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THE PENNEY-FUJIWARA PLANCHEREL FORMULA FOR GELFAND PAIRS

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ABSTRACT. This paper is concerned with the Penney-Fujiwara Plancherel formula for the quasi-regular representation of a homogeneous space G/H. The specific spaces considered are semidirect products $G = H \ltimes N$, where H is compact and N is simply connected nilpotent. An explicit description is given for the Plancherel formula, and the attendant Penney distributions associated with it, in the important special case that (G, H) is a *Gelfand pair*. The description is in orbital parameters, and thereby supplies an important instance of the validity of the Kirillov orbit method that is outside the usual context of exponential solvable groups. Other results that are found in the paper include: (i) a proof of the fact that a distribution-theoretic form of Frobenius reciprocity holds for Gelfand pairs; (ii) the demonstration that the well-known form of the Penney distributions in terms of *real* polarizations continues to be valid when the representations are realized by *complex* polarizations; (iii) an explicit formula for the intertwining operator for the direct integral decomposition of the quasi-regular representation of G/H; and (iv) an orbital criterion for the quasi-regular representation to be multiplicity-free, as well as a criterion for an irreducible to occur in the spectrum.

0. Introduction. This paper is devoted to the Plancherel formula which describes the direct integral decomposition of a quasi-regular representation. The latter is the natural representation of a Lie group G on the square-integrable functions on a homogeneous space of G. More precisely, if H is a closed subgroup of G, the corresponding quasi-regular representation is nothing more than the representation τ of G obtained by inducing the identity representation of H up to G. The *abstract* or "soft" Plancherel formula is a statement of unitary equivalence between τ and a direct integral of irreducible unitary representations

$$\tau \cong \int_{\mathcal{X}}^{\oplus} n(\pi)\, \pi \, d\nu_H(\pi). \tag{0.1}$$

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ν_H is the Plancherel measure *class*, $\mathcal{X}$ is the spectrum, and $n(\pi)$ is the multiplicity. The formula is considered soft because only the class of the measure is specified and because the intertwining operator which affects the unitary equivalence may not be specified. To "harden" the formula, one must deal with a distribution-theoretic version [**19**]. Such a version is called a Penney-Fujiwara Plancherel formula (abbreviated **PFPF**). The specific ingredients of the **PFPF** are reviewed below in Section 1. Suffice it to say in this introduction that the Kirillov orbit method has played a governing role in its development (also discussed in Section 1). Because of that, all instances but one (namely, reductive symmetric spaces) in which the **PFPF** has been computed explicitly have fallen within the realm of monomial spectrum—that is, when the spectrum consists of representations induced from a character. According to the orbit method, irreducible representations are parameterized by coadjoint orbits. Monomial spectrum means that the orbits have real polarizations. The polarizing groups themselves are critical ingredients in the formulas for the Penney distributions that enter the **PFPF**—see formula (1.3) below. The reader may find extensive discussions of these ideas in [**7, 14, 15** and **16**].

But, of course, monomial spectrum scenarios do not exhaust all of the interesting situations. As mentioned earlier, reductive symmetric spaces constitute a prime example. For those, the spectrum is highly nonmonomial. Even the next most sophisticated kind of induction, namely, holomorphic induction via a positive complex polarization, does not suffice to describe the spectrum. We would like to consider situations in which holomorphically induced representations do constitute the spectrum. This is important because, up to now, no one has had any idea how to extend the definition of the Penney distributions to the situation in which the polarizing groups lie in the complexification of G. Accomplishing that is one of the main goals of this paper.

The simplest and most natural place to consider holomorphically induced representations is in the context of solvable groups. At least in the algebraic case, those are semidirect products of tori with simply connected nilpotent groups. We shall be concerned primarily with a natural analog of those, namely groups $G = K \ltimes N$, where K is compact and N is normal and simply connected nilpotent. The homogeneous spaces G/K have been studied recently in [**3**] and [**2**], in particular for the case where (G, K) is a Gelfand pair (see Section 5 for the definition). In this paper we shall compute the Penney distributions (in two different guises) and the **PFPF** for *all* Gelfand pairs. We note that the case where N is abelian, in which case G/H is an abelian symmetric space, has been studied in [**12**] and [**15**].

The organization of the paper is as follows. In Section 1 we describe explicitly the **PFPF** for a general homogeneous space G/H. In Section 2 we specialize to the case of compact H. In Section 3 we further specialize to the situation in which the multiplicity function is identically 1. For now the reader may take that to be the definition of a Gelfand pair. In Section 4 we review the representation theory of Lie groups with co-compact nilradical. We apply that theory to Gelfand pairs in Section 5. In Section 6, we prove the **PFPF** for a Gelfand pair in the context of Mackey parameters. Finally, in Section 7, we reformulate and refine the results of Section 6 by recasting the formulas into an orbital configuration. The main results of the paper are: Theorem 3.1, which asserts the uniqueness of the Penney distributions when H is compact; Theorem 6.2, the actual **PFPF** for a Gelfand pair; and Theorem 7.1, the orbital realization of the Penney distributions for a holomorphically induced representation.

Some of the results were announced in [**17**].

1. The Penney-Fujiwara Plancherel formula. For simplicity, we shall assume that G and its closed subgroup H are both unimodular. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. We write $\mathcal{H}_\pi^\infty$ to denote the Frechet space of C^∞ vectors of π. Its antidual space is denoted by $\mathcal{H}_\pi^{-\infty}$. Each of $\mathcal{H}_\pi^\infty$, $\mathcal{H}_\pi^{-\infty}$ is acted upon by G, therefore also by $\mathcal{D}(G) := C_c^\infty(G)$. It is well-known that

$$\pi(\mathcal{D}(G))\mathcal{H}_\pi^{-\infty} \subset \mathcal{H}_\pi^\infty.$$

We shall always denote the quasi-regular representation $\mathrm{Ind}_H^G 1$ by τ. It acts (by right translation) on the space $\mathcal{H}_\tau$ of Borel functions f on G which are left H-invariant and square-integrable with respect to the G-invariant measure on G/H. The canonical cyclic distribution $\alpha_\tau \in \mathcal{H}_\tau^{-\infty}$ is defined by $\langle \alpha_\tau, f\rangle = \bar{f}(1)$. The matrix coefficients of α_τ are computed in [**15**]:

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau\rangle = \omega_H(1), \qquad \omega \in \mathcal{D}(G),$$

where

$$\omega_H(g) = \int_H \omega(g^{-1}h^{-1})\,dh. \tag{1.1}$$

The abstract **PFPF** is formula (1.2) in the following

Theorem 1.1. *Suppose that τ is type* I *and that in the direct integral decomposition of τ (formula* (0.1)*), the multiplicity function is finite almost everywhere. Then, for ν_K – a.a. $\pi \in \hat{G}$, there are $n(\pi)$ linearly independent, H-invariant distributions $\beta_j^\pi \in (\mathcal{H}_\pi^{-\infty})^H$, and a measure μ_H in the class ν_H so that*

$$\omega_H(1) = \int_{\hat{G}} \sum_{j=1}^{n(\pi)} \langle \pi(\omega)\beta_j^\pi, \beta_j^\pi \rangle \, d\mu_H(\pi), \qquad \omega \in \mathcal{D}(G). \tag{1.2}$$

The β_j^π are called the *Penney distributions*. It is explained in [**14**] and [**15**] that one consequence of this formula is the fact that the map

$$\omega_H \hookrightarrow \{\pi(\omega)\beta_j^\pi\}$$

is the intertwining operator affecting the direct integral decomposition of τ. It is also quite well known that the dimension of the space $(\mathcal{H}_\pi^{-\infty})^H$ of invariant distributions may exceed $n(\pi)$; but, in this regard, see Theorem 3.1 below.

