behavior of the trajectories of the QSD given in Lemma 2.1 near the origin is given in the next result.

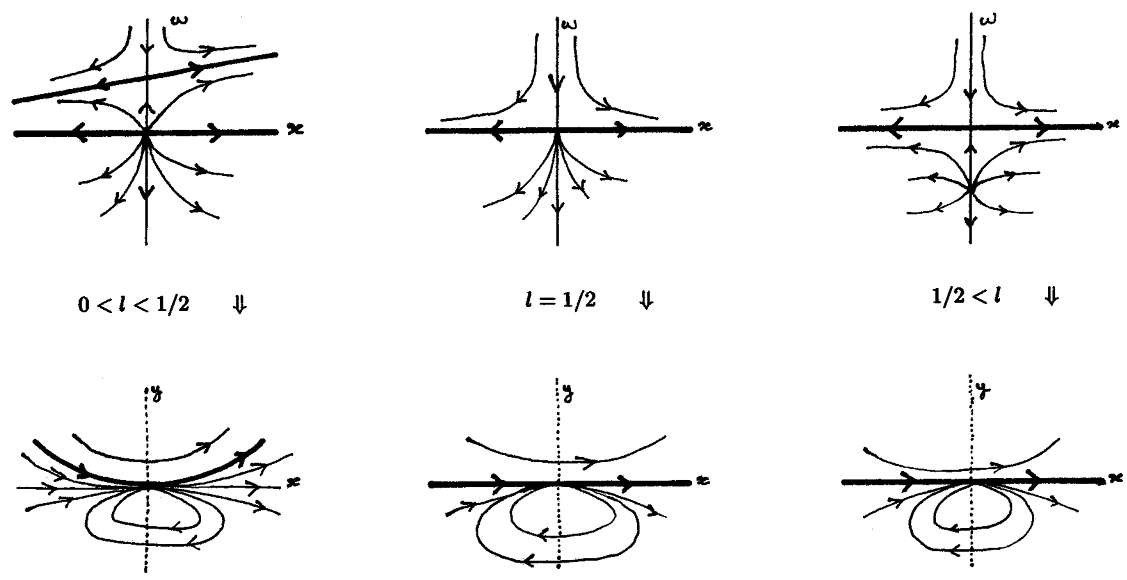


FIGURE 2.1. Local phase portrait of the $(0,0)$ of system (b) of Lemma 2.1 in accordance with l.

Lemma 2.2. (i) *The* $(0,0)$ *is a saddle-node point, a saddle point, the union of a hyperbolic and an elliptic sector or a cusp point if the* QSD *is written in the forms* (a), (b) *with* $l<0$, (b) *with* $l>0$ *or* (c), *respectively.*

(ii) *The straight line* $y=0$ *is invariant by the flow of systems* (a) *and* (b) *of Lemma* 2.1. *Furthermore, this straight line is always a separatrix through* $(0,0)$ *for system* (a), *and a separatrix for system* (b) *if and only if either* $l<0$ *or* $l\geq 1/2$ (*see Figure* 2.1).

Proof. (i) Follows from Theorem 66 and 67 of [**2**].

(ii) The separatrix problem is usually treated by using blow-up methods, see also [**2**, pp. 333–334]. For system (a) we obtain that $y=0$ is a separatrix. The straight line $y=0$ is not always a separatrix for system (b). From (i) of this lemma, for system (b) with $l<0$, and because $X(x,0)=(lx^2,0)$, we have that $y=0$ is a separatrix in such a case. When $0<l$, using the two consecutive blow-ups $x=x$, $y=w_1x$, and $x=x$, $w_1=wx$, (after omitting the common factor x) we obtain that it is converted into

$$\begin{cases} \dot{x}=lx+xw+mx^2w+nx^3w^2, \\ \dot{w}=(1-2l)w-2w^2-2mxw^2-2nx^2w^3. \end{cases}$$

The critical points for this system are $(0,0)$ and $(0,(1-2l)/2)$. So the behavior of the trajectories of this system in a neighborhood of the x-axis is obtained from Chapter IV and Theorem 65 of [**2**]. Consequently $y=0$ is not a separatrix through $(0,0)$ for system (b), if and only if $0<l<1/2$, see Figure 2.1. $\square$

Coppel has proved that a QSD has at most one limit cycle and that, when it exists, it is hyperbolic, see [**6**]. We can rewrite Coppel's result as follows.

Theorem 2.3. (i) *If a* QSD *has a degenerate singularity and it is not a cusp point, then the* QSD *has no limit cycles.*

(ii) *If a* QSD *has a degenerate singularity and it is a cusp point, then the* QSD *has, at most, one limit cycle, and when it exists, it is hyperbolic.*

Proof. (i) From Lemmas 2.1 and 2.2, if a QSD has a degenerate singularity that is not a cusp point, it is affine equivalent, to either system (a) or system (b) or system (c) with a degenerate critical point different from $(0,0)$.

Obviously system (a) has no limit cycles.

System (b) has no limit cycles either. The proof follows either from Bamon's paper [**4**] (by studying the stability of the possible limit cycle) or from Jager's paper [**10**] (by using the Bendixson-Dulac criterion).

It is not difficult to show that if system (c) has another degenerate critical point, then it has only these two singularities. Hence, from standard results on QS it has no limit cycles.

(ii) It follows from Coppel's paper. □

From Lemma 2.1 it is easy to show that a QSD has, at most, three critical points. As we have explained, our work deals with QSD with exactly two critical points and, whenever possible, they are studied when they define a semi-complete family of rotated vector fields. Next, lemmas are given in this sense.

From Lemma 2.1 we have the following classification.

Lemma 2.4. *A* QSD*, with exactly two critical points, is affine equivalent, scaling the variable t if necessary, to one of the following*:

$$\text{(I)}\quad \begin{cases} \dot{x} = y + Lx^2 + Mxy + Ny^2, \\ \dot{y} = xy, \end{cases} \qquad \text{with } L < 0,\ M \in \{0,1\},\ N \neq 0,$$

$$\text{(II)}\quad \begin{cases} \dot{x} = y + Lx^2 + Mxy + Ny^2, \\ \dot{y} = xy, \end{cases} \qquad \text{with } L > 0,\ M \in \{0,1\},\ N \neq 0,$$

$$\text{(III)}\quad \begin{cases} \dot{x} = y + Lx^2 + Mxy + y^2, \\ \dot{y} = x^2, \end{cases}$$

$$\text{(IV)}\quad \begin{cases} \dot{x} = y + Lx^2 + (L+M)xy, \\ \dot{y} = x^2 + xy, \end{cases} \qquad \text{with } M \neq 0.$$

Proof. From Lemma 2.1, we have to consider only systems (a), (b) and (c).

System (a) has only one critical point, the $(0,0)$. From system (b) we obtain (I) and (II) where $l < 0$ or $l > 0$, respectively. n may not be zero (if $n = 0$, $(0, 0)$ is its only critical point), and we can consider that m is either 0 or 1 taking the new coordinates $x_1 = mx$, $y_1 = m^2 y$ and $t_1 = m^{-1}t$ (if $m \neq 0$).

Now consider system (c). Either l or b can be considered greater than or equal to zero by taking the new coordinates $x_1 = x$, $y_1 = -y$ and $t_1 = -t$. If $c \neq 0$, in the coordinates $x_1 = |c|x$, $y_1 = |c|^{3/2}y$ and $t_1 = |c|^{-1/2}t$, we obtain that c is either 1 or -1 according with $c > 0$ and $c < 0$. Furthermore, when $c = 0$, if system (c) has two critical points, then $b \neq 0$, and in the new coordinates $x_1 = b^2 x$, $x_2 = b^3 y$ and $t_1 = b^{-1}t$, we obtain $b = 1$.

It is not difficult to confirm that the conditions that ensure that system (c) has exactly two critical points are the following (we have taken into account the changes of coordinates given above):

(i) $l = 0$, $m \neq 0$, $c = 1$, $b = 2$.

(ii) $l \neq 0$, $m^2 - lbm + cl^2 \neq 0$, $c = 1$, $b = 2$.

(iii) $l \neq 0$, $m^2 - lbm + cl^2 = 0$, $2m - lb \neq 0$, $c = \pm 1$.

(iv) $c = 0$, $b = 1$, $lb - m \neq 0$.

