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FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO DIVISORS

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ABSTRACT. The purpose of this paper is to study factorization in commutative rings with zero divisors with particular emphasis on how the theory of factorization in integral domains is similar to and different from the theory for commutative rings. Various notions of "associate" are considered. Each form of "associate" leads to a type of "irreducible" element, and each type of "irreducible" element leads to a form of atomicity (elements being products of that type of irreducible element) and unique factorization. Numerous examples are given, including an example of an atomic LCM ring which does not satisfy ACCP or have unique factorization. Factorization in polynomial rings and power series rings is considered.

1. Introduction. A fundamental theme in algebra is the factorization of elements into irreducible elements. The setting is usually a commutative integral domain R with identity. A nonzero, nonunit element a of R is said to be irreducible if for any factorization $a = bc$, either b or c is a unit. There are then two natural questions: (1) What integral domains have the property that every nonzero, nonunit element has a factorization into irreducible elements? and (2) What uniqueness properties, if any, do factorizations into irreducible elements have? As to the first question, usually some chain condition, such as the ascending chain condition on principal ideals, is used to show that every element has a factorization into irreducible elements. The second question is more complicated. Factorizations can be highly nonunique. The class group of a Dedekind domain (or more generally, the divisor class group of a Krull domain) in some sense measures the deviation from having unique factorization. For a discussion of the possible lengths of factorizations, see Anderson and Anderson [**5**].

Much of the theory of factorization in an integral domain can be generalized to commutative rings with zero divisors, often in several ways. Some of this has already been done in a series of papers by

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Bouvier (see references). And several authors, including Anderson and Markanda [**10**] and [**11**], Billis [**13**], Bouvier, Fletcher [**28**] and [**29**], and Galovich [**31**] have considered unique factorization in commutative rings with zero divisors.

Section 2 investigates the notion of two elements being associates. Let R be a commutative ring with identity. Elements a and b of R are said to be associated (respectively, strongly associated) if $(a) = (b)$ (respectively, $a = ub$ for some unit u). A nonunit element a of R is irreducible (respectively, strongly irreducible) if $a = bc$ implies b or c is associated (respectively, strongly associated) with a. A third stronger notion of associate, called very strong associate, and the related notion of very strongly irreducible are considered along with m-irreducible elements, that is, elements that generate maximal principal ideals. It is shown that an element is irreducible if and only if the ideal it generates is maximal among the principal ideals contained in some prime ideal.

Each type of irreducibility leads to a form of atomicity and this is covered in Section 3. As expected, if R satisfies the ascending chain condition on principal ideals (ACCP) every nonunit of R is a product of irreducibles. Each type of associate and irreducible element leads to a form of unique factorization ring. These are investigated in Section 4. The previous work of Bouvier, Fletcher, and Galovich is compared and unified.

Section 5 consists of examples. An easy method to construct atomic rings without ACCP is given. An example of a one-dimensional quasi-local atomic LCM ring not satisfying ACCP and which is not a unique factorization ring is given. Section 6 studies factorization in the polynomial ring $R[X]$ and the power series ring $R[[X]]$. It is shown that $a \in R$ is irreducible in R if and only if a is irreducible in $R[X]$, but that the other forms of irreducible do not behave so nicely.

2. Associates and irreducibles.

For a commutative integral domain D with identity, the terminology concerning divisibility and factorization is more or less standard. Two elements $a, b \in D$ are *associates* if $a \mid b$ and $b \mid a$, and this is equivalent to $aD = bD$ or to $a = ub$ for some $u \in U(D)$ where $U(D)$ is the group of units of D. A nonzero, nonunit element $a \in D$ is *irreducible* (or an *atom*) if $a = bc$ implies b or c is a unit, or, equivalently, if $a = bc$ implies b or c is an associate of a. In terms of ideals, a is irreducible if and only if aD is a maximal element of the set of proper principal ideals of D. The domain D is *atomic* if every nonzero, nonunit element of D can be written as a finite product of atoms. If D satisfies the ascending chain condition on principal ideals (ACCP), then D is atomic. But Grams [**33**] showed that the converse is false. Further such examples were given by Zaks [**47**]. For a survey of factorization in an integral domain, the reader is referred to Anderson, Anderson and Zafrullah [**7**] and [**8**].

In the case where D is an integral domain, there is really no difference between studying factorization of elements or of principal ideals. Let K be the quotient field of D and $K^* = (K - \{0\}, \cdot)$. Then $G(D) = K^*/U(D)$ is called the *group of divisibility of* D. The group $G(D)$ is partially ordered by $aU(D) \leq bU(D) \Leftrightarrow a \mid b$ in $D \Leftrightarrow bD \subseteq aD$. Thus, the map $G(D) \to P(D)$ given by $aU(D) \to aD$, where $P(D)$ is the group of nonzero principal fractional ideals of D under multiplication ordered by reverse inclusion, is an order isomorphism. Here $G_+(D)$, the positive cone of $G(D)$, is naturally (order) isomorphic to the monoid $P_+(D)$ of nonzero principal integral ideals of D or to the congruence monoid $D^*/\sim$ where $D^* = (D - \{0\}, \cdot)$ and $a \sim b \Leftrightarrow aD = bD$.

Suppose now we decide to look at factorization and divisibility in the context of commutative rings with zero divisors. Before we can even consider what parts of the theory of factorization for integral domains carry over to commutative rings with zero divisors, we need to decide what our definitions will be. Surprisingly, even the definitions of "associate" and "irreducible" are not obvious.

So, let R be a commutative ring (always with identity) having total quotient ring $T(R)$, $R^* = R - \{0\}$, and group of units $U(R)$. As usual, $a \mid b$ means $b = ac$ for some $c \in R$, or equivalently, $bR \subseteq aR$. A nonunit $a \in R$ is a *prime element* or is *prime* if aR is a prime ideal, that is, $a \mid bc \Rightarrow a \mid b$ or $a \mid c$. We first give three different definitions for "associate."

Definition 2.1. Let R be a commutative ring, and let $a, b \in R$. Then a and b are *associates* if $a \mid b$ and $b \mid a$, that is, $aR = bR$. If $a = ub$ for some $u \in U(R)$, we say that a and b are *strong associates*. Finally, we say that a and b are *very strong associates* if (i) a and b are associates and (ii) either $a = b = 0$ or $a \neq 0$ and $a = rb \Rightarrow r \in U(R)$.

We write $a \sim b$, respectively $a \approx b$, $a \cong b$, if a and b are associates respectively, strong associates, very strong associates. Certainly, $a \cong b \Rightarrow a \approx b \Rightarrow a \sim b$ and $\sim$ and $\approx$ are equivalence relations, even congruences on the monoid $(R, \cdot)$. However, $\cong$ need not be an equivalence relation on R. For if $e \in R$ is an idempotent with $e \neq 0, 1$, then $e = e^2$ shows that $e \ncong e$. Note that $a \cong a$ says that $a = 0$ or $a \neq 0$ and $a = ra \Rightarrow r \in U(R)$. Thus, $\cong$ is reflexive on R if and only if for $x, y \in R$, $x = xy \Rightarrow x = 0$ or $y \in U(R)$. A commutative ring satisfying this last condition has been called *présimplifiable* by Bouvier [**18**]. It is not hard to prove that R is présimplifiable $\Leftrightarrow Z(R) \subseteq 1 - U(R) = \{1 - u \mid u \in U(R)\} \Rightarrow Z(R) \subseteq \operatorname{rad}(R)$ where $Z(R)$ and $\operatorname{rad}(R)$ are the set of zero divisors of R and the Jacobson radical of R, respectively. For results on présimplifiable rings, the reader is referred to Bouvier [**18, 20, 22, 23** and **24**]. Fletcher [**29**] considered commutative rings satisfying a property equivalent to being présimplifiable. For $r \in R$, he defined $U(r) = \{\beta \in R \mid \beta\gamma r = r$, there exists a $\gamma \in R\}$, so $U(r) = \{\beta \in R \mid \beta(r) = (r)\}$. We always have $U(R) \subseteq U(r)$ and Fletcher called R a *pseudo-domain* if $U(R) = U(r)$ for each $0 \neq r \in R$. Note that for $r \neq 0$, $r \cong r \Leftrightarrow U(r) = U(R)$. Thus, R is a pseudo-domain if and only if R is présimplifiable. Our first theorem shows that $\cong$ is an equivalence relation if and only if $\cong$ is reflexive, or equivalently, if R is présimplifiable.