We close this section by citing the explicit orbital formulas for all of these ingredients which are known when G and H are simply connected nilpotent Lie groups [**4, 7**] and [**13**]. A substantial amount of recent research has centered around the question of extending the applicability of these formulas to more general groups and homogeneous spaces. The facts are these:

1. The spectrum of τ is parameterized by the orbits that meet $\mathfrak{h}^\perp = \{\phi \in \mathfrak{g}^* : \phi(\mathfrak{h}) = 0\}$;

2. The multiplicity is finite precisely when the orbits in (1) have an intersection with $\mathfrak{h}^\perp$ whose dimension agrees generically with that of the H-orbits on $\mathfrak{h}^\perp$;

3. In that case, the multiplicity of a representation associated to such an orbit Λ is $\#\left(\Lambda \cap \mathfrak{h}^{\perp}\right)/H$.

4. The Penney distribution for the representation corresponding to an H-orbit $H \cdot \phi$, $\phi \in \mathfrak{h}^{\perp}$ inside $G \cdot \phi \cap \mathfrak{h}^{\perp}$ is

$$f \longrightarrow \int_{H \cap B \backslash H} \bar{f}. \tag{1.3}$$

Here B is a real polarizing group for ϕ, $\pi = \mathrm{Ind}_B^G \chi_\phi$, $\chi_\phi(\exp X) = e^{i\phi(X)}$, and the space $\mathcal{H}_\pi$ consists of Borel functions, transforming on the left under B by χ_ϕ, and square-integrable on G/B.

We shall investigate the truth of these properties for Gelfand pairs in Section 7.

2. The PFPF for $\mathbf{Ind}_K^G 1$. We assume now that G is a connected unimodular Lie group and $K \subset G$ is a compact subgroup. We are concerned with the **PFPF** for the quasi-regular representation $\tau = \mathrm{Ind}_K^G 1$.

Proposition 2.1. *For any irreducible unitary representation π of G, acting in a Hilbert space $\mathcal{H}_\pi$, we set*

$$\mathcal{H}_\pi^K = \{\xi \in \mathcal{H}_\pi : \pi(k)\xi = \xi, \forall\, k \in K\}$$

and $n(\pi) = \dim \mathcal{H}_\pi^K$. Suppose $\tau = Ind_K^G 1$ is type I. *Then there is a unique Borel measure class ν_K on $\hat{G}$, concentrated in $\hat{G}_K = \{\pi \in \hat{G} : \mathcal{H}_\pi^K \neq \{0\}\}$, such that*

$$\tau = \int_{\hat{G}}^{\oplus} n(\pi)\, \pi \, d\nu_K(\pi). \tag{2.1}$$

Proof. We first observe that the multiplicities $n(\pi)$ do not depend on the realization of π; in fact, $n(\pi)$ depends only on the (unitary equivalence) class of π. Next we observe that the technique of Anh reciprocity applies here [**1**]. In particular, since K is compact, the identity representation $1 \in \hat{K}$ has positive Plancherel measure. Thus the conclusion follows immediately from [**1**, Corollary 1.10]. □

Henceforth, we assume that K is *large enough* in G to guarantee that for

$$\nu_K - a.a.\ \pi \in \hat{G}, \quad \dim \mathcal{H}_\pi^K < \infty.$$

In that case the multiplicities $n(\pi)$ that appear in (2.1) are (almost) all finite. The abstract **PFPF** enunciated in Theorem 1.1 therefore asserts that: for $\nu_K - a.a.\ \pi \in \hat{G}$, we can find $n(\pi)$ distributions $\beta_1^\pi, \dots, \beta_{n(\pi)}^\pi \in (\mathcal{H}_\pi^{-\infty})^K$, and a choice μ_K of Plancherel measure (in the class ν_K), so that

$$\omega_K(1) = \int_{\hat{G}} \sum_{j=1}^{n(\pi)} \langle \pi(\omega)\beta_j^\pi, \beta_j^\pi \rangle \, d\mu_K(\pi), \qquad \omega \in \mathcal{D}(G). \tag{2.2}$$

Remark. The Plancherel measure μ_K in the class ν_K is not yet uniquely specified by formula (2.2). This is because the choice of the β_j^π may vary, e.g., by scalars depending on π. Canonical choices of the distributions will yield a specific μ_K, as happens in the usual Plancherel formula for a unimodular type I group [**5**, Chapter 18], where the existence of a canonical trace specifies the distributions. (See the remark in Section 6.)

We can be somewhat more precise about the choice of the distributions β_j^π. For any $\pi \in \hat{G}_K$, we may choose a family of orthonormal vectors

$$\xi_1^\pi, \dots, \xi_{n(\pi)}^\pi \in \mathcal{H}_\pi^K.$$

Then the β_j^π may be specified by the equation

$$\langle \beta_j^\pi, \xi \rangle = \langle \xi_j^\pi, \xi \rangle, \qquad 1 \le j \le n(\pi).$$

At this point we make two observations. First, it is conceivable that the space spanned by the β_j^π, $1 \le j \le n(\pi)$, does not exhaust $(\mathcal{H}_\pi^{-\infty})^K$. Second, even if it does, there is typically no canonical choice of the vectors $\{\xi_j^\pi\}$. We can move toward canonical choices if we specialize to

3. The multiplicity-free situation. We continue with the scenario of Section 2, but now we impose the additional hypothesis: for

$$\nu_K - a.a.\ \pi \in \hat{G}, \qquad n(\pi) = 1.$$

Then the quasi-regular representation τ is *multiplicity-free*

$$\tau = \int_{\hat{G}}^{\oplus} \pi \, d\nu_K(\pi).$$

Therefore, according to Penney's result [**19**], we have in this case

$$1 \leq \dim \left(\mathcal{H}_\pi^{-\infty}\right)^K, \qquad \nu_K - a.a.\ \pi \in \hat{G}.$$

But in fact, one of the main results of this paper is

Theorem 3.1. *If $\tau = Ind_K^G 1$ is multiplicity-free, then*

$$\dim \left(\mathcal{H}_\pi^{-\infty}\right)^K = 1, \qquad \nu_K - a.a.\ \pi \in \hat{G}.$$