We note that conditions (i) and (ii) give system (c) written as

$$\dot{x} = y + lx^2 + mxy, \qquad \dot{y} = (x + y)^2, \quad \text{with } l \neq m,$$

hence, after the two changes of variables $x + y = x_1$, $y = y_1$, plus $x_2 = (l - m)^{2/3}x_1$, $y_2 = (l - m)y_1$, and $t_1 = (l - m)^{-1/3}t$, they produce system (III). If conditions (iii) or (iv) hold, we obtain system (IV) in the following way. If $c = 0$, it is only necessary to rename m with $L + M$. If $c \neq 0$, then $c = (m^2 - lbm)/m^2$ and, so, system (c) can be written as

$$\begin{cases} \dot{x} = y + lx^2 + mxy, \\ \dot{y} = \left(x + \frac{m}{l}y\right)\left(x + \frac{lb-m}{l}y\right). \end{cases}$$

Then, by means of the changes: $x_1 = x + ((lb - m)/l)y$, $y_1 = y$, plus $x_1 = ((2m - lb)/l)^2 x$, $y_1 = ((2m - lb)/l)^3 y$, $t_1 = ((2m - lb)/l)^{-1}t$, we transform it into

$$\begin{cases} \dot{x} = y + L'x^2 + (L' + N')xy + N'y^2, \\ \dot{y} = x^2 + xy, \end{cases}$$

for certain L' and N' depending on l, m and b. Now, in the coordinates $x_1 = x + y$, $y_1 = -y$, $t_1 = -t$, we get system (IV), where $L = -L' - 1$ and $M = N'$. $\square$

Lemma 2.5. (a) *Systems* (III) *and* (IV) *are semi-complete families of rotated vector fields, with parameter* $L \pmod{x^2}$ *in system* (III) *and with parameter* $L \pmod{x^2 + xy}$ *in system* (IV).

(b) *System* (III) *is a semi-complete family of rotated vector fields, with parameter* $M \pmod{x^3 y}$, *in the fourth quadrant.*

Proof. (a) It is because $x^2(\partial/\partial L)(y + Lx^2 + Mxy + y^2) + (y + Lx^2 + Mxy + y^2)(\partial/\partial L)x^2 = x^4 \geq 0$, for system (III), and because of $(x^2 + xy)(\partial/\partial L)(y + Lx^2 + (L + M)xy + Ny^2) + (y + Lx^2 + (L + M)xy + Ny^2)(\partial/\partial L)(x^2 + xy) = (x^2 + xy)^2 \geq 0$, for system (IV); see [**13**].

(b) The proof follows in the same way as in the previous one. $\square$

Proposition 2.6. *If a* QSD*, with exactly two critical points, can have limit cycles then it is affine equivalent, scaling the variable* t, *if necessary, to system* (IV) *of Lemma* 2.5.

Proof. From Lemma 2.4 we only have to consider the systems from (I) to (IV). By using Lemma 2.2 and Theorem 2.3, we know that systems (I) and (II) have no limit cycles. System (III) also, cannot have limit cycles because $\dot{y} \geq 0$. System (IV) can have limit cycles. In fact, the finite critical point difference from (0,0), can be a weak focus of the first order. $\square$

3. QSD without limit cycles. From Lemma 2.4 and Proposition 2.6 the families of QSD with exactly two critical points and which have no limit cycles run from (I) to (III) of Lemma 2.4. Now we will study them case by case.

Family (I). The $(0, 0)$ is a saddle point with the invariant straight line $y = 0$ through it (see Lemma 2.2). The other critical point is $(0, -1/N)$, and it is a saddle point or a nondegenerate point of index 1 where $N > 0$ or $N < 0$, respectively.

The directions s associated with the infinite critical points are given by the roots of

$$s(-Ns^2 - Ms + (1 - L)) = 0;$$

hence system (I) has 3, 2 or 1 infinite critical points according to $\Delta = M^2 + 4N(1 - L) > 0$, $\Delta = 0$ or $\Delta < 0$, respectively. It is not difficult to study the type of these critical points.

If $M = 0$, the vector field remains invariant by the change of variables $x_1 = -x$, $y_1 = y$ and $t_1 = -t$. From this symmetry and the above results, we obtain Figure 3.1. When $M = 1$, the above considerations give Figure 3.2.

Remark 3.1. There are five different phase portraits for system (I). Phase portrait corresponding with the case $M = 0$, $N > 0$, coincides with the one numbered 1 in the case $M = 1$.

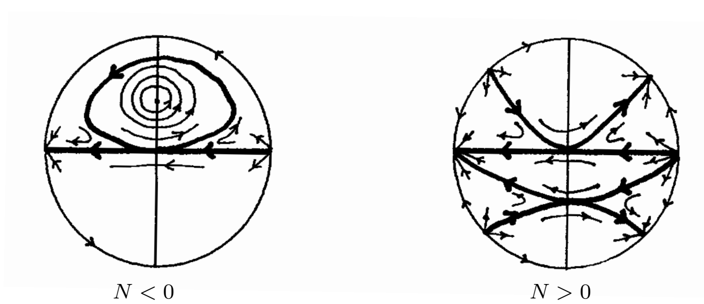


FIGURE 3.1. Phase portraits of system (I) when $M = 0$.

Family (II). The study of the infinite critical points and of the finite critical point different from $(0, 0)$ is similar to the study made in the preceding family. The major difference appears when we study whether the invariant straight line $y = 0$ is a separatrix for the origin, see Lemma 2.2(ii).

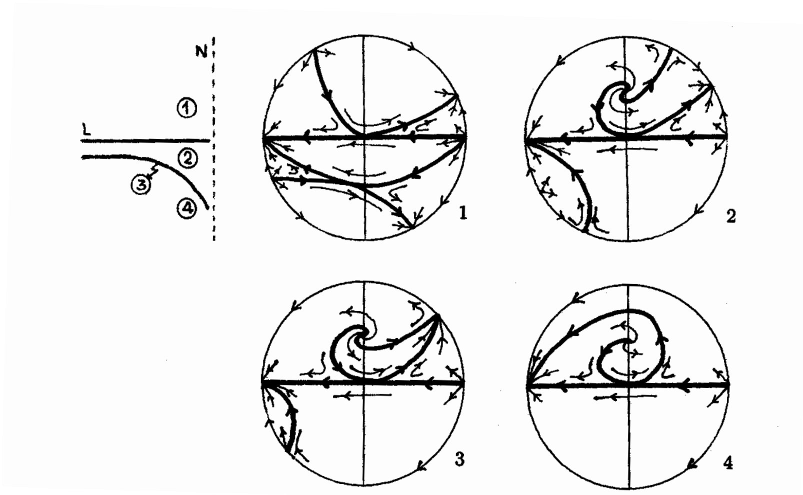


FIGURE 3.2. (L,N)-plane of parameters and phase portraits of system (I) when $M=1$ and $N\neq 0$. The curve in the (L,N)-plane is $M^2+4N(1-L)=0$.

If we consider L as a parameter, three special values appear, $L=1/2$, $L=L^*$ and $L=1$. For $L=1/2$ the straight line, $y=0$ changes its behavior with respect to the flow (see Lemma 2.2(ii) again). If we denote the value by $L=L^*$ so that $\Delta=M+4N(1-L^*)=0$, for this value of L, system (II) has an infinite double critical point. If $L=1$, the infinite critical point associated with the direction $y=0$ is also a double critical point if $M\neq 0$, or a triple critical point if $M=0$.

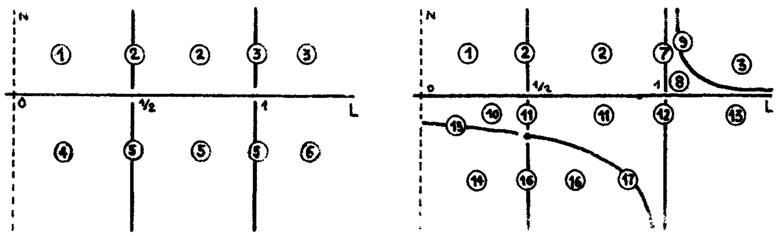
If $M=0$, under the change of variables $x_1=-x$, $y_1=y$ and $t_1=-t$, the vector field remains invariant.

From the above considerations, the 17 phase portraits of system (II), depicted in Figure 3.4 in accordance with Figure 3.3, follow. There is only a particular difficulty in drawing picture 14 of Figure 3.4. In this case $y=0$ is not a separatrix through $(0,0)$, and so we do not know if the two separatrices of $(0,0)$ coincide or not.