Theorem 2.2. *Let R be a commutative ring.*

(1) *For $a, b \in R$, $a \cong b \Rightarrow b \cong a$, so $\cong$ is symmetric.*

(2) *For $a, b, c \in R$, $a \cong b$ and $b \sim c \Rightarrow a \cong c$. Hence, $\cong$ is transitive. If $a \cong a$ and $a \sim b$, then $a \cong b$.*

(3) *The following conditions are equivalent*:

(i) *$\cong$ is a congruence on $(R, \cdot)$,*

(ii) *$\cong$ is an equivalence relation on R,*

(iii) *$\cong$ is reflexive on R,*

(iv) *$\sim$, $\approx$, and $\cong$ coincide on R,*

(v) *R is présimplifiable.*

Proof. (1) Suppose that $a \cong b$. Then a and b are associates. Suppose $a \neq 0$. Let $b = sa$; we need $s \in U(R)$. Since a and b are associates, $a = tb$. Thus, $a = tb = t(sa) = (ts)a = (ts)tb = (tst)b$. Since $a \cong b$, $tst \in U(R)$ and hence $s \in U(R)$. Thus, $b \cong a$.

(2) Suppose $a \cong b$ and $b \sim c$. Then $a \sim c$. We may assume $a \neq 0$. Suppose $a = rc$. Now $c = sb$ so $a = rsb$. Thus, $rs \in U(R)$ and hence $r \in U(R)$. Hence, $a \cong c$.

(3) Certainly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The third statement of (2) shows that if $\cong$ is reflexive, then $\sim$ and $\cong$ coincide. Hence, $\sim, \approx$ and $\cong$ coincide. And certainly, if $\sim$, $\approx$ and $\cong$ coincide, then $\cong$ is reflexive. Thus (iii) $\Leftrightarrow$ (iv). We have already remarked that (iii) $\Leftrightarrow$ (v). It remains to show that (iii) $\Rightarrow$ (i). Suppose that $\cong$ is reflexive. Then by (1) and (2), $\cong$ is actually an equivalence relation. We must show that $a \cong b \Rightarrow ca \cong cb$. Now certainly $ca \sim cb$. We may assume that $ca \neq 0$. Suppose that $ca = r(cb)$. Now $a \cong b$ gives that $a = ub$ for some $u \in U(R)$. Then $ca = r(cb) = r(cu^{-1}a) = (ru^{-1})ca$. Hence ru^{-1} is a unit and thus r is also a unit. □

If R is an integral domain or quasi-local, then R is présimplifiable and hence $\sim$, $\approx$ and $\cong$ all coincide. If $a \in R$ is regular (that is, $a \notin Z(R)$), then $a \cong a$ so $a \sim b \Rightarrow a \cong b$ for any $b \in R$. As earlier remarked, if $e \neq 0, 1$ is idempotent, then $e \approx e$, but $e \not\cong e$. It is also possible to have $a \sim b$ but $a \not\approx b$. The following example is given without proof in Fletcher [**28**]. A second example is given in Example 6.1.

Example 2.3. Let $R = F[X, Y, Z]/(X - XYZ)$ where F is a field. Denoting the images of X, Y, Z in R by x, y, z, we have $x = xyz$, so $x \sim xy$. But $x \not\approx xy$. For suppose that $\bar{f}x = xy$ where $f \in F[X, Y, Z]$. Then $fX - YX \in X(1 - YZ)$, so $f - Y \in (1 - YZ)$ and hence $f = Y + h(1 - YZ)$ for some $h \in F[X, Y, Z]$. To show that $\bar{f}$ is not a unit, it suffices to show that $(Y + h(1 - YZ), X) \neq F[X, Y, Z]$. Setting $Y = Z$ and $X = 0$, we see that this is indeed the case.

Different authors have taken different definitions for "associate" and "irreducible." For a discussion of this, see Anderson and Markanda [**10**]. Using the three different definitions for "associate," we get three different definitions for "irreducible."

Definition 2.4. Let R be a commutative ring, and let $a \in R$ be a nonunit. Then a is *irreducible*, respectively, *strongly irreducible*, *very strongly irreducible*, if $a = bc \Rightarrow b$ or c is associated, respectively, strongly associated, very strongly associated, with a.

In the preceding definition, we do allow $a = 0$. It is easily seen that the following are equivalent: (1) R is an integral domain, (2) 0 is very strongly irreducible, (3) 0 is strongly irreducible, and (4) 0 is irreducible. Clearly, a very strongly irreducible $\Rightarrow$ a is strongly irreducible $\Rightarrow$ a is irreducible, but we will give examples at the end of this section to show that none of these implications can be reversed. Also, if R is présimplifiable (for example, if R is an integral domain or is quasi-local), then $\sim$, $\approx$, and $\cong$ concide, and hence so do the notions of irreducible, strongly irreducible, and very strongly irreducible. We next give alternative characterizations of our various forms of irreducibility.

Theorem 2.5. *Let R be a commutative ring. For a nonzero, nonunit $a \in R$, the following conditions are equivalent.*

(1) *a is very strongly irreducible, that is, $a = bc \Rightarrow a \cong b$ or $a \cong c$.*

(2) *$a = a_1 \cdots a_n \Rightarrow a \cong a_i$ for some i.*

(3) *$a = bc \Rightarrow b$ or c is a unit.*

(4) *$a = a_1 \cdots a_n \Rightarrow$ every a_i except one is a unit.*

(5) *$a \cong a$ and $a \cong bc \Rightarrow a \cong b$ or $a \cong c$.*

(6) *$a \cong a$ and $a \cong a_1 \cdots a_n \Rightarrow a \cong a_i$ for some i.*

Proof. (1) $\Rightarrow$ (3). Suppose $a = bc$. By (1), say $a \cong b$. But then $a \cong b$ and $a = bc$ forces c to be a unit.

(3) $\Rightarrow$ (5). We first show $a \cong a$. Certainly $a \sim a$ and $a = ra \Rightarrow r$ or a is a unit. But by hypothesis a is not a unit, so r must be a unit. Suppose $a \cong bc$. Then $a = u(bc) = (ub)c$ for some unit u. By hypothesis ub or c is a unit and hence $a \sim c$ or $a \sim b$. But then $a \cong a$ and Theorem 2.2(2) gives $a \cong c$ or $a \cong b$.

(5) $\Rightarrow$ (1). Suppose that $a = bc$. Now $a \cong a$ gives $a \cong bc$. Hence, $a \cong b$ or $a \cong c$.

The proof that (2) $\Rightarrow$ (4) $\Rightarrow$ (6) $\Rightarrow$ (2) is similar. It is clear that (6) $\Rightarrow$ (5) and (5) $\Rightarrow$ (6) follows by induction on n. □

Note that if a is very strongly irreducible and $a \sim b$, then $a \cong b$ and b is also very strongly irreducible. For if $a = 0$, then $b = 0$ while if $a \neq 0$, then $a \cong a$. Thus, $a \sim b$ forces $a \cong b$ by Theorem 2.2(2). Then Theorem 2.5(5) shows that b is very strongly irreducible. In Theorem 2.5 we cannot add the condition $a \cong bc \Rightarrow a \cong b$ or $a \cong c$ or even the condition that $a \cong a_1 \cdots a_n \Rightarrow a \cong a_i$ for some i. Indeed, $e = (1, 0) \in \mathbf{Z}_2 \times \mathbf{Z}_2$ satisfies these two conditions vacuously, but e is not very strongly irreducible. The statement that $a \cong a$ in conditions (5) and (6) of Theorem 2.5 is essential.

Galovich [**31**] used the notion of very strongly irreducible (condition (3) of Theorem 2.5) in his study of unique factorization rings. In studying unique factorization, Fletcher [**28**] defined "irreducible" in yet another way as follows. A *refinement* of a factorization $a = a_1 \cdots a_n$ is obtained by factoring one or more of the a_i's. A nonunit a was said to be "irreducible" if each factorization of a has a refinement containing a. Thus a is "irreducible" if whenever $a = a_1 \cdots a_n$, then for some i, $a_i = aa_i'$. In this case $a_iR = aR$. So Fletcher's "irreducible" is equivalent to the condition: $a = a_1 \cdots a_n$ implies $a \sim a_i$ for some i. Our definition of irreducible is just the case for $n = 2$. We next show that these two definitions are equivalent.

Theorem 2.6. *Let R be a commutative ring. For a nonunit $a \in R$, the following conditions are equivalent.*

(1) *a is irreducible, that is, $a = bc \Rightarrow a \sim b$ or $a \sim c$.*

(2) *a is irreducible in the sense of Fletcher, that is, $a = a_1 \cdots a_n \Rightarrow a \sim a_i$ for some i.*

(3) $(a) = (b)(c) \Rightarrow (a) = (b)$ or $(a) = (c)$.