Proof. Aside from a ν_K-null set in $\hat{G}_K$, every representation $\pi \in \hat{G}_K$ satisfies $n(\pi) = \dim \mathcal{H}_\pi^K = 1$. For any of these representations π, corresponding to any non-zero $\xi_0 \in \mathcal{H}_\pi^K$, we may define $\beta = \beta_0$ by $\langle \beta_0, \xi \rangle = \langle \xi_0, \xi \rangle, \xi \in \mathcal{H}_\pi^\infty$. Then β_0 is a non-zero element of $(\mathcal{H}_\pi^{-\infty})^K$. On the other hand, suppose $\beta \in (\mathcal{H}_\pi^{-\infty})^K$, $\beta \neq 0$. Then there is $\xi_1 \in \mathcal{H}_\pi^\infty$, $\langle \beta, \xi_1 \rangle \neq 0$. Next define a projection P on $\mathcal{H}_\pi$ by

$$P = \int_K \pi(k) \, dk.$$

We note that $P \neq 0$ since $P\xi = \xi$ for any $\xi \in \mathcal{H}_\pi^K$. In fact P is precisely the orthogonal projection of $\mathcal{H}_\pi$ onto the one-dimensional space $\mathcal{H}_\pi^K$. Next we observe that P preserves $\mathcal{H}_\pi^\infty$, and so by duality also maps $\mathcal{H}_\pi^{-\infty}$ to itself. β being K- fixed, it follows that P fixes β. Therefore $0 \neq \langle \beta, \xi_1 \rangle = \langle P\beta, \xi_1 \rangle = \langle \beta, P\xi_1 \rangle$. In particular $P\xi_1 \neq 0$. Moreover $P\xi_1 \in \mathcal{H}_\pi^\infty \cap \mathcal{H}_\pi^K$. Hence $\mathcal{H}_\pi^K \subset \mathcal{H}_\pi^\infty$. Now it is clearly no loss of generality to assume $\xi_1 \in \mathcal{H}_\pi^K$, replacing ξ_1 by $P\xi_1$ if necessary. Moreover, we may assume $\langle \beta, \xi_1 \rangle = 1$. Thus to summarize, for any nonzero $\beta \in (\mathcal{H}_\pi^{-\infty})^K$ we have produced a non-zero element $\xi_1 \in \mathcal{H}_\pi^K \subset \mathcal{H}_\pi^\infty$ satisfying $\langle \beta, \xi_1 \rangle = 1$.

Now set $\mathcal{V}_\pi = \left(\mathcal{H}_\pi^K\right)^\perp = \ker P$, $\mathcal{H}_\pi = \mathcal{H}_\pi^K \oplus \mathcal{V}_\pi$. This decomposition respects the C^∞ vectors. That is, if we set $\mathcal{V}_\pi^\infty = \mathcal{H}_\pi^\infty \cap \mathcal{V}_\pi$, then

$$\mathcal{H}_\pi^\infty = \mathcal{H}_\pi^K \oplus \mathcal{V}_\pi^\infty. \tag{3.1}$$

In fact, if $\xi \in \mathcal{H}_\pi^\infty$, then it is written uniquely

$$\xi = c\xi_1 + \eta, \quad c \in \mathbf{C}, \qquad P\eta = 0.$$

Since $\xi_1 \in \mathcal{H}_\pi^K \subset \mathcal{H}_\pi^\infty$, then $\eta \in \mathcal{V}_\pi^\infty$. Note also that $\mathcal{V}_\pi^\infty = \ker P|_{\mathcal{H}_\pi^\infty}$.

Claim. $\mathcal{V}_\pi^\infty = \ker \beta$. *In fact, if* $\eta \in \mathcal{V}_\pi^\infty$ *and* $\langle \beta, \eta \rangle \neq 0$, *then*

$$0 \neq \langle \beta, \eta \rangle = \langle P\beta, \eta \rangle = \langle \beta, P\eta \rangle = 0.$$

The reverse inclusion is evident from (3.1).

The argument concludes quickly now. Any $\xi \in \mathcal{H}_\pi^\infty$ is written uniquely $\xi = c\xi_1 + \eta, c \in \mathbf{C}$, $\eta \in \mathcal{V}_\pi^\infty = \ker \beta$. Then $\langle \beta, \xi \rangle = \langle \beta, c\xi_1 \rangle = \bar{c} = \langle \xi_1, \xi \rangle$. That is, any $\beta \in (\mathcal{H}_\pi^{-\infty})^K$ is uniquely determined by an element in the one-dimensional space $\mathcal{H}_\pi^K$; or alternatively the map $\xi \to \beta_\xi$, $\mathcal{H}_\pi^K \to (\mathcal{H}_\pi^{-\infty})^K$ is a (conjugate-linear) bijection. $\square$

We shall use Theorem 3.1 in a critical way in the last section of the paper. We close this section with two

Remarks. 1. If we rewrite formula (2.2) (i.e. the **PFPF**) in the multiplicity-free situation, we obtain

$$\omega_K(1) = \int_{\hat{G}_K} \langle \pi(\omega)\beta^\pi, \beta^\pi \rangle \, d\mu_K(\pi), \qquad \omega \in \mathcal{D}(G). \tag{3.2}$$

If we replace β^π by $c(\pi)\beta^\pi$, the Plancherel measure μ_K is replaced by $|c(\pi)|^{-2}\mu_K$. Therefore, taking into account Theorem 3.1, if we choose β^π to correspond to an invariant vector of norm 1, β^π is only uniquely specified up to a scalar of modulus 1, but μ_K is then uniquely determined.

2. It follows from [**9**, IV.3] that if τ is multiplicity-free, then

$$\dim \mathcal{H}_\pi^K \equiv \begin{cases} 1 & \pi \in \hat{G}_K \\ 0 & \pi \notin \hat{G}_K. \end{cases}$$

That is, $n(\pi)$ cannot do anything pathological on a set of measure zero.

4. Groups with co-compact nilradical. Next we specialize to connected Lie groups G with co-compact and simply connected nilradical N. That is, the quotient G/N of the group by its nilradical is presumed compact. The harmonic analysis of such groups was studied thoroughly in [**11**]. In particular, both orbital and Mackey parameters for the irreducible unitary representations of G were derived therein. The Mackey parameters start with the fact that the nilradical must split. That is, there exists a compact (connected) subgroup $K \subset G$ such that $G = K \ltimes N$. Now let $\gamma \in \hat{N}$, K_γ the stability group. Since K_γ is compact, there is no obstruction to extending γ [**6, 11**]. That is, there exists an ordinary representation $\tilde{\gamma}$ of K_γ on the space of γ satisfying

$$\tilde{\gamma}(k)\gamma(k^{-1}nk) = \gamma(n)\tilde{\gamma}(k), \qquad k \in K_\gamma,\ n \in N. \tag{4.1}$$

Equation (4.1) only specifies $\tilde{\gamma}$ up to a character of K_γ. The irreducible unitary representations of G are given by

$$\pi_{\gamma,\sigma} = \mathrm{Ind}_{K_\gamma N}^{G} \sigma \otimes \tilde{\gamma} \times \gamma, \qquad \sigma \in \hat{K}_\gamma.$$

Two of these π_{γ_1,σ_1}, π_{γ_2,σ_2} are equivalent if and only if there exists a $k \in K$ such that $k \cdot \gamma_1 = \gamma_2$, $k \cdot \sigma_1 \cong \sigma_2$. Thus the Mackey parameterization of $\hat{G}$ is summarized by the fiber diagram

$$\begin{array}{ccc} \hat{K}_\gamma & \longrightarrow & \hat{G} \\ & & \downarrow \\ & & \hat{N}/G. \end{array}$$