Assuming that they coincide for some values of L and N, consider the system

$$\begin{cases} \dot{x}=y+Lx^2+(1+\varepsilon)xy+Ny^2,\\ \dot{y}=xy.\end{cases}$$

This system is a semi-complete family of rotated vector fields with parameter ε (mod $xy=0$). Hence, for ε with suitable sign, a limit



(a) (b)

FIGURE 3.3. Different behaviors of the flow of system (II) in the (L, N)-plane of parameters, for cases: (a) $M = 0$, (b) $M = 1$, in accordance with Figure 3.4.

cycle appears from the separatrix loop (see [**9, 13**]) and this fact contradicts Theorem 2.3(i). So the separatrices do not coincide and, again from this theorem, phase portrait of this system must be like that in picture 14 of Figure 3.4.

Family (III). The critical points of system (III) are $(0,0)$ and $(0,-1)$. Using Lemma 2.2 the first one is a cusp point. From Theorems 65 and 67 of [**2**], $(0,-1)$ is either a cusp point or a saddle-node point, according to $M = 0$ or $M \neq 0$, respectively.

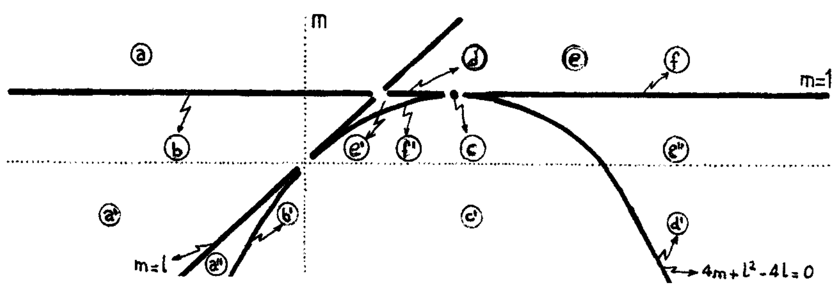


FIGURE 3.5. Different behaviors of the flow of system (3) at infinity, in the (l, m)-plane of parameters, with $m \neq l$.

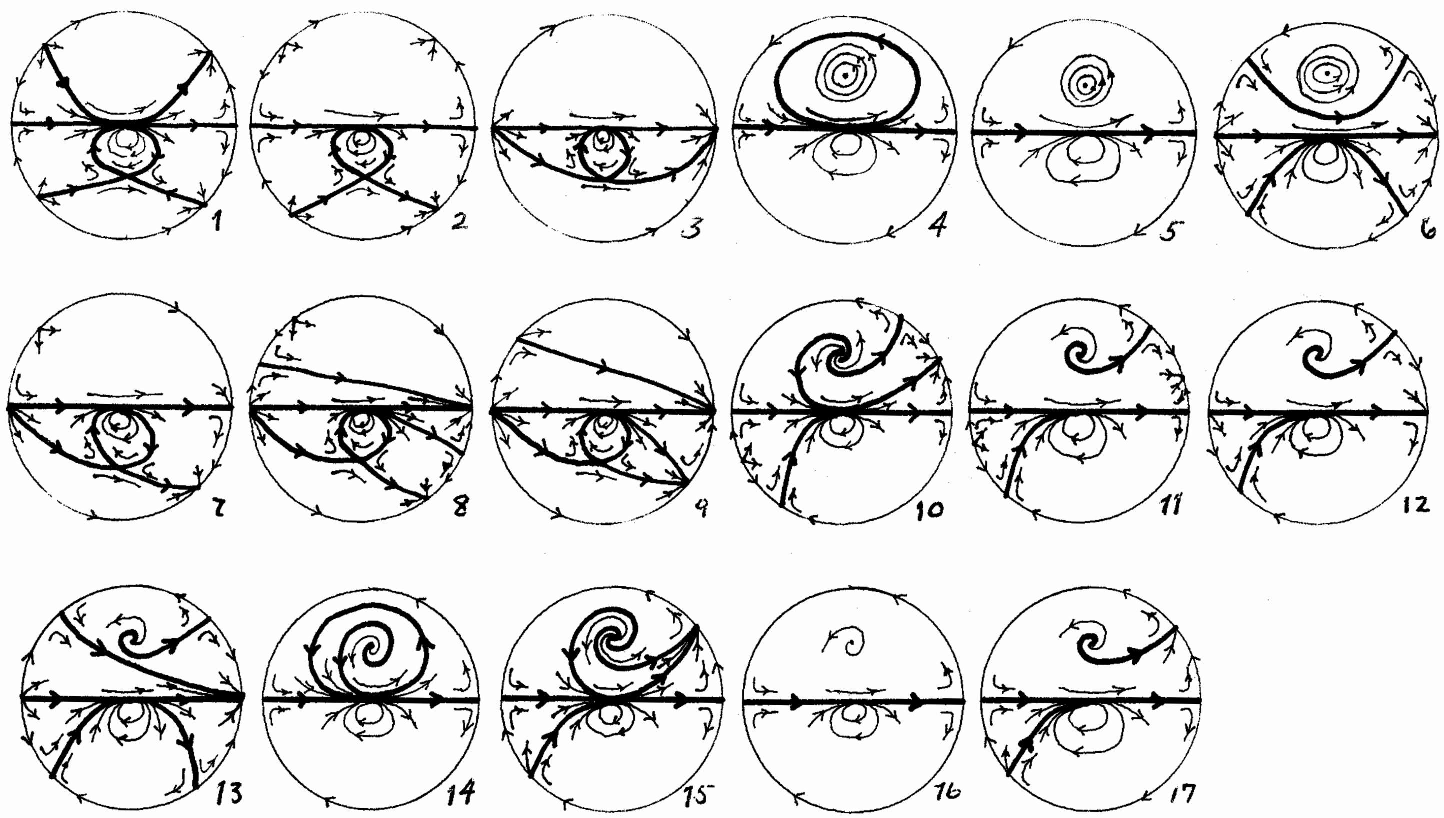


FIGURE 3.4. Phase portraits of system (II) in accordance with Figure 3.3.

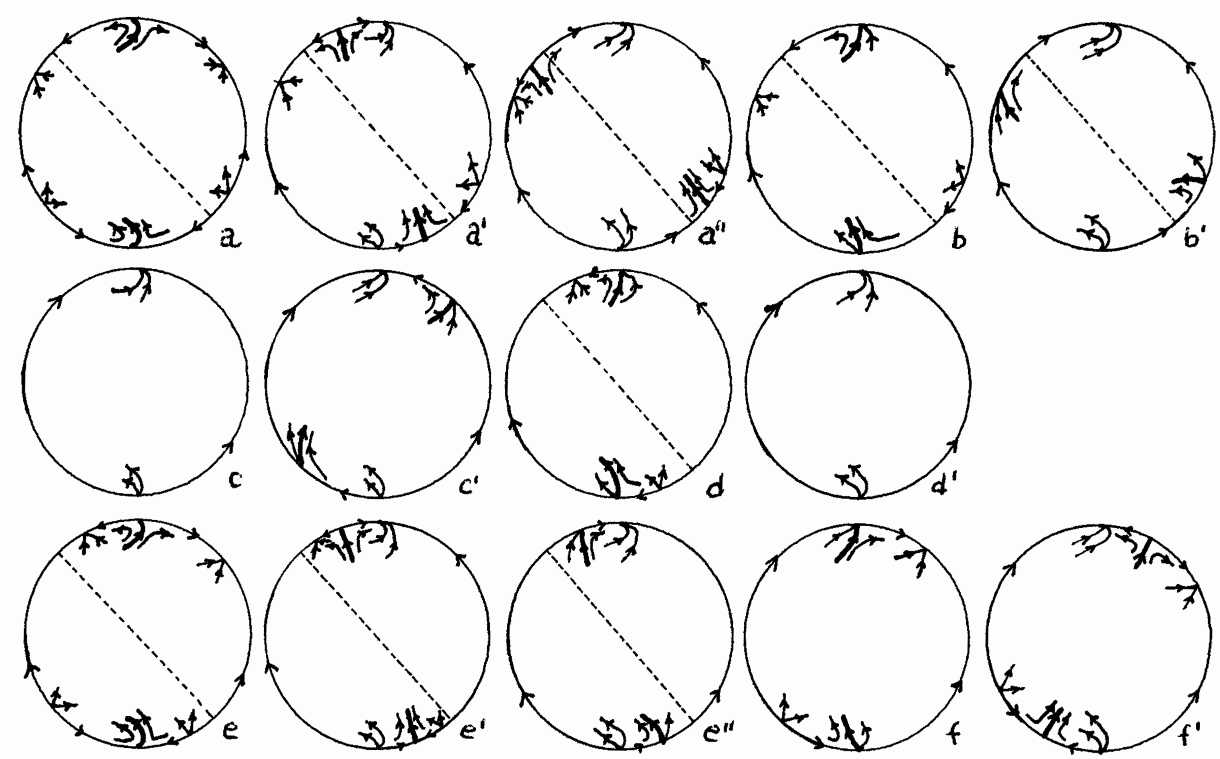


FIGURE 3.6. Phase portraits of system (3) in a neighborhood of infinity, according to the regions in the (l, m)-plane given in Figure 3.5.