(4) $(a) = (a_1) \cdots (a_n) \Rightarrow (a) = (a_i)$ *for some i.*

(5) $a \sim bc \Rightarrow a \sim b$ *or* $a \sim c$.

(6) $a \sim a_1 \cdots a_n \Rightarrow a \sim a_i$ *for some i.*

Proof. Clearly, (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (3), and (6) $\Rightarrow$ (5). Also, (3) $\Rightarrow$ (4)

and (5) $\Rightarrow$ (6) easily follow by induction. It is easily proved that (4) $\Rightarrow$ (2) and (3) $\Leftrightarrow$ (5). So it suffices to prove that (1) $\Rightarrow$ (3). Suppose that $(a) = (b)(c)$. Then $a = rbc$ for some $r \in R$. Now $a = (rb)c = b(rc)$ gives that $(a) = (rb)$ or $(a) = (c)$ and $(a) = (b)$ or $(a) = (rc)$. We may assume that $(a) = (rb)$ and $(a) = (rc)$ for otherwise we are done. Now $(a) = (rb)$ gives $(ac) = (rbc) = (a)$ and $(a) = (rc)$ gives $(ab) = (rcb) = (a)$. Hence $(a) = (a)(b) = (ac)(b) = (a)(bc) = (a)^2$. So $a = sa^2$ for some $s \in R$. Thus $e = sa$ is idempotent and $(e) = (a)$. Write $R = R_1 \times R_2$ where $R_1 = eR$ and $R_2 = (1-e)R$ with $e = (1,0)$ and $a = (\alpha, \beta)$. Then $aR = eR$ gives $\alpha \in U(R_1)$ and $\beta = 0$. (Hence $a = ue$ for some $u \in U(R)$.) Now a irreducible forces $\beta = 0$ to be irreducible in R_2, that is, R_2 must be a domain. Thus, aR is prime and hence $aR = bRcR$ gives $aR = bR$ or $aR = cR$. □

Corollary 2.7. *If $a \in R$ is irreducible and $a \sim a'$, then a' is also irreducible. For $e \in R$ idempotent, e is irreducible $\Leftrightarrow$ e is strongly irreducible $\Leftrightarrow$ e is prime. However, e is very strongly irreducible $\Leftrightarrow e = 0$ is prime.*

Proof. The proof of the first statement follows from (1) $\Rightarrow$ (5) of the theorem. The proof of (1) $\Rightarrow$ (3) of the theorem shows that if e is idempotent, then e must be prime, and it is easily seen that if $e = (1,0)$ is prime, then it is actually strongly irreducible. □

Theorem 2.8. *Let R be a commutative ring, and let a be a nonunit of R. Then the following statements are equivalent.*

(1) *a is irreducible.*

(2) $(a) \subset (b), (c) \Rightarrow (a) \subset (b)(c)$.

(3) *The set $S = \{b \in R \mid (a) \subset (b)\}$ is a saturated multiplicatively closed subset of R.*

Proof. (1) $\Rightarrow$ (2). Suppose that a is irreducible and $(a) \subset (b)$ and $(a) \subset (c)$. Now $(a) \subseteq (b)$, so $(a) = (d)(b)$ for some $d \in R$. Since a is irreducible and $(a) \subset (b)$, we must have $(a) = (d)$ and hence $(a) = (a)(b)$. Similarly, $(a) = (a)(c)$. Hence, $(a) = (a)(c) = (a)(b)(c) \subseteq (b)(c)$. If $(a) = (b)(c)$, then $(a) = (b)$ or $(a) = (c)$, a contradiction.

Hence, $(a) \subset (b)(c)$.

(2) $\Rightarrow$ (3). Clearly S is multiplicatively closed by (2). Also, since $xy \in S \Rightarrow (x) \supseteq (xy) \supset (a)$, we see that $x \in S$, and hence S is saturated.

(3) $\Rightarrow$ (1). Suppose $(a) = (b)(c)$ where $(a) \neq (b)$ and $(a) \neq (c)$. Then $(a) \subset (b)$ and $(a) \subset (c)$. By (3), $bc \in S$ and hence $(a) \subset (bc)$, a contradiction. □

While Corollary 2.7 shows that an idempotent principal ideal (x) is irreducible if and only if it is prime, the next result shows that the conditions that $(x) = (x)^2$ and that x is irreducible are closely related.

Theorem 2.9. *Let R be a commutative ring, and let $x \in R$.*

(1) $(x) = (x)^2$ *if and only if* $(x) \subseteq (y) \Rightarrow (x) = (x)(y)$.

(2) *If x is irreducible, then* $(x) \subset (y) \Rightarrow (x) = (x)(y)$.

(3) *If* $(x) \subset (y) \Rightarrow (x) = (x)(y)$, *then either x is irreducible or* $(x) = (x)^2$.

Proof. (1) $(\Leftarrow)(x) \subseteq (x)$ gives $(x) = (x)(x)$. $(\Rightarrow)(x) = (x)^2 =\subseteq (x)(y) \subseteq (x)$, so $(x) = (x)(y)$.

(2) Suppose $(x) \subset (y)$. Now $(x) = (y)(z)$ for some z, and since x is irreducible, $(x) = (z)$. Hence, $(x) = (y)(z) = (x)(y)$.

(3) Suppose that $(x) \subset (y)$ and $(x) \subset (z)$. By hypothesis, $(x) = (x)(y)$ and $(x) = (x)(z)$. Hence, $(x) = (xy)(z) \subseteq (yz)$. If x is not irreducible, then by Theorem 2.8, we get $(x) = (y)(z)$ for some such y, z. But then $(x)^2 = (x)((y)(z)) = ((x)(y))(z) = (x)(z) = (x)$. □

We next give some conditions equivalent to an element being strongly irreducible.

Theorem 2.10. *Let R be a commutative ring. For a nonunit $a \in R$, the following conditions are equivalent.*

(1) *a is strongly irreducible, that is,* $a = bc \Rightarrow a \approx b$ *or* $a \approx c$.

(2) $a = a_1 \cdots a_n \Rightarrow a \approx a_i$ *for some i.*

(3) *$a \approx bc \Rightarrow a \approx b$ or $a \approx c$.*

(4) *$a \approx a_1 \cdots a_n \Rightarrow a \approx a_i$ for some i.*

If a is strongly irreducible and $b \sim a$, then $b \approx a$, and hence b is strongly irreducible.

Proof. It is clear that (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) is easily proved by induction. Certainly (3) $\Rightarrow$ (1). If (1) holds and $a \approx bc$, then $a = (ub)c$ for some unit u. Then $a \approx ub$, and hence also $a \approx b$, or $a \approx c$.

Suppose that a is strongly irreducible and $a \sim b$. Then $(a) = (b)$ and so $a = bc$ for some c. If $a \approx b$, we are done; so suppose $a \approx c$. Then $(a) = (c)$, so $(a) = (b)(c) = (a)^2$. Then in the notation of the proof of (1) $\Rightarrow$ (3) of Theorem 2.6, $a = ue$ where u is a unit and e is idempotent. But then $bR = aR = eR$ gives that $b = ve$ for some unit v. Hence, $b = u^{-1}va$, so $b \approx a$. Thus, b is strongly irreducible. □

If a is a nonzero element of an integral domain R, then a is irreducible if and only if (a) is a maximal element in the set of proper principal ideals of R. This leads to another form of irreducibility, the form which was used by Bouvier (see references).

Definition 2.11. Let R be a commutative ring. A nonunit $a \in R$ is *m-irreducible* if (a) is a maximal element in the set of proper principal ideals of R.

Theorem 2.12. *Let R be a commutative ring, and let $a \in R$ be a nonunit.*

(1) *a is m-irreducible if and only if $a = bc \Rightarrow b \in U(R)$ or $a \sim b$.*

(2) *If $a \neq 0$ is very strongly irreducible, then a is m-irreducible.*

(3) *If a is m-irreducible, then a is strongly irreducible.*

(4) *If a is m-irreducible and $a \sim b$, then $a \approx b$ and b is m-irreducible.*

(5) *Let a be irreducible so $S = \{b \mid (a) \subset (b)\}$ is multiplicatively closed. Then $a/1$ is m-irreducible in R_S.*

Proof. (1) ($\Rightarrow$). Here $a = bc$ gives $(a) \subseteq (b)$. If $(b) \neq R$, $(a) = (b)$ by the maximality of (a).

($\Leftarrow$). Suppose $(a) \subseteq (b)$. Then $a = bc$ for some c. By hypothesis, $(b) = R$ or $(a) = (b)$. So (a) is maximal in the set of proper principal ideals.