Here is the orbital parameterization. Let $\mathfrak{A}(G)$ denote the *allowable* linear functionals in the real dual of the Lie algebra

$$\mathfrak{A}(G) = \{\phi \in \mathfrak{g}^* : \exists \text{ a unitary character } \chi_\phi \text{ of } G_\phi \ni d\chi_\phi = i\phi|_{\mathfrak{g}_\phi}\}.$$

The stability groups G_ϕ will not be connected in general. But for the application we have in mind, they are connected. So now assume G_ϕ is connected for all $\phi \in \mathfrak{A}(G)$. Then χ_ϕ is unique if it exists. Now a key role in [**11**, Lemma 4.2] is played by the *Alignment Lemma.* It asserts that within any G-orbit (over all of $\mathfrak{g}^*$ actually), there exists a functional ϕ satisfying

$$G_\phi = K_\phi N_\phi, \qquad G_\theta = K_\theta N_\theta, \qquad \theta = \phi|_{\mathfrak{n}}.$$

Any such functional is called *aligned*. The alignment property is preserved under the action of K, hence any functional can be aligned by means of an element from N. If two functionals ϕ and $n \cdot \phi$ are both aligned, then $n \in N_\theta Z_N(K_\theta)$ [**11**]. Now let $\phi \in \mathfrak{A}(G)$ be aligned, $\theta = \phi|_{\mathfrak{n}}$. Let $\gamma_\theta \in \hat{N}$ be the corresponding representation. Then it is possible to make a canonical choice of the extension $\tilde{\gamma}_\theta$–see [**11**] (or more generally [**6**]). Briefly, we select a K_θ-invariant positive polarization $\mathfrak{h}$ for θ, realize γ_θ by holomorphic induction via $\mathfrak{h}$, and then define $\tilde{\gamma}_\theta(k)f(n) = f(k^{-1}nk)$, when f is realized in the natural space of equivariant functions that comprise the space of the holomorphically induced representation. (The reader can find the details in [**6**], including the independence of polarization argument.) Next, let $\xi = \phi|_{\mathfrak{k}_\theta} \in \mathfrak{A}(K_\theta)$. Then $(K_\theta)_\xi = K_\phi$ and thus [**11**] a unique representation $\sigma_\xi \in \hat{K}_\theta$ is determined (we are using the connectivity of the stabilizer here.) The representation of G corresponding to ϕ is then $\pi_\phi := \pi_{\gamma_\theta,\sigma_\xi}$. The Mackey parameters can be completely expunged from the picture by using holomorphic induction on G itself. For example, augment the above choice of $\mathfrak{h}$ by the choice $\mathfrak{a}$ of a positive polarization for ξ. Then the algebra $\mathfrak{b} = \mathfrak{a} + \mathfrak{h}$ is a positive polarization for ϕ satisfying the Pukanszky condition, and the corresponding holomorphically induced representation of G, usually denoted $\mathfrak{b}\text{-Ind}_{G_\phi}^G \chi_\phi$, is equivalent to π_ϕ. (See [**11**].) Thus the orbital parameterization of $\hat{G}$ (with connected stabilizers) can be "diagramed" via

$$\begin{array}{ccc} \mathfrak{A}(K_\theta)/K_\theta & \longrightarrow & \hat{G} = \mathfrak{A}(G)/G \\ & & \downarrow \\ & & \mathfrak{n}^*/G. \end{array}$$

5. Gelfand pairs. Now specialize further to Gelfand pairs (G, K). We take $G = K \ltimes N$ as in the previous section, but we suppose $\tau = \text{Ind}_K^G 1$ is multiplicity-free. One of the basic results of [**3**] is that, for all practical purposes, N must be a Heisenberg group. We assume henceforth therefore that N is a $(2r+1)$-dimensional Heisenberg group. Since K is connected (and compact) it must fix $Z :=$ Cent N. It acts naturally then on $N/Z = \mathbf{C}^r$, and so we may assume $K \subset U(r)$. In somewhat more detail, we shall realize N as in [**3**] or [**17**]; namely,

$$N = V \oplus \mathbf{R}, \qquad V = \mathbf{C}^r$$

with multiplication

$$nn' = (v,t)(v',t') = \left(v + v', t + t' + \frac{1}{2}\operatorname{Im} v \cdot \bar{v}'\right).$$

The Lie algebra $\mathfrak{n}$ can be parameterized by the same coordinates, except that the Lie bracket is specified by

$$[(v,t),(v',t')] = (0, \operatorname{Im} v \cdot \bar{v}').$$

We can therefore denote elements $\theta \in \mathfrak{n}^*$ by the coordinates $\theta = (\psi, \lambda)$, $\psi \in V^* = \operatorname{Hom}_{\mathbf{R}}(V, \mathbf{R})$, $\lambda \in \mathbf{R}$ where

$$\theta(v,t) = \psi(v) + \lambda t.$$

It is easy to compute the adjoint and co-adjoint actions in these coordinates: for $n = (v,t) \in N$, $X = (w,s) \in \mathfrak{n}$, $\theta = (\psi, \lambda) \in \mathfrak{n}^*$, we have

$$\begin{aligned} \operatorname{Ad} n\, X &= (w, s + \operatorname{Im} v \cdot \bar{w}) \\ \operatorname{Ad}^* n\, \theta &= (\psi - \hat{v}, \lambda), \end{aligned}$$

where $\hat{v} \in V^*$ is defined by $\hat{v}(v_1) = \operatorname{Im} v \cdot v_1$.

Now any compact (connected) subgroup $K \subset U(r)$ acts naturally on N by $k \cdot (v,t) = (k \cdot v, t)$, where the dot on the right denotes the natural action of $U(r)$ on $\mathbf{C}^r$. The semidirect product group $G = K \ltimes N$ is our object of concern.

Various equivalent criteria for the representation $\tau = \operatorname{Ind}_K^G 1$ to be multiplicity-free are known. The most important ones are found in [**3**], [**10**] and [**2**]. They are:

1. the algebra $L^1(G//K)$ of K-bi-invariant integrable functions under convolution is abelian;

2. the algebra $D(G/K)$ of G-invariant differential operators on G/K is abelian;

3. the action of K_c, the complexification of K, on the algebra $\mathcal{P}(V)$ of holomorphic polynomials is without multiplicity;

4. the algebra $PD(V)^K$ of K-invariant differential operators with polynomial coefficients is abelian;

5. a Borel subgroup of K_c has a Zariski-open orbit on V.

In case K acts irreducibly on V, these actions have been classified by V. Kac. This is tabulated elegantly in [**3**].

The "soft" representation-theoretic decomposition of τ can be read off from [**13**,Theorem 7.1]. The details are as follows. G has three kinds of irreducible representations: (i) those trivial on N, i.e., irreducibles of K; (ii) those trivial on Z, but not N, i.e., the infinite-dimensional irreducibles of the motion group $K \ltimes (N/Z)$; and (iii) those non-trivial on Z. The representations in (i) are finite-dimensional. Those in (ii) constitute the spectrum of the abelian symmetric space $K \ltimes (N/Z)/K$, which is well understood [**12**]. The representations in (iii) are the generic representations of G, while those in (i)–(ii) are degenerate and do not appear in τ. We shall deal only with those in (iii) in what follows.