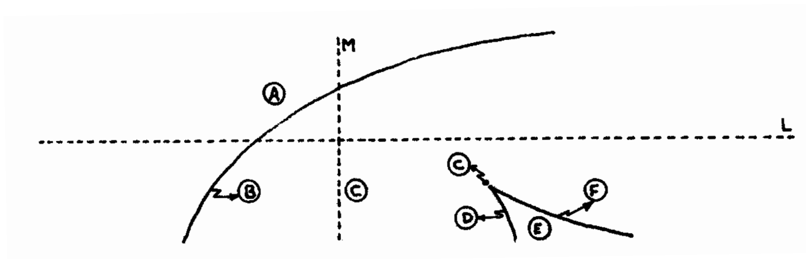


FIGURE 3.7. Different behaviors of the flow of system (III) at infinity. Phase portraits are depicted in Figure 3.8. The plotted curve is $\mathcal{C}$.

Proposition 3.2. *The behavior of the trajectories of system* (III) *of Lemma* 2.4*, in a neighborhood of infinity, is given in Figures* 3.7 *and* 3.8*.*

Proof. To study the infinite critical of system (III), it is more convenient to use its expression before applying the two changes of coordinates given in the proof of Lemma 2.4, that is, system

$$(3) \qquad \begin{cases} \dot{x} = y + lx^2 + mxy, \\ \dot{y} = (x+y)^2, \qquad \text{with } l \neq m. \end{cases}$$

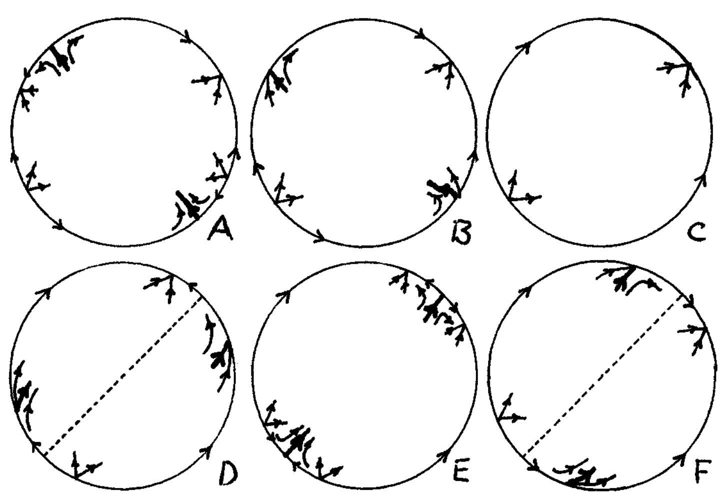


FIGURE 3.8. Phase portraits of system (III) in a neighborhood of infinity according to the regions in the (L, M)-plane given in Figure 3.7.

The infinite critical points are given by the zeros of $x(x+y)^2 - y(lx^2 + mxy) = x(x^2 + (2-l)xy + (1-m)y^2) = 0$. Note that there are double critical points at infinity if either $4m + l^2 - 4l = 0$ or $m = 1$. These curves are the ones depicted in Figure 3.5.

Figures 3.5 and 3.6 follow, again, using [**2**].

To obtain the behavior of the trajectories of system (III) in a neighborhood of the infinity in terms of the parameters (L, M), it is enough to study the relation between (l, m) and (L, M) that comes from the proof of Lemma 2.4. This relation is

$$L = (l+1)(l-m)^{-1/3}, \qquad M = (m-2l)(l-m)^{-2/3}.$$

The results obtained are summarized in Figures 3.7 and 3.8. From now on, we will call $\mathcal{C}$ the curve $-4M^3 - L^2M^2 + 18LM + 4L^3 + 27 = 0$, that separates different behaviors, at infinity, for the flow of system (III), in the (L, M)-plane of parameters.

The relationship between the notations used in Figures 3.6 and 3.8 is the following: phase portraits, in the (l, m)-plane of parameters, with the same small letter correspond with the same phase portrait, in the (L, M)-plane, with that capital letter. $\square$

From the above considerations, and using the following facts:

(i) the curve $y = -Mx - 1$ is a straight line without contact (respectively, invariant) when $LM + 1 \neq 0$ (respectively $LM + 1 = 0$),

(ii) if A gives the slope of an infinite critical point, and $(M+A)(M+2A) \neq 0$, then the straight line $y = Ax - A/(M+2A)$ is without contact and, when $A < 0$ and $M \leq 0$, we have, in addition, that $-1 < -A/(M+2A) < 0$.

(iii) for phase portraits 21 and 22 in Figure 3.16, we have $M + A < 0$ and $M + 2A < 0$,

(iv) the scalar product of the vector field given by (III), on a straight line $y = Ax$, where A is the slope of an infinite critical point, with the vector $(-A, 1)$ is $-A^2x$. (Note that this is a general property of any quadratic system, see [**16**]).

We can obtain most of the phase portraits of system (III). To be more precise, we can get the global phase portrait in all the (L, M)-plane of

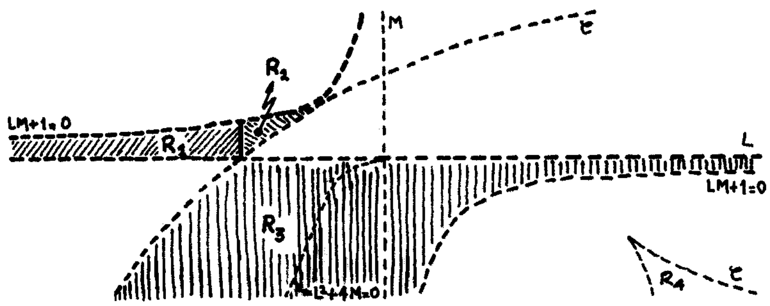


FIGURE 3.9. Zones R_1, R_2 and R_3 where the study of the phase portraits of system (III) is more complicated.

parameters except in the regions:

$$R_1 = \{(L, M) \mid L \in (-\infty, -3 \cdot 2^{-2/3}], M \in (0, -1/L)\},$$
$$R_2 = \{(L, M) \mid L \in (-3 \cdot 2^{-2/3}, -1], M \in (C(L), -1/L)\},$$

and

$$\begin{aligned} R_3 = {} & \{(L, M) \mid L \in (-\infty, -3 \cdot 2^{-2/3}], M \in (C(L), -\infty)\} \\ & \cup \{(L, M) \mid L \in (-3 \cdot 2^{-2/3}, 0), M \in (0, -\infty)\} \\ & \cup \{(L, M) \mid L \in (0, +\infty), M \in (0, -1/L)\}, \end{aligned}$$

where $(L, C(L))$ is a parametrization of the curve $\mathcal{C}$. See Figure 3.9 for more details.

As an example of the above assertion, we will prove that the phase portrait of (III) in the region R_4 of Figure 3.9 is 21 of Figure 3.16. In this case by using all the results obtained above, we get Figure 3.10(a). From the Poincare-Bendixson theory, we have Figure 3.10(b), that is phase portrait 21 of Figure 3.16. All phase portraits obtained in this way are summarized in Figure 3.16.

To finish our classification we must study regions R_1, R_2 and R_3.