(2) Suppose $a \neq 0$ is very strongly irreducible. Let $(a) \subseteq (b)$, so $a = bc$ for some $c \in R$. If $a \cong b$, then $(a) = (b)$. If $a \cong c$, then $c = ua$ for some unit u. Then $a = bc = bua$. Thus, $a \neq 0$ gives $bu \in U(R)$ and hence $b \in U(R)$. So $(b) = R$. Thus a is m-irreducible.

(3) Suppose a is m-irreducible. Let $a = bc$. Then $(a) \subseteq (b), (c)$. If either (b) or $(c) = R$, then $a \approx c$ or $a \approx b$. So suppose $(a) = (b) = (c)$. Thus (a) is idempotent. Therefore (a) is a maximal ideal. Then Corollary 2.7 gives that a is strongly irreducible.

(4) Suppose that a is m-irreducible. Then by (3), a is strongly irreducible and hence by Theorem 2.10, $b \sim a$ gives that $b \approx a$. Clearly, b is m-irreducible.

(5) By Theorem 2.8, S is multiplicatively closed. Now $(a) \cap S = \phi$, so $(a)_S$ is a proper principal ideal of R_S. Suppose that $(a)_S \subseteq (c)_S$. Then $a/1 = (r/s)(c/1)$ where $r \in R$ and $s \in S$. So there exists $t \in S$ with $tsa = trc$. Now $ts \in S$, so $(a) \subset (ts)$. Since a is irreducible, $(tsa) = (a)$. Thus, $(a) = (tsa) = (trc) \subseteq (c)$. If $(a) = (c)$, then $(a)_S = (c)_S$, while if $(a) \subset (c)$, then $c \in S$, so $(c)_S = R_S$. □

We have seen that for $a = 0$, a is irreducible $\Leftrightarrow$ a is strongly irreducible $\Leftrightarrow$ a is very strongly irreducible $\Leftrightarrow$ a is prime. But $a = 0$ is m-irreducible $\Leftrightarrow$ 0 is a maximal ideal, that is, R is a field. Suppose that $a \neq 0$. Then a very strongly irreducible $\Rightarrow$ a is m-irreducible $\Rightarrow$ a is strongly irreducible $\Rightarrow$ a is irreducible. But none of these implications can be reversed. Even in an integral domain, an irreducible element (and hence very strongly irreducible element) need not be prime. The element x in Example 2.3 is prime and hence irreducible, but is not strongly irreducible. For $x = (xy)z$, but $x \not\approx xy$, as we have already seen, and $x \not\approx z$ since this gives $(x) = (x)^2$ which is also false. (For $(x) = (x)^2$ gives $(X) + (X)(1 - YZ) = (X)^2 + (X)(1 - YZ)$ which implies $F[X, Y, Z] = (X, 1 - YZ)$, a contradiction.) The element $(1, 0) \in \mathbf{Z} \times \mathbf{Z}$ is strongly irreducible but not m-irreducible, and the element $(1, 0) \in \mathbf{Z}_2 \times \mathbf{Z}_2$ is m-irreducible but not very strongly irreducible. In Section 5 we give an example (Example 5.7) of an irreducible element that is neither prime nor strongly irreducible. We summarize these implications in the next theorem.

Theorem 2.13. *Let a be a nonzero element of a commutative ring R. Then a very strongly irreducible $\Rightarrow$ a is m-irreducible $\Rightarrow$ a is strongly irreducible $\Rightarrow$ a is irreducible and a prime $\Rightarrow$ a is irreducible. Moreover, none of these implications can be reversed. For a commutative ring R, 0 is very strongly irreducible $\Leftrightarrow$ 0 is strongly irreducible $\Leftrightarrow$ 0 is irreducible $\Leftrightarrow$ 0 is prime. However, 0 is m-irreducible $\Leftrightarrow$ R is a field.*

The next theorem gives the exact relationship between irreducible and m-irreducible elements. An m-irreducible element is an element that generates an ideal that is maximal in the set of all proper principal ideals while an irreducible element is one that generates an ideal that is maximal in the set of principal ideals contained in some fixed prime ideal.

Theorem 2.14. *Let R be a commutative ring. Let P be a prime ideal of R. Suppose that $a \in P$ is such that $(a) \subseteq (b) \subseteq P$ implies $(a) = (b)$. Then a is irreducible. Conversely, suppose that $a \in R$ is irreducible. Then there is a prime ideal P with $a \in P$ such that $(a) \subseteq (b) \subseteq P$ implies $(a) = (b)$.*

Proof. Suppose that (a) is maximal in the set of principal ideals contained in P. Let $a = bc$. Now $bc = a \in P$, so say $b \in P$. Then $(a) \subseteq (b) \subseteq P$, so $(a) = (b)$. Thus, a is irreducible.

Conversely, suppose that a is irreducible. Then by Theorem 2.8 the set $S = \{b \in R \mid (b) \supset (a)\}$ is a saturated multiplicatively closed set. Since $a \notin S$, there is a prime ideal P with $a \in P$ and $P \cap S = \phi$. If $(a) \subset (b) \subseteq P$, then $b \in P \cap S$, a contradiction. $\square$

We end this section by showing how the various forms of "associate" and "irreducible" behave in direct products of commutative rings. This result may be used to give further examples showing that the implications given in Theorem 2.13 cannot be reversed.

Theorem 2.15. *Let* $\{R_\alpha\}_{\alpha\in\Lambda}$ *be a family of commutative rings, and let* $R = \Pi R_\alpha$. *Let* $a = (a_\alpha)$, $b = (b_\alpha) \in R$.

(1) $a \sim b \Leftrightarrow a_\alpha \sim b_\alpha$ *for each* $\alpha \in \Lambda$, $a \approx b \Leftrightarrow a_\alpha \approx b_\alpha$ *for each* $\alpha \in \Lambda$, *and* $a \cong b \Leftrightarrow a_\alpha \cong b_\alpha$ *for each* $\alpha \in \Lambda$ *and if some* $a_{\alpha_0} = 0$, *then all* $a_\alpha = 0$.

(2) *a is irreducible, respectively, strongly irreducible, m-irreducible prime* $\Leftrightarrow$ *each* $a_\alpha \in U(R_\alpha)$ *except for one* $\alpha_0 \in \Lambda$ *and that* a_{α_0} *is irreducible, respectively, strongly irreducible, m-irreducible, prime in* R_{α_0}.

(3) *a is very strongly irreducible* $\Leftrightarrow$ *each* $a_\alpha \in U(R_\alpha)$ *except for one* $\alpha_0 \in \Lambda$ *and that* a_{α_0} *is very strongly irreducible in* R_{α_0} *but is not* 0 *unless* $|\Lambda| = 1$ *and* R_{α_0} *is a domain.*

Proof. The proofs of (1) and (2) are left to the reader.

(3) ($\Rightarrow$). Now a very strongly irreducible implies that a is irreducible and hence by (2) all a_α except one, say a_{α_0}, are units. Clearly, a_{α_0} must be very strongly irreducible. Suppose that $a_{\alpha_0} = 0$. Then R_{α_0} is a domain and since $a \cong a$ (Theorem 2.5) by (1) each $a_\alpha = 0$, which is only possible if $|\Lambda| = 1$.

($\Leftarrow$). The case where $|\Lambda| = 1$ is clear so we assume that $|\Lambda| > 1$. Thus $a_{\alpha_0} \neq 0$ is very strongly irreducible and for $\alpha \neq \alpha_0$, $a_\alpha \in U(R_\alpha)$. Then $a = bc$ where $b = (b_\alpha)$, $c = (c_\alpha) \in \Pi R_\alpha$ gives $b_\alpha, c_\alpha \in U(R_\alpha)$ for $\alpha \neq \alpha_0$ and $a_{\alpha_0} = b_{\alpha_0} c_{\alpha_0}$, so $a_{\alpha_0} \cong b_{\alpha_0}$ or $a_{\alpha_0} \cong c_{\alpha_0}$. Hence, $a \cong b$ or $a \cong c$. $\square$

3. Atomicity.

Each form of irreducibility leads to a form of atomicity.

Definition 3.1. A commutative ring R is *atomic*, respectively, *strongly atomic*, *very strongly atomic*, *m-atomic*, *p-atomic* if every nonzero, nonunit element of R is a finite product of irreducible, respectively, strongly irreducible, very strongly irreducible, m-irreducible, prime elements.