Fix a non-zero element $Z_0 \in \operatorname{Cent}\mathfrak{n}$. If $\phi \in \mathfrak{g}^*$ is generic, then $\lambda = \phi(Z_0) \neq 0$. Corresponding to each $\lambda \neq 0$, the flat variety $\Lambda_\lambda = \{\theta \in \mathfrak{n}^* : \theta(Z_0) = \lambda\} = \{(\psi, \lambda) : \psi \in V^*\}$ is a single N-orbit. Hence, there is a unique irreducible $\gamma_\lambda \in \hat{N}$ corresponding. For any $\theta \in \Lambda_\lambda$, there is a unique $n \in V$ so that $n \cdot \theta = (0, \lambda)$ is aligned. This is because the set $N_\theta Z_N(K)$ equals Z. Indeed, $N_\theta = Z$ and any element in $Z_N(K)$ would give rise to a K-invariant vector in V. In that case the action of K on $\mathcal{P}(V)$ would have uniform infinite multiplicity. (Note that we are using $K_\theta = K$ here.)

The representation $\tilde{\gamma}_\lambda = \tilde{\gamma}_\theta$, $\theta \in \Lambda_\lambda$ was defined canonically earlier. We have

Lemma 5.1. *The class of $\tilde{\gamma}_\lambda$ as a representation of K depends at most on* $\operatorname{sgn}(\lambda)$.

Proof. It is enough to prove the result for $K = U(r)$. Actually this is a well-known fact. The scalars act on N by $a\cdot(v,t)=(av,a^2t), a\in\mathbf{R}^\times$. Moreover, that action commutes with the action of K. The result is clear from those observations. □

Thus for any λ, $\lambda'\neq 0$, we have $\tilde{\gamma}_\lambda\cong\tilde{\gamma}_{\lambda'}$ if $\operatorname{sgn}(\lambda)=\operatorname{sgn}(\lambda')$. $\tilde{\gamma}_1$ may or may not be equivalent to $\tilde{\gamma}_{-1}$ depending upon the size of K. For example, they are not equivalent if $K=U(r)$ or $\mathbf{T}^r$, but they are equivalent for $K=SU(2)$.

Now suppose $\tilde{\gamma}_\lambda=\sum^{\oplus}_{\sigma\in\hat{K}} n_\lambda(\sigma)\,\sigma$. Our assumption is that $n_\lambda(\sigma)\leq 1$. Write

$$\begin{aligned}\hat{K}^+ &= \text{ spec of } \tilde{\gamma}_\lambda, \qquad \lambda>0\\ \hat{K}^- &= \text{ spec of } \tilde{\gamma}_\lambda, \qquad \lambda<0.\end{aligned}$$

Then, according to [**13**], we have

$$\tau=\int^{\oplus}_{\lambda>0}\sum^{\oplus}_{\bar{\sigma}\in\hat{K}^+}\pi_{\lambda,\sigma}\,d\lambda\oplus\int^{\oplus}_{\lambda<0}\sum^{\oplus}_{\bar{\sigma}\in\hat{K}^-}\pi_{\lambda,\sigma}\,d\lambda. \tag{5.1}$$

We have used the obvious abbreviation $\pi_{\lambda,\sigma}=\pi_{\gamma_\lambda,\sigma}$. The Lebesgue measure *class* is indicated. But of course one of our main desires is to specify the precise measure within the class; see Theorem 6.2 below.

It is also worth mentioning that when the multiplicity-free hypothesis is suspended, several new (and perhaps unexpected) features emerge. The features are illustrated in the following

Examples. In all of the following $r=2$ and $K=\mathbf{T}$, but we take three different embeddings of K into $U(2)$. The first two are discussed in [**13**], the last in [**2**].

1. Let $K=\{u\in\mathbf{C}:|u|=1\}$ act on $V=\mathbf{C}^2=\{v=(v_1,v_2): v_j\in\mathbf{C}\}$ by $u\cdot(v_1,v_2)=(uv_1,u^{-1}v_2)$. One computes readily that the allowable G-orbits which meet $\mathfrak{k}^\perp$ parameterize the spectrum of τ. Furthermore, the generic orbit intersections $G\cdot\phi\cap\mathfrak{k}^\perp$, $\phi\in\mathfrak{A}(G)\cap\mathfrak{k}^\perp$, are of dimension 3, 2 bigger than the generic K-orbit dimensions. And, in fact, the multiplicity in τ is uniformly infinite.

2. Let K act on V by $u\cdot(v_1,v_2)=(uv_1,uv_2)$. Once again, the allowable G-orbits that meet $\mathfrak{k}^\perp$ parameterize the spectrum. And

the orbit dimension statements are unchanged. But in this case, the multiplicity is finite, albeit unbounded.

3. Let K act on V by $u \cdot (v_1, v_2) = (u^2 v_1, u^3 v_2)$. Again we have finite unbounded multiplicity. Moreover, one can compute that although the representation $\pi_{1,1}$ fails to appear in the spectrum of τ, its corresponding orbit does meet $\mathfrak{k}^{\perp}$.

The import of these examples will become fully clear after the orbital presentation of the **PFPF** for τ in the last section.

6. The PFPF for Gelfand pairs. Now we want the Penney distributions and an explicit Plancherel measure for Gelfand pairs. The material in Section 3 tells us how to get the data. We shall work in this section in the Mackey parameters. In the next section we turn to the orbital configuration.

From formula (5.1), we see that the spectrum of τ consists of representations

$$\{\pi_{\lambda,\sigma} : \lambda > 0 \text{ and } \bar{\sigma} \in \hat{K}^{+};\ \text{or } \lambda < 0 \text{ and } \bar{\sigma} \in \hat{K}^{-}\}. \tag{6.1}$$

Hence, in what follows, we always assume the pair (λ, σ) satisfies one of the conditions in the set (6.1). Then the representation $\pi_{\lambda,\sigma} = \sigma \otimes \tilde{\gamma}_\lambda \times \gamma_\lambda$ acts on $\mathcal{H}_\sigma \otimes \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda := \mathcal{H}_{\gamma_\lambda}$. Moreover, $\pi_{\lambda,\sigma}|_K = \sigma \otimes \tilde{\gamma}_\lambda$. Let $\mathcal{H}_\lambda(\sigma)$ stand for the unique K-invariant finite-dimensional subspace of $\mathcal{H}_\lambda$ which transforms under the action of K according to $\bar{\sigma}$. We then realize $\mathcal{H}_\sigma$ as a copy of $\mathcal{H}_\lambda(\sigma)$ with the conjugate structure. Then we need to identify a vector in $\mathcal{H}_\sigma \otimes \mathcal{H}_\lambda(\sigma)$ which is fixed by K. But that is classical. Let $\{\xi_l^\sigma\}$ be any orthonormal basis in $\mathcal{H}_\sigma$. Denote the corresponding family in the conjugate space $\mathcal{H}_\lambda(\sigma)$ by $\{\bar{\xi}_l^\sigma\}$. Then an invariant vector is given by $v_{\lambda,\sigma} = \sum_l \xi_l^\sigma \otimes \bar{\xi}_l^\sigma$. Correspondingly, we can define an invariant distribution by

$$\beta_{\lambda,\sigma} : \xi \to \langle v_{\lambda,\sigma}, \xi \rangle. \tag{6.2}$$

Of course a different choice of basis might lead to a different invariant vector (and so a different distribution), but only up to a scalar (by Theorem 3.1). In accordance with the first remark after Theorem 3.1, we can uniquely specify the matrix coefficients of $\beta_{\lambda,\sigma}$ by normalizing.