Region R_1. In this region the α-limit and ω-limit set of all the separatrices is determined, except for one of the stable separatrices of

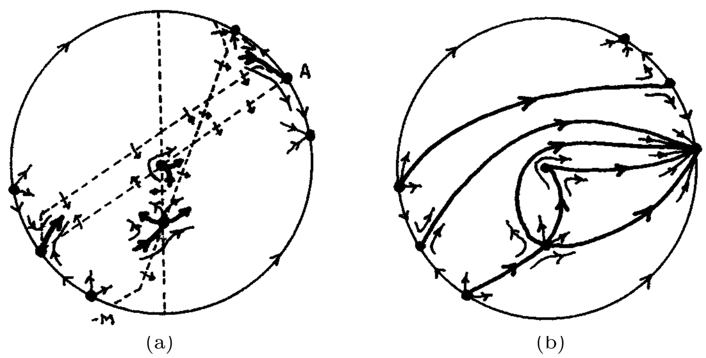


FIGURE 3.10. (a) The flow of system (III) on the straight lines: $y = Ax$, $y = -Mx - 1$ and $y = Ax - A/(M + 2A)$, for phase portrait 21. (b) Phase portrait 21.

the saddle-node point and the unstable separatrix of the infinite saddle. In Figure 3.11 we can see the three possible phase portraits that can occur.

In the boundaries of the set R_1, $\{(L, M) \mid L \in (-\infty, -3 \cdot 2^{-2/3}), M = -1/L\}$ and $\{(L, M) \mid L \in (-\infty, -3 \cdot 2^{-2/3}), M = 0\}$, we know that the phase portraits of (III) are 1 and 12 of Figure 3.16, respectively. Therefore, using the continuous dependence with respect to parameters, for the vector field (III), we get that, for a fixed L, there exists at least one M such that (III) has phase portrait 9 of Figure 3.16. Furthermore, using Lemma 2.5(b) and the fact that, in a neighborhood of phase portrait 9, the movement of the infinite critical points in the fourth quadrant is as we show in Figure 3.11, we have that the value of M obtained above is unique (we call it $M_1(L)$). Hence, the information showed for R_1 in Figures 3.15 and 3.16, follows.

Region R_2. The study of phase portraits in this region is more complicated than the above one. The reason is that we do not know the phase portrait of (III) on the boundary of R_2 given by $R_2^1 = \{(L, M) \mid L \in (-3 \cdot 2^{-2/3}, -1], M = C(L)\}$. More explicitly, in the boundary of R_2^1, that is, at the points $(-3 \cdot 2^{-2/3}, 0)$ and $(-1, 1)$, in the (L, M)-plane, the phase portraits of (III) are 13 and 6 of Figure 3.16, respectively. By using similar ideas to those in the above case, we can get the information summarized in Figure 3.12(a).

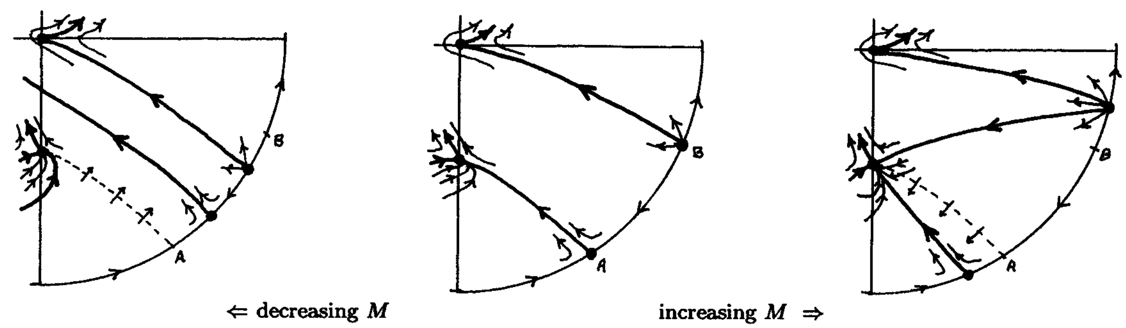


FIGURE 3.11. Behavior of the flow and movement of the infinite critical points, of system (III), in the fourth quadrant, in a neighborhood of the phase portrait (9) (at the center of the figure).

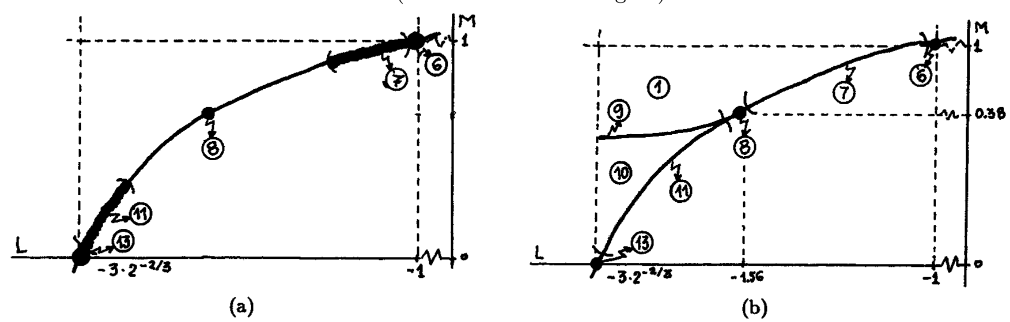


FIGURE 3.12. Region R_2 of system (III), (a) a first approximation, (b) conjectured diagram.

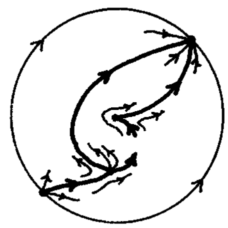


FIGURE 3.13. Phase portrait of system (III) in the region R_3.

We have not been able to prove that the phase portrait 8 of Figure 3.16 only occurs at one point of R_2^1. The major difficulty is that on R_2^1, system (III) is not a semi-complete family of rotated vector fields. A numerical study made using [**3**], seems to show that, in fact, the phase portrait 8 of Figure 3.16 only occurs for the point $(L, M) \approx (-1.56, 0.38)$ and that on R_2^1 phase portraits of (III) are like we indicate in Figure 3.12(b).

Assuming the above information on R_2^1, we may conclude that the function $M_1(L)$, found in the study of R_1, can be continued for $L \in [-3 \cdot 2^{-2/3}, -1/56\dots]$, and that $M_1(-1.56\dots) = 0.38\dots$. Therefore, we have the information summarized in Figure 3.12(b). We want to indicate that, using numerical derivation with eight points, we get that the common point of $(L, M_1(L))$ with $(L, C(L))$ is a $\mathcal{C}^0$ contact point.

Region R_3. In this region previous results allow us to establish the information given in Figure 3.13.

In order to finish the phase portrait, we proceed in the following way. If we assume that $M < 0$, $L < 0$ and $L^2 + M \geq 0$, it is easy to prove that there exists $s < 0$ such that the flow associated with the vector field (III) on the segment $\{(sy, y) \mid -1 \leq y < 0\}$ is as we show in Figure 3.14.

Therefore, in this subregion of R_3 the phase portrait of (III) is 17 of Figure 3.16. On the other hand, on the branch of the hyperbola $\{(L, M) \mid LM + 1 = 0, M < 0\}$ the phase portrait of (III) is 19 of the same Figure.

Hence, arguing as in the previous regions and using Lemma 2.5(a), we have that, for $M < 0$, there exists a unique continuous curve $L_3(M)$

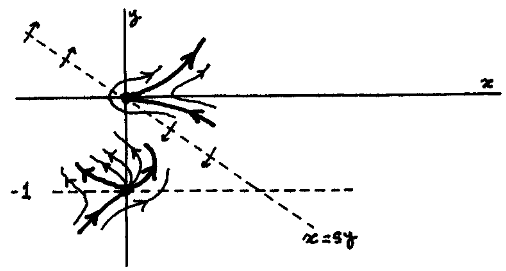


FIGURE 3.14. Vector field (III) on the straight line $x = sy$, for $L < 0$, $M < 0$, $L^2 + 4M \geq 0$ and some $s < 0$.