Notice that in our definitions of atomicity we have only required nonzero, nonunit elements to be finite products of irreducibles of the appropriate type. If R is not an integral domain, then $0 = ab$ where a and b are nonzero, nonunit elements. Writing a and b as finite products of irreducibles of the appropriate type shows that 0 is also such a product. Next suppose that R is an integral domain. Then since 0 is very strongly irreducible and prime, in the definitions of atomic, strongly atomic, very strongly atomic, and p-atomic we could equivalently require every nonunit to be a product of irreducibles of the appropriate type. But for m-atomic the situation is different as 0 is m-irreducible $\Leftrightarrow$ 0 is a maximal ideal of $R \Leftrightarrow R$ is a field.

Note that for a commutative ring R, R very strongly atomic $\Rightarrow R$ is m-atomic $\Rightarrow R$ is strongly atomic $\Rightarrow R$ is atomic and R p-atomic $\Rightarrow R$ is atomic. As in the domain case, if R satisfies ACCP, then R is atomic.

Theorem 3.2. *Let R be a commutative ring. If R satisfies* ACCP, *then R is atomic.*

Proof. Suppose that some (nonzero) nonunit of R is not a product of irreducibles. Let $S = \{(a) \mid 0 \neq a$ is a nonunit of R that is not a product of irreducibles$\}$. (So $(x) \in S$ means that some associate of x is not a product of irreducibles.) Let (a) be a maximal element of S. Then a is not irreducible. (Note that we have used the fact that an associate of an irreducible is irreducible.) So $a = bc$ where $(a) \neq (b)$ and $(a) \neq (c)$. So $(a) \subset (b), (c)$. By maximality, b and c are products of ireducibles, and hence so is a. This contradiction shows that R is atomic. □

We next investigate how the various forms of atomicity behave with respect to direct products.

Theorem 3.3. *Let R be a commutative ring. Suppose that* 0 *is a finite product of irreducible elements. Then R is a finite direct product of indecomposable rings.*

Proof. Suppose that $0 = a_1 \cdots a_n$ where each a_i is irreducible. If $n = 1$, then R is an integral domain and is itself indecomposable. So assume that $n > 1$. Suppose that $R = R_1 \times \cdots \times R_m$ where R_i is a not necessarily indecomposable commutative ring. Since by Theorem 2.15(2) an irreducible element of $R = R_1 \times \cdots \times R_m$ is a unit in all but one of its coordinates, we see that 0 cannot be expressed as a product of fewer than m irreducible elements. Thus $m \leq n$. Hence R is a direct product of at most n indecomposable rings. □

Theorem 3.4. *Let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a family of commutative rings, and let $R = \Pi R_\alpha$. If R satisfies* ACCP *or any of the forms of atomicity, then Λ is finite.*

Let $R_1, \dots, R_n$ be commutative rings and $R = R_1 \times \cdots \times R_n$.

(1) *R satisfies* ACCP*, respectively, is atomic, strongly atomic, p-atomic, if and only if each R_i satisfies* ACCP*, respectively, is atomic, strongly atomic, p-atomic.*

(2) *R is m-atomic if and only if each R_i is m-atomic and if $n > 1$ and some R_i is a domain, then R_i must be a field.*

(3) *R is very strongly atomic if and only if each R_i is very strongly atomic and if some R_i is a domain we must have $n = 1$.*

Proof. Suppose that $R = \Pi R_\alpha$. If R satisfies ACCP or any of the forms of atomicity, then 0 is a finite product of, say, n irreducible elements. But then the proof of Theorem 3.3 shows that $|\Lambda| \leq n$.

Suppose $R = R_1 \times \cdots \times R_n$.

(1) That R satisfies ACCP if and only if each R_i satisfies ACCP follows from the fact that every principal ideal of $R_1 \times \cdots \times R_n$ has the form $I_1 \times \cdots \times I_n$ where I_i is a principal ideal of R_i. The remaining statements of (1) easily follow from Theorem 2.15(2).

(2) We may assume $n > 1$. Suppose that R is m-atomic. Let a be a nonunit of R_i. Then $(1, \dots, 1, a, 1, \dots, 1) \in R_1 \times \cdots \times R_n$ is a product of m-irreducible elements of R. This yields a factorization $a = a_1 \cdots a_k$ where each $a_j \in R_i$ is m-irreducible (Theorem 2.15(2)). Thus, each nonunit of R_i is a product of m-irreducibles. If R_i is a domain, then 0 must be m-irreducible and hence R_i is a field. Conversely, the hypothesis shows that each element of R of the form $(1, \dots, 1, a, 1, \dots, 1)$ where a is a nonunit of R_i is a product of m-irreducibles. Since each nonunit of R is a product of elements of this type (for various i), R is m-atomic.

(3) Suppose that R is very strongly atomic. As in (2), we see that each R_i is very strongly atomic. Suppose that some R_i is a domain. If $n > 1$, then $(1, \dots, 1, 0, 1, \dots, 1)$ (where the ith slot is 0) is irreducible, but not very strongly irreducible. Hence, if some R_i is a domain, we must have $n = 1$. Conversely, suppose that each R_i is very strongly atomic. If some R_i is a domain and $n = 1$, the result is trivial, so suppose that each R_i is not a domain. Then for $a \in R_i$ very strongly irreducible, $a \neq 0$ and hence $(1, \dots, 1, a, 1, \dots, 1)$ is very strongly irreducible (Theorem 2.15(3)). Since every nonunit of R is a product of elements of this type (for various i), R is very strongly atomic. □

Corollary 3.5. *Let R be a commutative ring. If R satisfies* ACCP, *respectively, is atomic, strongly atomic, m-atomic, very strongly atomic, p-atomic, then R is a finite direct product of indecomposable rings satisfying* ACCP, *respectively, which are atomic, strongly atomic, m-atomic, very strongly atomic, p-atomic.*

Proof. Just combine Theorems 3.3 and 3.4. □

Mori in a series of four papers [**41, 42, 43** and **44**] characterized the rings, called *π-rings*, with the property that every proper principal ideal is a product of prime ideals. A commutative ring R is a π-ring if and only if R is a finite direct product of special principal ideal rings (SPIRs) and π-domains, that is, domains which are π-rings. There are many characterizations of π-domains; for example, R is a π-domain if and only if R is a locally factorial Krull domain [**4**]. The next theorem characterizes p-atomic rings. It shows that an indecomposable p-atomic ring is either an SPIR or a UFD and that a p-atomic ring satisfies ACCP and is strongly atomic.

Theorem 3.6. *For a commutative ring R, the following statements are equivalent.*

(1) *R is p-atomic, that is, every* (*nonzero*) *element of R is a product of prime elements.*

(2) *R is a finite direct product of* SPIRs *and* UFDs.

(3) *Every* (*nonzero*) *proper principal ideal of R is a product of*

principal prime ideals.

Proof. Clearly (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) follow from the previously mentioned result of Mori. □

Let R be a commutative ring. Then R very strongly atomic $\Rightarrow$ R is m-atomic $\Rightarrow$ R is strongly atomic $\Rightarrow$ R is atomic, R p-atomic $\Rightarrow$ R is strongly atomic, and R p-atomic $\Rightarrow$ R satisfies ACCP $\Rightarrow$ R is atomic. However, none of these implications can be reversed. We have already remarked that even an atomic domain need not satisfy ACCP. Any Noetherian domain that is not a UFD is strongly atomic and satisfies ACCP, but is not p-atomic. The ring $R = F[X, Y, Z]/(X - XYZ)$ given in Example 2.3 is Noetherian and hence satisfies ACCP, but is not strongly atomic. For, as we have already seen, x is a principal prime that is not strongly irreducible and hence x cannot be written as a product of strongly irreducible elements. It is interesting to note that R is indecomposable, but is not présimplifiable. Theorem 3.4 shows that $\mathbf{Z} \times \mathbf{Z}$ is strongly atomic but not m-atomic, and that $\mathbf{Z}_2 \times \mathbf{Z}_2$ is m-atomic but not very strongly atomic. We summarize these results in the next theorem.

Theorem 3.7. *Let R be a commutative ring. Then R is very strongly atomic $\Rightarrow$ R is m-atomic $\Rightarrow$ R is strongly atomic $\Rightarrow$ R is atomic*; *R p-atomic $\Rightarrow$ R is strongly atomic and R satisfies* ACCP; *and R satisfies* ACCP $\Rightarrow$ *R is atomic. However, none of these implications can be reversed.*

Anderson, Anderson and Zafrullah [**7**] defined an integral domain R to be a *bounded factorization domain* (BFD) if for each nonzero, nonunit element $a \in R$, there exists a natural number $N(a)$ such that for any factorization $a = a_1 \cdots a_n$ where each a_i is a nonunit we have $n \leq N(a)$. They showed that a BFD satisfies ACCP and that a Noetherian domain or Krull domain is a BFD.