Hence we modify the choice as follows

$$(6.3) \qquad v_{\lambda,\sigma} = d_\sigma^{-1/2} \sum_{l=1}^{d_\sigma} \xi_l^\sigma \otimes \bar{\xi}_l^\sigma, \qquad d_\sigma = \dim \sigma.$$

$\beta_{\lambda,\sigma}$ is defined as before in terms of $v_{\lambda,\sigma}$ by formula (6.2).

Before stating the explicit **PFPF** for (G, K) we recall the well-known Plancherel theorem for the unimodular group N. It asserts

Theorem 6.1 (Plancherel theorem for the Heisenberg group). *There is a scalar c_r so that for any $f \in \mathcal{D}(N)$, we have*

$$f(1) = c_r \int_{\mathbf{R}} Tr\, \gamma_\lambda(f)\, |\lambda|^r \, d\lambda,$$

where $\gamma_\lambda(f) = \int_N \gamma_\lambda(n) f(n)\, dn$.

Remark. As is well known, this can be interpreted as a **PFPF** by considering $N \times N/\Delta$, $\Delta = \{(n,n) : n \in N\}$. The irreducibles are those of the form $\gamma_\lambda \otimes \bar{\gamma}_\lambda$, $\gamma_\lambda \in \hat{N}$, which act on $\mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_\lambda$. The latter Hilbert space is naturally isomorphic to $\mathcal{HS}(\mathcal{H}_\lambda)$, the Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. The canonical Δ-invariant distribution on the space $\mathcal{HS}(\mathcal{H}_\lambda)$ is precisely

$$\beta_\lambda : T \longrightarrow Tr(T)$$

and a standard computation reveals that if $f = \omega_\Delta$, $\omega \in \mathcal{D}(N \times N)$, then

$$\langle (\gamma_\lambda \otimes \bar{\gamma}_\lambda)(\omega)\beta_\lambda, \beta_\lambda \rangle = Tr\, \gamma_\lambda(f).$$

For this, see [**19**, Chapter 3].

Now we are ready for the **PFPF** for (G, K).

Theorem 6.2. *Let (G, K) be a Gelfand pair. Let*

$$\mathcal{H}_{\pm 1} = \sum_{\bar{\sigma} \in \hat{K}^{\pm}}^{\oplus} \mathcal{H}_{\pm 1}(\sigma)$$

be the decomposition of the Hilbert space $\mathcal{H}_{\pm 1}$ into inequivalent invariant irreducible unitary K-modules. By Lemma 5.1, *the decomposition* *for $\mathcal{H}_{\pm 1}$ is the same as that for any $\mathcal{H}_{\pm\lambda}$, $\lambda > 0$. Let $\{\bar{\xi}_l^\sigma\}$ be any orthonormal basis in $\mathcal{H}_{\pm 1}(\sigma)$. Define $v_{\lambda,\sigma}$ by (6.3) and $\beta_{\lambda,\sigma}$ by (6.2). Then the* **PFPF** *takes the form*

(6.4)

$$\omega_K(1) = c_r \int_{\lambda>0} \sum_{\bar{\sigma}\in\hat{K}^+} \langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \beta_{\lambda,\sigma}\rangle\, d_\sigma\, |\lambda|^r\, d\lambda + c_r \int_{\lambda<0} \sum_{\bar{\sigma}\in\hat{K}^-} \langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \beta_{\lambda,\sigma}\rangle\, d_\sigma\, |\lambda|^r\, d\lambda, \qquad \omega\in\mathcal{D}(G).$$

Proof. We compute the matrix coefficients $\langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \beta_{\lambda,\sigma}\rangle$. We begin by computing the vector $\pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}$. The representation $\pi = \pi_{\lambda,\sigma}$ acts in $\mathcal{H}_\sigma \otimes \mathcal{H}_\lambda$. Let $\xi \in \mathcal{H}_\sigma$, $\eta \in \mathcal{H}_\lambda$. Then, using the usual functional adjoint $\omega^*(g) = \bar{\omega}(g^{-1})$, we have

$$\begin{aligned}
\langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \xi\otimes\eta\rangle &= \langle \beta_{\lambda,\sigma}, \pi(\omega)^*\xi\otimes\eta\rangle \\
&= \langle \beta_{\lambda,\sigma}, \pi(\omega^*)\xi\otimes\eta\rangle \\
&= \int_G \omega(g^{-1})\langle \beta_{\lambda,\sigma}, \pi(g)\xi\otimes\eta\rangle\, dg \\
&= \int_{KN} \omega(n^{-1}k^{-1})\langle \beta_{\lambda,\sigma}, \sigma(k)\xi \otimes \tilde{\gamma}_\lambda(k)\gamma_\lambda(n)\eta\rangle\, dk\, dn \\
&= \int_{KN} \omega(n^{-1}k^{-1})\langle v_{\lambda,\sigma}, \sigma(k)\xi \otimes \tilde{\gamma}_\lambda(k)\gamma_\lambda(n)\eta\rangle\, dk\, dn \\
&= \int_{KN} \omega(n^{-1}k^{-1}) d_\sigma^{-1/2} \sum_{l=1}^{d_\sigma} \langle \xi_l^\sigma \otimes \bar{\xi}_l^\sigma, \sigma(k)\xi \otimes \tilde{\gamma}_\lambda(k)\gamma_\lambda(n)\eta\rangle\, dk\, dn \\
&= \int_{KN} \omega(n^{-1}k^{-1}) d_\sigma^{-1/2} \sum_{l=1}^{d_\sigma} \langle \xi_l^\sigma, \sigma(k)\xi\rangle \langle \bar{\xi}_l^\sigma, \tilde{\gamma}_\lambda(k)\gamma_\lambda(n)\eta\rangle\, dk\, dn \\
&= \int_{KN} \omega(n^{-1}k^{-1}) d_\sigma^{-1/2} \sum_{l=1}^{d_\sigma} \langle \sigma(k^{-1})\xi_l^\sigma, \xi\rangle \langle \tilde{\gamma}_\lambda(k^{-1})\bar{\xi}_l^\sigma, \gamma_\lambda(n)\eta\rangle\, dk\, dn.
\end{aligned}$$

But on $\mathcal{H}_\sigma \otimes \mathcal{H}_\lambda(\sigma)$ the vector $\sum \xi_l^\sigma \otimes \bar{\xi}_l^\sigma$ is K-invariant. Hence, the last formula is exactly

$$\int_N \omega_K(n) d_\sigma^{-1/2} \sum_{l=1}^{d_\sigma} \langle \xi_l^\sigma, \xi \rangle \langle \gamma_\lambda(n^{-1}) \bar{\xi}_l^\sigma, \eta \rangle \, dn$$
$$= \int_N \omega_K(n) d_\sigma^{-1/2} \langle \sum_{l=1}^{d_\sigma} \xi_l^\sigma \otimes \gamma_\lambda(n^{-1}) \bar{\xi}_l^\sigma, \xi \otimes \eta \rangle \, dn.$$

In particular, we have

$$\pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma} = \int_N \omega_K(n) d_\sigma^{-1/2} \sum_{l=1}^{d_\sigma} \xi_l^\sigma \otimes \gamma_\lambda(n^{-1}) \bar{\xi}_l^\sigma \, dn.$$