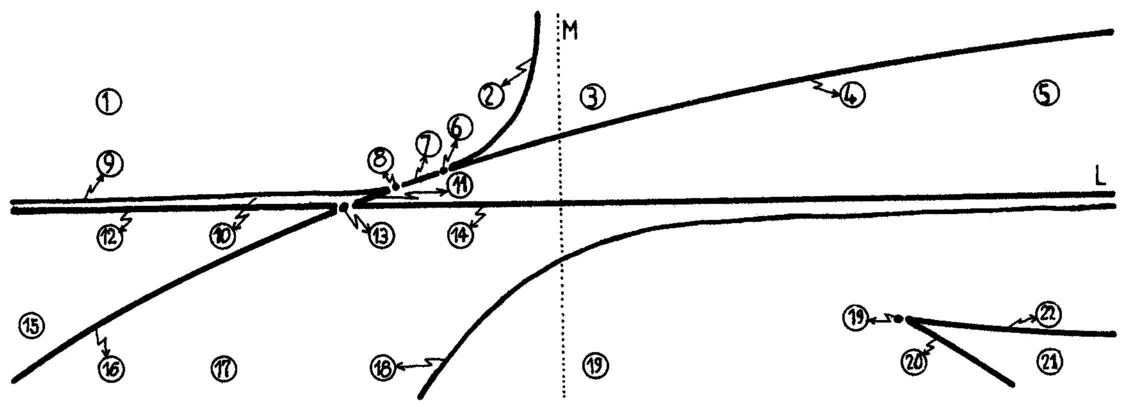


FIGURE 3.15. Different behaviors of the flow of system (III) in the (L, M)-plane.

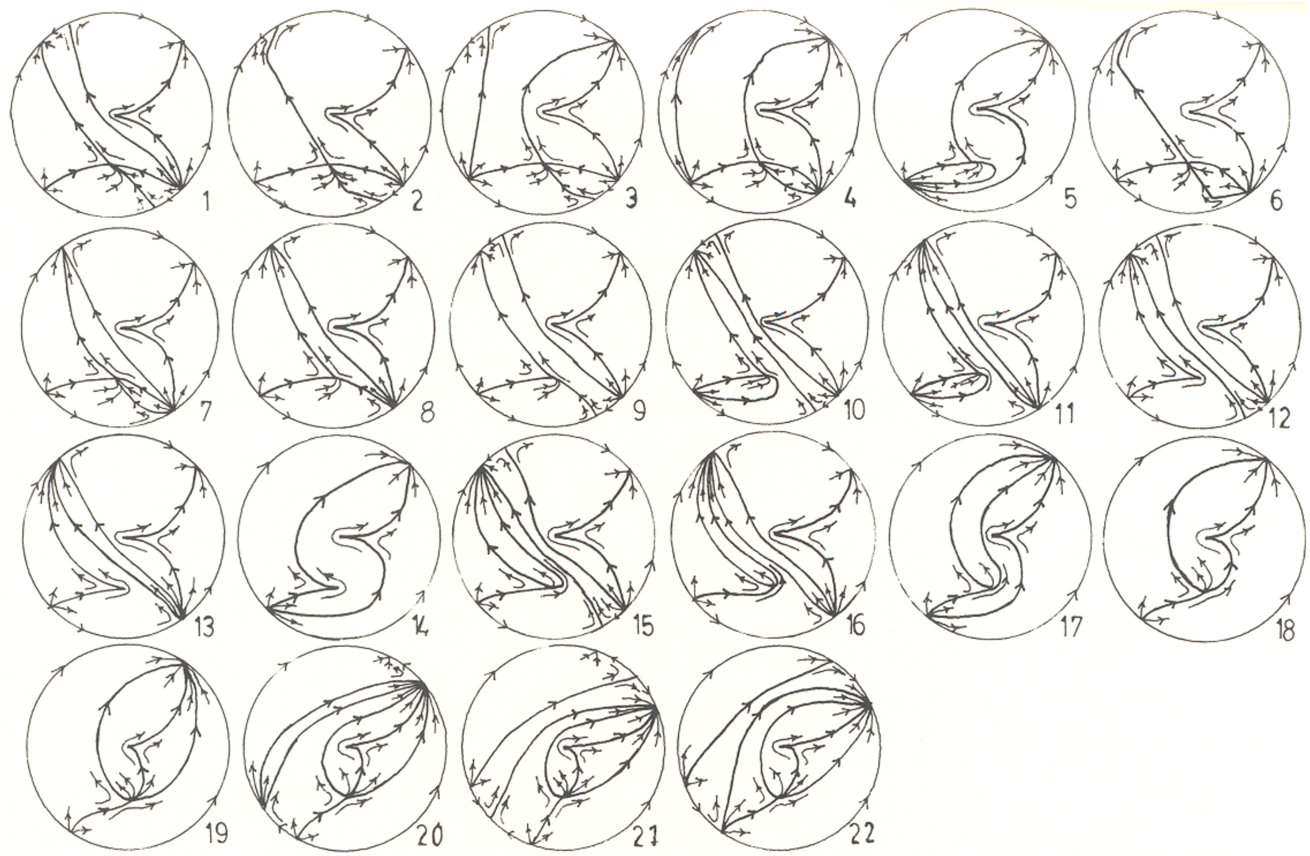


FIGURE 3.16. Phase portraits of system (III) in accordance with the regions in the (L, M)-plane given in Figure 3.15.

such that $-2\sqrt{M} < L_3(M) < -1/M$, on which the phase portrait of (III) is 18 of Figure 3.16. Therefore, the results of Figures 3.15 and 3.16, for the region R_3, follow.

From all the above considerations, we can summarize all the phase portraits of system (III). There are 22 of them, drawn in Figure 3.16 following Figure 3.15.

4. QSD with limit cycles. From Proposition 2.6 we have to consider system (IV) of Lemma 2.4.

First of all we study its infinite and finite critical points. From Lemma 2.2, (0,0) is a cusp point. The next result gives the nature of the other critical point.

Lemma 4.1. *The critical point of* (IV)*, different from the origin, is* $(-1/M, 1/M)$ *and it is*:

(i) *A saddle point if* $M > 0$.

(ii) *A nondegenerate critical point of index* 1 *if* $M < 0$. *More exactly*:

(a) *An attractor if* $M > L + 1$ (*node or focus*).

(b) *A weak repellor focus of first order if* $M = L + 1$.

(c) *A repellor if* $M < L + 1$ (*node or focus*).

Furthermore, the point is a node or a focus according to whether $(M - L - 1)^2 + 4M \geq 0$ *or* $(M - L - 1)^2 + 4M < 0$.

Proposition 4.2. *The behavior of the trajectories of system* (IV) *of Lemma* 2.4 *in a neighborhood of infinity is given in Figures* 4.2 *and* 4.3.

Proof. The directions $x = sy$ associated with the infinite critical points are the real roots of

$$s(s^2 - (L-1)s - (L+M)) = 0.$$

Note that there are double or triple critical points at infinity if either $(L-1)^2 + 4(L+M) = 0$ or $L + M = 0$. These curves are plotted in Figure 4.2.