Definition 3.8. A commutative ring R is called a *bounded factorization ring* (BFR) if for each nonzero nonunit $a \in R$, there exists a natural number $N(a)$ so that for any factorization $a = a_1 \cdots a_n$ of a

where each a_i is a nonunit we have $n \leq N(a)$.

As in the domain case, if R is a BFR, then R satisfies ACCP. Also, a BFR R is présimplifiable. For suppose that in R, $0 \neq x = xy$ with y a nonunit. Then $x = xy = xy^2 = \cdots$, so x has arbitrarily long factorizations. Note that since a BFR is présimplifiable, each irreducible is very strongly irreducible. Thus a BFR is very strongly atomic.

Theorem 3.9. *For a Noetherian ring R, the following conditions are equivalent.*

(1) *R is a* BFR.

(2) *R is présimplifiable.*

(3) $\cap_{n=1}^{\infty}(y^n) = 0$ *for each nonunit $y \in R$.*

(4) $\cap_{n=1}^{\infty} I^n = 0$ *for each proper ideal I of R.*

Proof. We have already observed that (1) $\Rightarrow$ (2). Certainly (4) $\Rightarrow$ (3) $\Rightarrow$ (2). By the Krull intersection theorem, $\cap_{n=1}^{\infty} I^n = 0_{1-I} = \{x \in R \mid xi = x \text{ for some } i \in I\}$. If R is présimplifiable, then $0_{1-I} = 0$, so (2) $\Rightarrow$ (4). We show that (4) $\Rightarrow$ (1). Let $0 \neq x \in R$ be a nonunit and let $Z(R/(x)) = P_1 \cup \cdots \cup P_n$ where each P_i is a prime ideal of R. Suppose that $x = ab$ where a is a nonunit. Note that $b \notin (x)$, for $b \in (x)$ gives $x \in a(x)$ and hence $(x) = (a)(x)$ so that $x \in \cap_{n=1}^{\infty}(a^n) = 0$, a contradiction. Hence $a \in P_i$ for some i. Suppose that x has arbitrarily long factorizations. If $x = a_1 \cdots a_m$ where $m \geq kn$ and each a_i is a nonunit, then each a_i is in some P_j and hence $x \in P_i^k$ for some $1 \leq i \leq n$. So for each k, there exists a $1 \leq i(k) \leq n$ with $P_{i(k)}^k$. Thus, for some $1 \leq l \leq n$, there are infinitely many k with $i(k) = l$. Then $x \in \cap_{m=1}^{\infty} P_l^m = 0$, a contradiction. □

Note that, in Theorem 3.9, the Noetherian hypothesis was used in only two places. First, in (2) $\Rightarrow$ (4) we used a form of the Krull intersection theorem and in (4) $\Rightarrow$ (1) we used the fact that $Z(R/(x))$ is a finite union of prime ideals. The form of the Krull intersection theorem used, that $\cap_{n=1}^{\infty} I^n = \{x \in R \mid x = xi \text{ for some } i \in I\}$ holds for locally Noetherian rings, or more generally, for rings in which

$\cap_{n=1}^{\infty} M_M^n = 0_M$ for each maximal ideal M of R. For a discussion of the Krull intersection theorem, see Anderson [**2**]. We isolate this portion of the previous theorem.

Theorem 3.10. *Let R be a commutative ring with the property that for each ideal I of R, $\cap_{n=1}^{\infty} I^n = \{x \in R \mid x = xi$ for some $i \in I\}$. Then the following statements are equivalent.*

(1) *$\cap_{n=1}^{\infty} I^n = 0$ for each proper ideal I of R.*

(2) *$\cap_{n=1}^{\infty} (y^n) = 0$ for each nonunit $y \in R$.*

(3) *$Z(R) \subseteq \text{rad}\,(R)$.*

(4) *R is présimplifiable.*

Proof. (1) $\Rightarrow$ (2). This is always true.

(2) $\Rightarrow$ (1). Let $z \in \cap_{n=1}^{\infty} I^n$. Then $z = zi$ for some $i \in I$, so $z \in \cap_{n=1}^{\infty} (i^n) = 0$.

(2) $\Rightarrow$ (4). Suppose that $xy = x$ and $y \notin U(R)$. Then $x \in \cap_{n=1}^{\infty} (y^n) = 0$.

(4) $\Rightarrow$ (2). Let $y \notin U(R)$ and $x \in \cap_{n=1}^{\infty} (y^n)$. Then $x = x(ry^n)$ for some $r \in R$ and $n \geq 1$. Then $ry^n \notin U(R)$ forces $x = 0$ and hence $\cap_{n=1}^{\infty} (y^n) = 0$.

(1) $\Leftrightarrow$ (3). [**2**, Proposition 1]. □

The proof of (4) $\Rightarrow$ (1) of Theorem 3.9 can be modified to show that in a Krull ring there is a bound on the lengths of factorizations of regular elements. For results on Krull rings the reader is referred to Huckaba [**35**], Kennedy [**37**], or Matsuda [**39**] and [**40**].

Theorem 3.11. *Let R be a Krull ring. For each regular element a of R, there exists a natural number $N(a)$ so that if $a = a_1 \cdots a_n$ where each a_i is a nonunit then $n \leq N(a)$.*

Proof. We maintain the notation of (4) $\Rightarrow$ (1) of Theorem 3.9. Let x be a regular nonunit of R, and suppose that $x = ab$ where a is a nonunit. Note that $b \notin (x)$, for otherwise $x \in \cap_{n=1}^{\infty} (a^n) \subseteq Z(R)$ since

R is completely integrally closed. Hence $a \in P_i$ for some i. As the proof of (4) $\Rightarrow$ (1) shows, if x has arbitrarily long factorizations, then $x \in \cap_{m=1}^{\infty} P_l^m$ for some $1 \leq l \leq n$. But this is a contradiction since $\cap_{m=1}^{\infty} P_l^m \subseteq Z(R)$. □

We remark that Theorem 3.11 may also be proved by observing that $n \leq V(a)$ where $V = \sum v_P$ and $\{v_P\}$ is the defining family of rank one discrete valuations for R.

Theorem 3.6 shows that a ring R is p-atomic if and only if every proper (nonzero) principal ideal is a product of principal prime ideals. This raises the question of whether the other types of atomicity can be defined using principal ideals rather than elements. For example, let us call R *i-atomic* if every proper nonzero principal ideal of R is a product of principal ideals generated by irreducible elements. Clearly, R atomic implies R is i-atomic. In a similar manner we can also define i-strongly atomic, i-very strongly atomic and i-m-atomic. In each case the form of atomicity implies the form of i-atomicity.

The notion of i-atomicity and factorization in general can be viewed in the context of a commutative monoid $(S, \cdot)$, possibly with 0. In S we can define $\sim$, $\approx$ and $\cong$ just as we did for rings (the monoid $(R, \cdot)$) in Definition 2.1. This leads to the notions of irreducible, strongly irreducible, and very strongly irreducible. The notions of prime element and m-irreducible may also be defined in S as for rings. Here a principal ideal in S is $aS = \{as \mid s \in S\}$. Note that a principal ring ideal of $(R, +, \cdot)$ is the same thing as a principal ideal of $(R, \cdot)$. Instead of working in the monoid $(R, \cdot)$ we could work in the monoid $R/\sim$, that is, we could work with principal ideals rather than elements. Now $a \in R$ is irreducible $\Leftrightarrow$ (a) is irreducible in $R/\sim$; this is the essence of Theorem 2.6. Thus if R is atomic, so is $R/\sim$. In fact, $R/\sim$ is atomic $\Leftrightarrow$ R is i-atomic. To say that $R/\sim$ is atomic says that, given a nonzero nonunit $a \in R$, there exist irreducible elements $a_1, \dots, a_n \in R$ with $(a) = (a_1) \cdots (a_n)$ or $a \sim a_1 \cdots a_n$, that is, each nonzero nonunit of R is associated with a product of irreducible elements.

Question 3.12. Let R be a commutative ring. Does $R/\sim$ atomic, respectively, strongly atomic, very strongly atomic, m-atomic, imply that R is atomic, respectively, strongly atomic, very strongly atomic, m-atomic?

Question 3.12 does have an affirmative answer for atomic rings with only finitely many nonassociate atoms as may be seen from (1) $\Leftrightarrow$ (5) of the next theorem. Such rings were characterized by Anderson [**3**] and the integral domains with this property were thoroughly investigated in Anderson and Mott [**12**].