This enables us to complete the computation of the matrix coefficient. Indeed

$$\begin{aligned}
\langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \beta_{\lambda,\sigma} \rangle &= \langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, v_{\lambda,\sigma} \rangle \\
&= \sum_{l'=1}^{d_\sigma} d_\sigma^{-1/2} \langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \xi_{l'}^\sigma \otimes \bar{\xi}_{l'}^\sigma \rangle \\
&= \sum_{l'=1}^{d_\sigma} d_\sigma^{-1} \int_N \omega_K(n) \\
&\qquad \cdot \sum_{l=1}^{d_\sigma} \langle \xi_l^\sigma \otimes \gamma_\lambda(n^{-1}) \bar{\xi}_l^\sigma, \xi_{l'}^\sigma \otimes \bar{\xi}_{l'}^\sigma \rangle \, dn \\
&= \int_N \omega_K(n) d_\sigma^{-1} \sum_{l=1}^{d_\sigma} \langle \gamma_\lambda(n^{-1}) \bar{\xi}_l^\sigma, \bar{\xi}_l^\sigma \rangle \, dn.
\end{aligned}$$

Next we observe that the family $\left\{ \{\bar{\xi}_l^\sigma\}_{l=1}^{d_\sigma} \right\}_{\bar{\sigma} \in \hat{K}^\pm}$ forms a complete orthonormal family inside $\mathcal{H}_{\pm\lambda}$. Therefore

$$\begin{aligned}
\sum_{\bar{\sigma} \in \hat{K}^\pm} \langle \pi_{\lambda,\sigma}(\omega)\beta_{\lambda,\sigma}, \beta_{\lambda,\sigma} \rangle \, d_\sigma &= \sum_{\bar{\sigma} \in \hat{K}^\pm} \int_N \omega_K(n) \sum_{l=1}^{d_\sigma} \langle \gamma_\lambda(n^{-1}) \bar{\xi}_l^\sigma, \bar{\xi}_l^\sigma \rangle \, dn \\
&= Tr\, \gamma_\lambda(\check{\omega}_K),
\end{aligned}$$

where $\check{\omega}_K(n) = \omega_K(n^{-1})$. And so finally the right side of (6.4) equals

$$c_r \int_{\mathbf{R}} Tr\,\gamma_\lambda(\check{\omega}_K)|\lambda|^r\,d\lambda = \check{\omega}_K(1) \qquad \text{by Theorem 6.1}$$
$$= \omega_K(1).$$

While this is nice, there is still, because of the choice of the orthonormal basis, too much ambiguity in the choice of the distributions. We want something even more canonical. We can achieve that by recasting the Plancherel formula and its distributions into an orbital mode. That is the content of the next section.

7. Orbital formulation of the Penney distributions. We wish to reformulate the results of the last section into an orbital configuration for two reasons. First of all, since there is an orbital presentation of $\hat{G}$, there should be one for $\hat{G}_K$ and the Penney distributions. Second, we have indicated that there is still a certain degree of "ambiguity of choices" in the definition of the Penney distributions (see (6.2) and (6.3)). An orbital formulation could clean that up. Thus the two specific problems we shall concentrate on in this section are:

1. Give an orbital description of the Penney distributions;

2. Give an orbital characterization of the spectrum $\hat{G}_K$ and multiplicity-free condition.

We shall settle the first and obtain a partial solution for the second. Problem 2 is treated in much greater detail in [**2**].

We take our cue from the data in Section 4 and the list of facts in Section 1. The orbital parameters for $\hat{G}_K$ should be the allowable G-orbits that meet $\mathfrak{k}^\perp$. Moreover, given $\phi \in \mathfrak{A}(G) \cap \mathfrak{k}^\perp$ and the corresponding representation π_ϕ, then, presuming it occurs in τ, its Penney distribution (which is unique up to scalar by Theorem 3.1) should be given by

$$\beta_\phi : f \longrightarrow \int_{K\cap B\backslash K} \bar{f}.$$

The only ambiguity would be the relatively trivial normalization of the invariant measure. The problem is that $\pi_\phi = \mathfrak{b} - \mathrm{Ind}_{G_\phi}^G \chi_\phi$ is holomorphically induced and $\mathfrak{b}$ is a subalgebra of the complexification $\mathfrak{g}_c$, not of $\mathfrak{g}$. How do we realize the distribution β_ϕ? The answer: Use the *exact same formula*, interpreting B as a complex polarizing group. We make that precise in Theorem 7.1 below.

Suppose a representation $\pi_{\gamma,\sigma}$ appears in the spectrum of τ. Then $\gamma = \gamma_\lambda$ for some λ and the pair (λ, σ) satisfy one of the conditions in the set (6.1). Let $\theta \in \Lambda_\lambda$, $\theta = (0, \lambda)$, and let $\xi \in \mathfrak{A}(K)$ correspond to σ. Then $\phi = \xi + \theta \in \mathfrak{A}(G)$ is aligned. We may construct a positive polarization for ϕ as in section 4, and then

$$\pi_{\gamma,\sigma} = \pi_\phi = \mathfrak{b} - \mathrm{Ind}_{G_\phi}^G \chi_\phi.$$

We shall prove below (in Proposition 7.2) that there exists $n \in N$ such that $\phi_1 = n \cdot \phi \in \mathfrak{k}^\perp$. (This is the partial answer to (2) mentioned above.) If we also conjugate $\mathfrak{b}$ by n, then, writing $\mathfrak{b}_1 = n \cdot \mathfrak{b}$, we have

$$\pi_\phi \cong \pi_{\phi_1} = \mathfrak{b}_1 - \mathrm{Ind}_{G_{\phi_1}}^G \chi_{\phi_1}, \qquad \phi_1 \in \mathfrak{A}(G) \cap \mathfrak{k}^\perp.$$

Changing notation slightly, we can state the main result of the section

Theorem 7.1. *Let $\phi \in \mathfrak{A}(G) \cap \mathfrak{k}^\perp$, and suppose the representation of G corresponding to the orbit $G \cdot \phi$ occurs in the spectrum of τ. Suppose $\mathfrak{b}$ is a positive complex polarization for ϕ satisfying the Pukanszky condition, and the admissibility condition: $\mathfrak{b} \cap \mathfrak{n}_c$ is a positive G_θ-invariant polarization for $\theta = \phi|_{\mathfrak{n}}$. Then $\pi_\phi = \mathfrak{b} - Ind_{G_\phi}^G \chi_\phi$ realizes the representation and the corresponding Penney distribution is given by*

$$\beta_\phi : f \longrightarrow \int_{K \cap \mathbf{B} \backslash K} \bar{f}, \qquad f \in \mathcal{H}_{\pi_\phi}^\infty. \tag{7.1}$$

Remarks. 1. The holomorphically induced representation π_ϕ acts by right translation in the space $\mathcal{H}_\phi = \mathcal{H}_{\pi_\phi}$ of Borel functions satisfying

$$f : G \longrightarrow \mathbf{C}$$

$$f(g_\phi g) = \chi_\phi(g_\phi) f(g), \qquad g_\phi \in G_\phi,\ g \in G \tag{7.2}$$

$$X \star f = -i\phi(X) f, \qquad X \in \mathfrak{b} \tag{7.3}$$

$$\int_{D \backslash G} |f|^2 \, d\dot{g} < \infty, \qquad D = \exp(\mathfrak{b} \cap \bar{\mathfrak{b}})$$

(see [**17**]). The complexified group $G_c = K_c \ltimes N_c$ is defined in an obvious manner. We interpret $\mathbf{B}$ as the analytic subgroup of G_c with Lie algebra $\mathfrak{b}$.