Except when $L + M = 0$, the results of Figures 4.2 and 4.3 follow

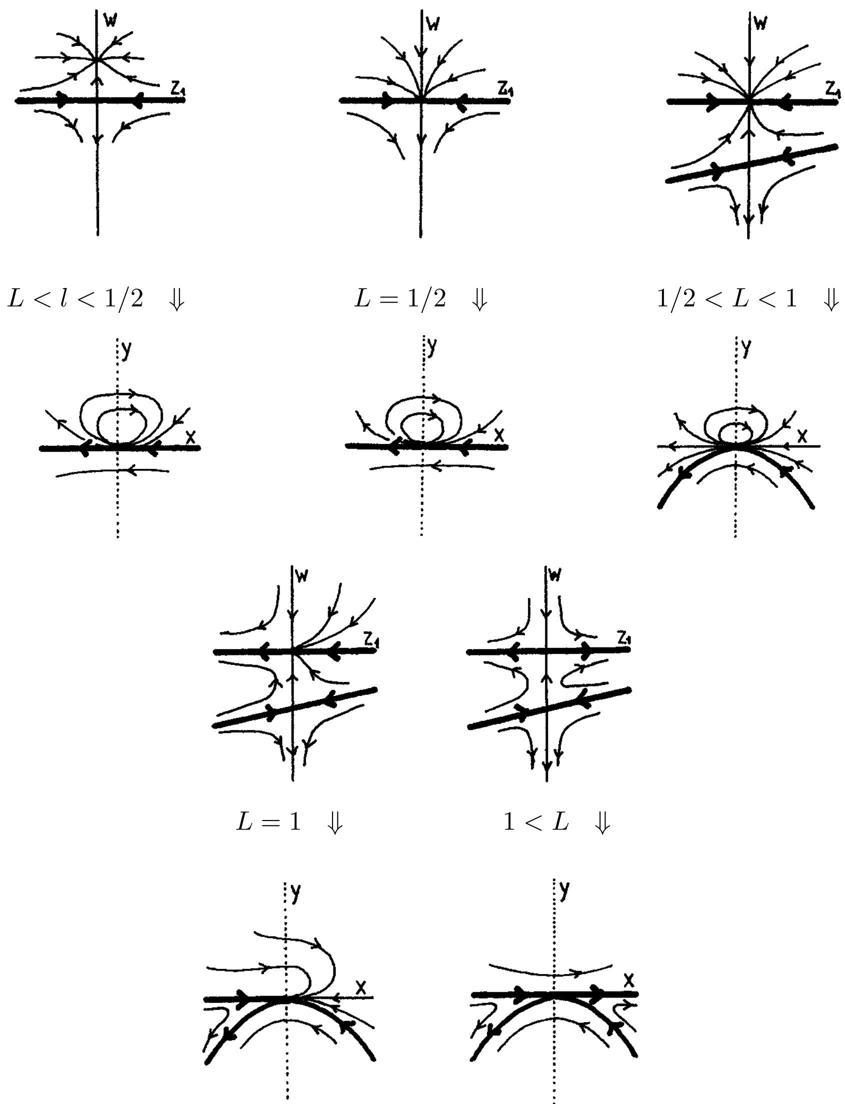


FIGURE 4.1. Behavior of the trajectories in a neighborhood of the infinite critical point $(0, 0)$ of U_2 for system (IV) when $L + M = 0$.

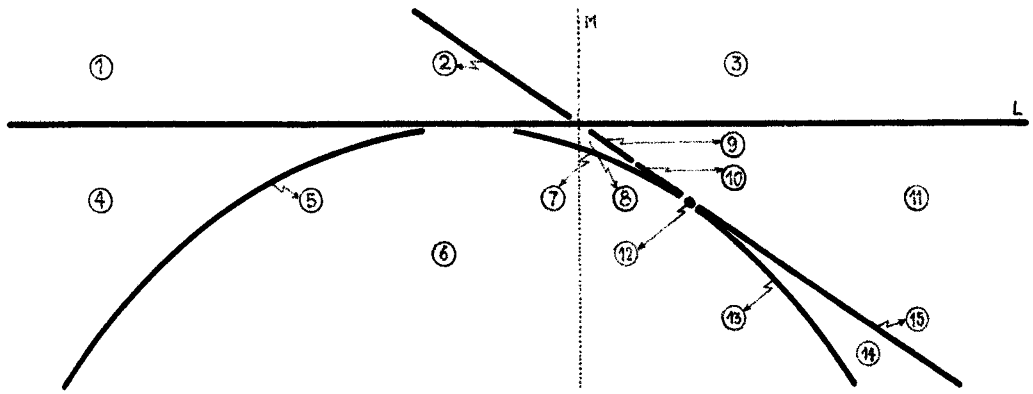


FIGURE 4.2. Different behaviors of the flow of system (IV) at infinity, in the (L, M)-plane of parameters, when $M \neq 0$. The two curves plotted are $(L-1)^2 + 4(L+M) = 0$ and $L + M = 0$. Phase portraits are given in Figure 4.3.

from standard use of Theorems 65 of [**2**] and the study of the simple equilibrium states (see [**2**, Chapter IV], for instance).

When $L+M=0$, the problem of determining if the circle at infinity, in the Poincare compactification, is a separatrix through the singularity associated with $s=0$ appears. We will make a more detailed study of this case. Consider the expression of (IV) in the local coordinates of chart U_2 (see [**15**]) when $L+M=0$.

$$\dot{z}_1 = z_2 + (L-1)z_1^2 - z_1^3, \qquad \dot{z}_2 = -z_1 z_2(1+z_1).$$

From Theorems 66 and 67 of [**2**], the $(0,0)$ is the union of a hyperbolic sector and an elliptic one, or a saddle-node point or a saddle point when $L<1$, $L=1$ or $L>1$, respectively. Applying the two successive blow-ups $z_1 = z_1$, $z_2 = z_1 w_1$ and $z_1 = z_1$, $w_1 = w z_1$, we obtain, after omitting a common factor z_1,

$$\dot{z}_1 = z_1(w - z_1), \qquad \dot{w} = w(-1 + z_1 - 2w).$$

Using similar considerations to those in the proof of Lemma 2.2(ii) we obtain Figure 4.1.

Note that the circle at infinity ($z_2 = 0$) is not a separatrix through $(0,0)$ when $1/2 < L < 1$. Hence, the proposition follows. $\square$

Proposition 4.3. *System* (IV) *has, at most, one limit cycle and, when it exists, it is hyperbolic. It exists if, and only if,* $M < 0$ *and* $L^*(M) < L < M-1$. *Here* $L^*(M)$ *is a function of* M *that satisfies* $(M-1) - 2\sqrt{-M} < L^*(M) < M-1$, *if* $M \le -4$, *and* $M - 1 - 2\sqrt{-M} < L^*(M) \le -1 - 2\sqrt{-M}$ *if* $M > -4$.

Proof. From Theorem 2.3, system (IV) has at most one limit cycle and, when it exists, it is hyperbolic.

It is well known (see [**5**], for instance) that a limit cycle for a QS must surround a focus. Hence, system (IV) can have limit cycles only when $M < 0$ and $(M-L-1)^2 + 4M < 0$. The origin is a repellor weak focus of order one when $L = M-1$ (see [**11**]), therefore, from the theory of rotated vector fields and Lemma 2.5(a), we have an unstable limit cycle arising from the origin when $L \lesssim M-1$, and growing when L decreases.

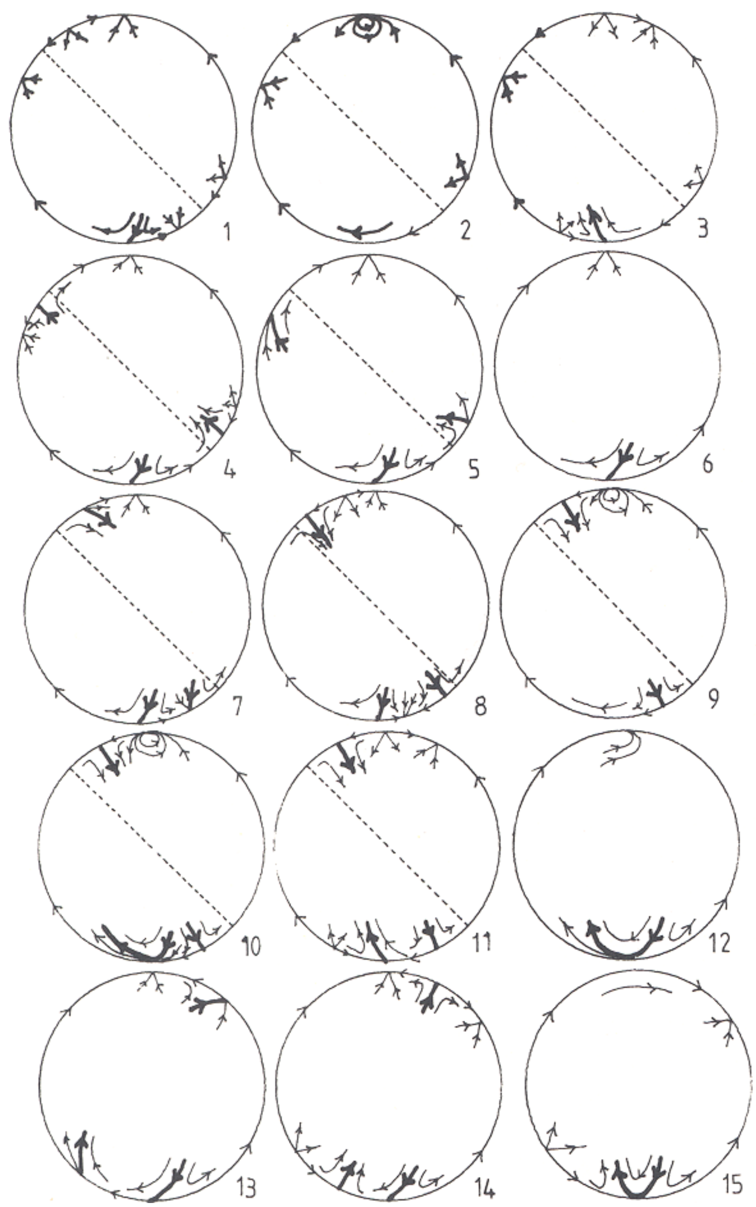


FIGURE 4.3. Phase portraits of system (IV) in a neighborhood of infinity, in accordance with the regions in the (L, M)-plane given in Figure 4.2.