Theorem 3.13. *For a commutative ring R, the following statements are equivalent.*

(1) *R is an atomic ring with only finitely many nonassociate irreducibles.*

(2) *$(P_+(R \cup \{0\}, \cdot)$ is finitely generated as a monoid.*

(3) *$R/\sim$ is finitely generated as a monoid.*

(4) *The monoid of all ideals of R under multiplication is finitely generated.*

(5) *R is a finite direct product of finite local rings,* SPIRs, *and one-dimensional semi-local domains D_i with the property that for each nonprincipal maximal ideal M of D_i, D_i/M is fnite and D_{iM} is analytically irreducible* (*that is, the M_M-adic completion of D_{iM} is an integral domain*).

We can consider the monoid $R/\approx$. It is easy to see that R satisfies any of the various forms of atomicity if and only if $R/\approx$ does.

The relations $\sim$, $\approx$ and $\cong$ could have been defined on $T(R)$ instead of R. To avoid confusion as to whether for $a, b \in T(R)$ $a \sim b$ means $aR = bR$ or $aT(R) = bT(R)$, we define $a \sim_R b$ to mean $aR = bR$. In a similar way $a \approx_R b$ means that $a = ub$ for some $u \in U(R)$ and $a \cong_R b$ means $a \sim_R b$ and either $a = b = 0$ or $a = rb(r \in R) \Rightarrow r \in U(R)$. Here $T(R)/\sim$ is a monoid partially ordered by $\bar{a} \leq \bar{b} \Leftrightarrow a \mid b$ in R. So $T(R)/\sim$ is order isomorphic to the monoid of principal fractional ideals of R ordered by reverse inclusion. The positive cone of $T(R)/\sim$ is $R/\sim$. For R an integral domain, $T(R)/\sim$ is (isomorphic to) the group of divisibility of R. For R a domain, factorization or divisibility questions in R are faithfully translated to factorization or divisibility questions in $G(R) \approx T(R)/\sim$. The question of whether R i-atomic implies R is atomic is part of the general question of whether for R not a domain, does $T(R)/\sim$ (or $R/\sim$) faithfully reflect the factorization or divisibility properties of R.

4. Unique factorization. Of course an integral domain D is a unique factorization domain if (1) every nonzero nonunit of D is a product of irreducibles, and (2) this factorization into irreducibles is unique up to order and associates. In the nondomain case we have a number of ways to define "associate" and "irreducible." The following definition extends a definition given by Allard [**1**].

Definition 4.1. Let R be a commutative ring and $a \in R$ a nonunit. Two factorizations of a into nonunits $a = a_1 \cdots a_n = b_1 \cdots b_m$ are said to be *isomorphic*, respectively, *strongly isomorphic*, *very strongly isomorphic* if $n = m$ and there exists a permutation $\sigma \in S_n$ such that $a_i \sim b_{\sigma(i)}$, respectively, $a_i \approx b_{\sigma(i)}$, $a_i \cong b_{\sigma(i)}$. Two factorizations of a into nonunits $a = a_1 \cdots a_n = b_1 \cdots b_m$ are said to be *homomorphic*, respectively, *strongly homomorphic*, *very strongly homomorphic*, if for each $i \in \{1, \ldots, n\}$ there exists a $j \in \{1, \ldots, m\}$ with $a_i \sim b_j$, respectively, $a_i \approx b_j$, $a_i \cong b_j$, and for each $i \in \{1, \ldots, m\}$, there exists a $j \in \{1, \ldots, n\}$ with $b_i \sim a_j$, respectively, $b_i \approx a_j$, $b_i \cong a_j$.

Each form of atomicity and "isomorphic" leads to a type of unique factorization ring as given by our next definition.

Definition 4.2. Let R be a commutative ring. Let $\alpha \in \{$atomic, strongly atomic, very strongly atomic, m-atomic, p-atomic$\}$ and $\beta \in \{$isomorphic, strongly isomorphic, very strongly isomorphic$\}$. Then R is an (α, β)-*unique factorization ring* if (1) R is α and (2) any two factorizations of a nonzero, nonunit element into irreducible elements of the type used to define α are β.

Note that for any choice of α and β, an (α, β)-unique factorization ring R is présimplifiable. Indeed, suppose that $xy = x$ where $x \neq 0$ and y is not a unit. Then x is not a unit. Factor x and y into irreducibles of the appropriate type, $x = a_1 \cdots a_n$, $y = c_1 \cdots c_m$; so

$a_1 \cdots a_n = x = xy = a_1 \cdots a_n c_1 \cdots c_m$ are two factorizations of x into irreducibles of the appropriate type that cannot be β, a contradiction. So R is présimplifiable. Alternatively, note that R is certainly a BFR and any BFR is présimplifiable.

Thus, in an (α, β)-unique factorization ring, the notions of associate, strongly associate, and very strongly associate coincide. Hence, the notions of irreducible, strongly irreducible, m-irreducible, and very strongly irreducible coincide as do the notions of isomorphic, strongly isomorphic, and very strongly isomorphic. Thus, with the exception of $\alpha = p$-atomic, the notions of (α, β)-unique factorization rings all coincide. Thus, we will use the term unique factorization ring for any of these rings.

Definition 4.3. Let R be a commutative ring. R is called a *unique factorization ring* if R is an (α, β)-unique factorization ring for some (and hence all) (α, β) (except $\alpha = p$-atomic).

Bouvier [**25**] proved that a commutative ring R is (in our terminology) an (m-atomic, isomorphic)-unique factorization ring if and only if R is either (1) a UFD, (2) an SPIR, or (3) quasi-local with $M^2 = 0$ where M is the unique maximal ideal of R. Galovich [**31**] gave a similar characterization of (very strongly atomic, strongly isomorphic) unique factorization rings. When reading the papers by Allard and Bouvier, notice that they used the term "associate" as we do, but they used "irreducible" to mean what we have called m-irreducible and "atomic" to mean what we have called m-atomic. Galovich used the term "associate" to mean what we have called strongly associate and the term "irreducible" to mean what we have called very strongly irreducible. We summarize these results in our next theorem.

Theorem 4.4. *For a commutative ring R the following conditions are equivalent.*

(1) *R is atomic and any two factorizations of a nonzero, nonunit element into irreducibles are isomorphic* (*that is, R is an* (*atomic, isomorphic*)*-unique factorization ring*).

(2) *R is a unique factorization ring in the sense of Bouvier* (*that is, R is an* (*m-atomic, isomorphic*)*-unique factorization ring*).

(3) *R is a unique factorization ring in the sense of Galovich* (*that is, R is a* (*very strongly atomic, strongly isomorphic*)*-unique factorization ring*).

(4) *R is very strongly atomic and any two factorizations of a nonzero, nonunit element into* (*very strong*) *irreducibles are isomorphic.*

(5) *R is either* (a) *a UFD,* (b) *an SPIR or* (c) *a quasi-local ring with* $M^2 = 0$ *where* M *is the unique maximal ideal of* R.

Bouvier [**18**] defined a commutative ring R to be a *D-atomic* if R is m-atomic and each m-irreducible is prime. Clearly R D-atomic implies R is p-atomic. The converse is false since $\mathbf{Z} \times \mathbf{Z}$ is p-atomic, but not D-atomic. In fact, combining Theorem 3.4 and Theorem 3.6 shows that R is D-atomic if and only if R is either a UFD or a finite direct product of SPIRs and fields. Bouvier [**19**] called R a *Gaussian ring* if R is a D-atomic unique factorization ring. Thus, we have the following theorem, the proof of which is left to the reader.

Theorem 4.5. *For a commutative ring R, the following conditions are equivalent.*

(1) *R is a Gaussian ring.*

(2) *R is a* UFD *or* SPIR.

(3) *R is a* (*p-atomic,* β)*-unique factorization ring where* $\beta \in \{isomorphic, stronglyisomorphic, verystronglyisomorphic\}$.

If, in the definition of a unique factorization ring, we replace the condition that factorizations be isomorphic by the condition that they be homomorphic, we get the following theorem. (This theorem could be stated with any of the forms of atomicity except very strongly atomic or p-atomic and with either homomorphic or strongly homomorphic.)

Theorem 4.6. *For a commutative ring, R, the following conditions are equivalent.*

(1) *R is either* (a) *a* UFD, (b) *a quasi-local ring* (R, M) *with* $M^2 = 0$, *or* (c) *a finite direct product of* SPIRs *and fields.*

(2) *R is atomic and any two factorizations of a nonzero, nonunit*

element into irreducibles are homomorphic.