2. The admissibility condition on the polarization is probably redundant; but that is not established in [**11**].

3. In [**15**] we demanded that HB be closed. Since K is compact, that is automatic here. But I would like to take this opportunity to correct an error in [**15**]. In fact, it may *not* be true that one can find a real polarization such that HB is closed. Without that, the implication $f \in C^\infty(G, H) \Rightarrow f|_H \in C^\infty(H, H \cap B)$ may be false. However, that does not affect the main results of [**15**] since one can in fact build into the induction argument the convergence of the integral in part (iii) of [**15**, Theorem 5.1].

Proof of Theorem 7.1. If $\phi \in \mathfrak{k}^\perp$, then $\chi_\phi|_{K \cap \mathbf{B}} \equiv 1$. Hence, because of (7.2) and (7.3), the integral in (7.1) makes sense. It is well known (see [**20**]) that the functions in $\mathcal{H}_\phi^\infty$ are smooth. Therefore, the integral converges and the distribution β_ϕ is well-defined as an element of $\mathcal{H}_\phi^{-\infty}$. But by the nature of the integral, it is also clear that β_ϕ is K-invariant. An application of Theorem 3.1 proves that it must be the Penney distribution corresponding to π_ϕ. □

Remark. The intersection $K \cap \mathbf{B}$ is no bigger than K_ϕ, but it may be smaller. To see that, note that if we start with an aligned functional ϕ_0 in the orbit of ϕ, then any polarization $\mathfrak{b}_0$ for ϕ_0 will be totally complex in the Levi part. In particular $K \cap \mathbf{B}_0 = K_{\phi_0}$. Then, let $n = n(v, 0)$ be the unique element conjugating ϕ_0 to ϕ. It is not difficult to check that

$$K \cap \mathbf{B} = (K_\phi)_v \,.$$

Very often this group is trivial. But it can also be large. For example, the trivial representation $\sigma = 1$ is in $\hat{K}^\pm$ for any $\lambda \neq 0$, since the constant polynomial on V is always K-invariant. In other words, the functionals $\phi = (0, \lambda)$ lie in $\mathfrak{k}^\perp \cap \mathfrak{A}(G)$, and in this case $K_\phi = K$ and $K \cap \mathbf{B} = K$. This is an instance in which the Penney distribution reduces to a point evaluation. Although the representations $\pi_\phi = \pi_{\lambda,0}$ constitute only a "small" portion of the spectrum of τ, they are generic.

This concludes the discussion of problem 1. Here is what we have to say regarding problem 2.

Proposition 7.2. *Suppose the representation π_ϕ, $\phi \in \mathfrak{A}(G)$, appears in the spectrum of τ. Then $G \cdot \phi \cap \mathfrak{k}^\perp \neq \varnothing$.*

Proof. Of course, it is no loss of generality to assume $\phi = \xi + \theta_\lambda$, $\theta_\lambda = (0, \lambda)$, is aligned. To say that π_ϕ occurs in the spectrum of τ is to say, by Section 5, that $\bar{\sigma}_\xi = \sigma_{-\xi} \in \operatorname{Spec} \tilde{\gamma}_\lambda$. Thus the statement of the proposition comes down to

$$\sigma_{-\xi} \in \operatorname{Spec} \tilde{\gamma}_\lambda \Longrightarrow N \cdot (\xi + \theta_\lambda) \cap \mathfrak{k}^\perp \neq \varnothing.$$

It is a straightforward computation to verify this fact when $K = U(r)$. (Indeed in that case the implication is valid in both directions.) Now we can rewrite the condition $N \cdot (\xi + \theta_\lambda) \cap \mathfrak{k}^\perp \neq \varnothing$. It says exactly that there is a vector $v \in V$ such that

$$\xi(X) = \frac{1}{2}\lambda[X \cdot v, v], \qquad \forall\, X \in \mathfrak{k}. \tag{7.4}$$

(The right side of (7.4) is an un-normalized moment map $\mu_\lambda : V \to \mathfrak{k}^*$, $\mu_\lambda(v)(X) = (\lambda/2)[X \cdot v, v]$, see [**21**] and [**22**]. But we shall not pursue that here.)

Now we know the precise spectrum of $\tilde{\gamma}_\lambda|_{U(r)}$. It consists of the irreducible representations of $U(r)$ acting on the spaces of homogeneous polynomials of degree m (in v or $\bar{v}$ depending on $\operatorname{sgn}(\lambda)$), $m \geq 0$. Let us write π_m for those representations and Ω_m for the corresponding coadjoint orbits in $\mathfrak{u}(r)^*$. Therefore, if $K \subset U(r)$ and $\sigma_{-\xi} \in \operatorname{Spec} \tilde{\gamma}_\lambda|_K$, then $\sigma_{-\xi} \in \operatorname{Spec} \pi_m|_K$ for some $m \geq 0$. Let $p : \mathfrak{u}(r)^* \to \mathfrak{k}^*$ denote the canonical projection. According to Heckman's result [**8**], we must have

$$-\xi = p(\alpha), \quad \text{some } \alpha \in \Omega_m. \tag{7.5}$$

Furthermore, referring back to the facts on $\mathfrak{u}(r)^*$, we see that there must exist a vector $v \in V$ satisfying

$$-\alpha(X) = \frac{1}{2}\lambda[X \cdot v, v], \qquad X \in \mathfrak{u}(r). \tag{7.6}$$

The proof is concluded by combining (7.5) and (7.6).

Notes. 1. The proof of Proposition 7.2 is valid for any semidirect product $G = K \ltimes N$, when N is a $(2r+1)$-dimensional Heisenberg group and $K \subset U(r)$, independent of whether (G, K) is a Gelfand pair.

2. Proposition 7.2 is also proven in [**2**, Theorem 4.3], but from the perspective of the moment map.

The concluding comments are the following. According to the facts in Section 4 and Examples 1–2 in Section 5, the condition that should characterize a Gelfand pair, i.e., the multiplicity-free situation, is

$$G \cdot \phi \cap \mathfrak{k}^{\perp} = K \cdot \phi, \quad \text{all } \phi \in \mathfrak{A}(G) \cap \mathfrak{k}^{\perp}.$$

In fact, these examples and the nilpotent theory [**4, 13**] strongly suggest that this condition is equivalent to *bounded* finite multiplicity. Indeed, the equivalence of these three properties, i.e., Gelfand pair, finite bounded multiplicity, and multiplicity-free, is proven in [**2**] when $K \subset U(r)$ is of full rank. Further evidence is offered in [**2**] to suggest that the equivalences are true without the rank condition. Finally, as Example 3 in Section 5 shows, the converse of Proposition 7.2 is *not* true in general, but I believe it is true generically for Gelfand pairs. Evidence for that belief is also supplied in [**2**].

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