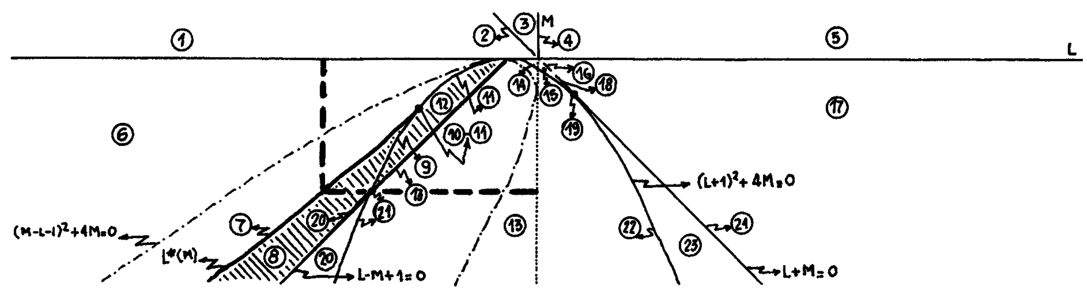


FIGURE 4.4. Different behaviors of the flow of system (IV) in the (L, M)-plane. See also Figure 4.6 to get more information in the box $(L^*(-4), 0) \times [-4, 0]$.

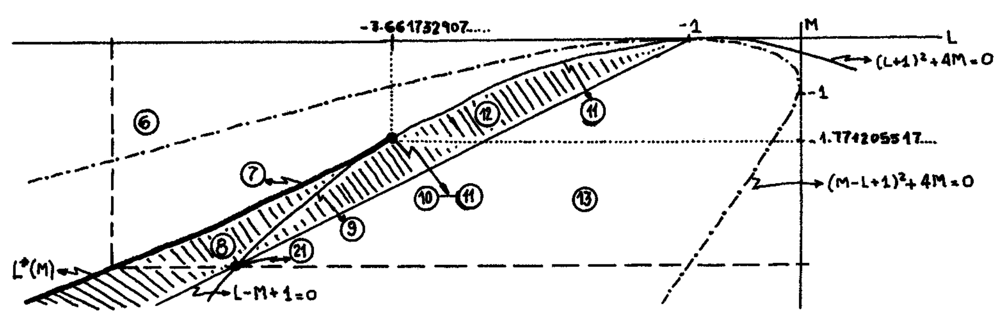


FIGURE 4.6. Conjectured behavior of the curve $L^*(M)$, of system (IV).

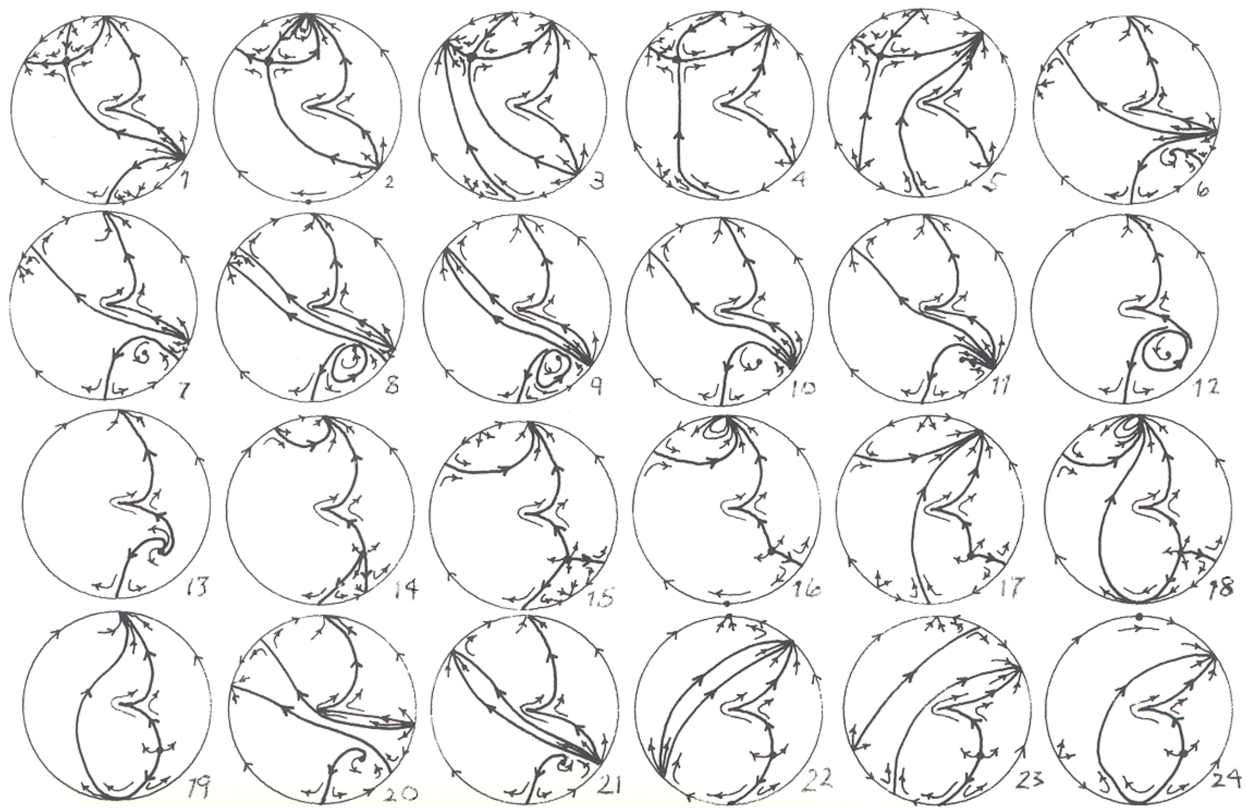


FIGURE 4.5. Phase portraits of system (IV) in accordance with the regions in the (L, M)-plane given in Figure 4.4.

Furthermore, the limit cycle must disappear for some value of L, called $L^*(M)$, which of course, must be such that $L^*(M) > M - 1 - 2\sqrt{-M}$. When $M > -4$, arguing in a similar way, we get $L^*(M) \leq -1 - 2\sqrt{-M}$. $\square$

In order to depict some of the 24 phase portraits of system (IV), as we show in Figure 4.5 in accordance with Figure 4.4, we need, in addition to the previous consideration, the following facts:

(i) on the straight line $y = Ax$, the scalar product of the vector field associated with (IV) and $(-A, 1)$, is $-A^2x$,

(ii) if A gives the slope on an infinite critical point (a saddle or a saddle-node point), and $1 - (L + M)A \neq 0$, then the straight line $y = Ax + A^2/(1-(L+M)A)$ is without a contact line and, in addition, when $M < 0$ and $L < 0$ we have that $1/M < -A/M + A^2/(1 - (L + M)A)$,

(iii) when $L = 0$, $x = -1/M$ is an invariant straight line.

When $-4 < M < 0$, we do not know whether or not the curve $(L^*(M), M)$ touches the parabola $\mathcal{P} = \{(L, M) \mid (L + 1)^2 + 4M = 0\}$ (of course this curve cannot cross $\mathcal{P}$). Furthermore, assuming that both curves have a common point, we neither know if at this point the phase portrait is either 10 or 11 of Figure 4.5.

Numerical calculations, made using [**3**], suggest that when $M \in (-4, -1.771205517)$ (respectively, $M \in (-1.771205517, 0)$) the phase portraits of (IV) on the parabola $\mathcal{P}$ are like 9 (respectively, 11) of Figure 4.5. Therefore, it seems that, for $M \in [-1.771205517\ldots, 0)$, $\mathcal{P}$ and the curve $(L^*(M), M)$ coincide and there the phase portrait is 11. We think that, when $M \in [-1.771205517\ldots, 0)$, the phase portrait that occurs is 11 because phase portrait 10 looks as a codimension 2 system, but we have not been able to prove that assertion. We point out that using numerical derivation and six points we obtain that the common point between the parabola $\mathcal{P}$ and the curve $(L^*(M), M)$ is a $\mathcal{C}^1$ contact point.

Hence, we conjecture that the bifurcation diagram, of system (IV), in the box $(L^*(-4), 0) \times (-4, 0)$ of Figure 4.4 is as we depict in Figure 4.6, where we have dashed the area with limit cycles.

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