(3) *R is m-atomic and any two factorizations of a nonzero, nonunit element into m-irreducibles are strongly homomorphic.*

Proof. Clearly (1) $\Rightarrow$ (2), (3) and (3) $\Rightarrow$ (2). So it suffices to show that (2) $\Rightarrow$ (1). Write $R = R_1 \times \cdots \times R_n$ where each R_i is indecomposable (Theorem 3.3). Clearly, each R_i also satisfies (2). So we may assume that R is indecomposable. Suppose that R is not a domain. So there is a nonzero irreducible element r that is a zero divisor. Hence, there is a $0 \neq t \in R$ with $rt = 0$. Factor $t = t_1 \cdots t_n$ into irreducibles where we can assume that no t_i is regular. Suppose that there is a regular irreducible element s. Then $0 \neq sr = (s+t)r$. Factoring $s+t$ into irreducibles yields (since factorizations are homomorphic) that some irreducible factor of $s+t$, and hence $s+t$ itself, is in (s), so $t \in (s)$. But this gives a factorization of t which contains the regular element s and hence is not homomorphic to the factorization $t = t_1 \cdots t_n$. This contradiction shows that every irreducible is a zero divisor and hence every nonunit of R is a zero divisor. Let $r \in R$ be irreducible and let $rx = 0$ for some $x \neq 0$. Letting $r = r_1$ and factoring x into irreducibles, $x = r_2 \cdots r_n$, we have $r_1 r_2 \cdots r_n = 0$ which can be rewritten in the form $r_1^{a_1} \cdots r_m^{a_m} = 0$ where the r_i's are nonassociate irreducibles and each $a_i \geq 1$. If $m = 1$, r_1 is nilpotent. If $m > 1$, put $z = r_1^{a_1} + r_2^{a_2} \cdots r_m^{a_m}$. As in the proof of Galovich [**31**, Lemma 3], we have $r_1^{a_1} z = r_1^{2a_1}$. If $r_1^{2a_1} \neq 0$, then z must be a unit. But then $(r_1^{a_1})$ is idempotent and hence must be 0 since R is indecomposable. Hence, every nonunit of R is nilpotent. Let M be the set of nilpotents of R, so (R, M) is a quasi-local ring. If R has only one nonassociate irreducible element r, then $M = (r)$ and R is an SPIR. Suppose that r and s are nonassociate irreducibles of R. A careful reading of the proof of Galovich [**31**, Proposition 6] shows that $r^2 = s^2 = rs = 0$. So $M^2 = 0$.

Thus, $R = R_1 \times \cdots \times R_n$ where each R_i is a UFD, quasi-local with $M^2 = 0$, or an SPIR. Suppose that $n > 1$. Assume that some R_i is quasi-local with $M^2 = 0$, but not an SPIR or field. Let r, s be nonassociate irreducibles of R_i. Then $r^2 = s^2 = 0$ and $(1, \ldots, 1, r, 1, \ldots, 1)$, $(1, \ldots, 1, s, 1, \ldots, 1)$ are irreducible in R with $(0, \ldots, 0) \neq (1, \ldots, 1, 0, 1, \ldots, 1) = (1, \ldots, 1, r, 1, \ldots, 1)(1, \ldots, 1, r, 1, \ldots, 1) = (1, \ldots, 1, s, 1, \ldots, 1)(1, \ldots, 1, s, 1, \ldots, 1)$, but the two factorizations are not homomorphic. So in this case $n = 1$. Suppose that some

R_i is a domain but not a field. Let $0 \neq r \in R_i$ be irreducible. Then $(1, \ldots, 1, 0, 1, \ldots, 1) = (1, \ldots, 1, 0, 1, \ldots, 1)\ (1, \ldots, 1, r, 1, \ldots, 1)$ are two non-homomorphic factorizations of $(1, \ldots, 1, 0, 1, \ldots, 1)$. Hence again $n = 1$. Thus, if $n > 1$, each R_i must be a field or SPIR. $\square$

Fletcher [**28**] defined a "unique factorization ring" in yet another way. He took as his definition of "irreducible" a form which we have seen (Theorem 2.6) to be equivalent to our definition of irreducible. He then defined what he called a *U-decomposition* of an element: $a = (a_1 \cdots a_k)(b_1 \cdots b_n)$ where each a_i, b_j is irreducible, $a_i \in U(b_1 \cdots b_n)$ for each $i = 1, \ldots, k$ and $b_j \notin U(b_1 \cdots \hat{b}_j \cdots b_n)$ for $j = 1, \ldots, n$. He defined R to be a "unique factorization ring" (which we will call a *Fletcher unique factorization ring*) if (1) every nonunit of R has a U-decomposition, and (2) If $(a_1 \cdots a_k)(b_1 \cdots b_n) = (a'_1 \cdots a'_{k'})(b' \cdots b'_{n'})$ are two U-decompositions of a nonunit element, then $n = n'$ and after reordering, if necessary, $b_i \sim b'_i$ for $i = 1, \ldots, n$. Notice in (2) we require 0 to have a "unique" U-decomposition. As any factorization of an element into irreducibles can be "refined" into a U-decomposition, it is (2) that is essential. Fletcher [**29**] showed that R is a Fletcher unique factorization ring if and only if R is a finite direct product of UFDs and SPIRs, that is, R is p-atomic. See Anderson and Markanda [**10**] for an alternative discussion of Fletcher unique factorization rings. Fletcher's definition of a unique factorization ring may seem somewhat artificial. A more natural formulation may be given in terms of principal ideals as follows.

Definition 4.7. Let R be a commutative ring and let $a \in R$ be a nonunit. The representation of $(a) = (a_1) \cdots (a_n)$ is called a *reduced product representation* for (a) if (1) each a_i is irreducible and (2) $(a) \subset (a_1) \cdots (\widehat{a_i}) \cdots (a_n)$ for each i. Two reduced product representations for (a), $(a) = (a_1) \cdots (a_n) = (b_1) \cdots (b_m)$ are *isomorphic* if $n = m$ and after reordering, if necessary, $(a_i) = (b_i)$ for each i and they are *homomorphic* if $\{(a_1), \ldots, (a_n)\} = \{(b_1), \ldots, (b_m)\}$.

Lemma 4.8. *Let R be an atomic ring in which any two reduced product representations of a proper principal ideal are homomorphic. Then every irreducible element of R is prime.*

Proof. Let $r \in R$ be irreducible. We may suppose $r \neq 0$. First suppose that r is not regular. Then there is an $a \neq 0$ with $ar = 0$. Let $(a) = (a_1) \cdots (a_n)$ be a reduced product representation for (a). Now $0 = (a_1) \cdots (a_n)(r)$ and (r) cannot be deleted for then $(a) = 0$. Thus, (r) must appear in any reduced product representation for 0. Suppose $ef \in (r)$, say $ef = cr$. Let $(c) = (c_1) \cdots (c_l)$ be a reduced product representation for (c). If (r) is a factor in a reduced product representation for (cr), then taking representations for (e) and (f), multiplying them together, and reducing, shows that (r) is a factor in the reduced product representation for e or f; so $e \in (r)$ or $f \in (r)$. Thus we can assume (r) is not a factor in a reduced product representation for (cr). But then $(cr) = (c)$ since if (r) is deleted from $(cr) = (c_1) \cdots (c_l)(r)$ we have a reduced product representation for (c), namely, $(c_1) \cdots (c_l)$ and we cannot have $(c) \supseteq (rc) = (c_1) \cdots (\widehat{c_i}) \cdots (c_l)$. Hence, $(cr) = (c)$. Now $(a_1) \cdots (a_n)(c_1) \cdots (c_l) = (a)(c) = (a)(cr) = 0$. Then reducing $(a_1) \cdots (a_n)(c_1) \cdots (c_l)$ we see we can delete any $(a_i) = (r)$. But then we get a reduced product representation of 0 without an (r), a contradiction.

Since each zero divisor must have an irreducible factor r which is a zero divisor and since as we have observed that (r) is a prime and is one of the factors in a reduced product representation for 0, we must have $Z(R) = (p_1) \cup \cdots \cup (p_n)$, a finite union of principal primes. So R has few zero divisors.

Let r be a regular irreducible element of R, and suppose that $ab \in (r)$. Since R has few zero divisors, there exist $s, t \in R$ with $a + sr$ and $b + tr$ regular (see Gilmer [**32**, p. 78] or Huckaba [**35**, Theorem 7.2]. Then $(a + sr)(b + tr) \in (r)$ and $a + sr \in (r)$, respectively, $b + tr \in (r) \Leftrightarrow a \in (r)$, respectively, $b \in (r)$. Thus, we can assume that a and b are regular. Then $ab = cr$ for some $c \in R$. Since c is regular, r must occur in a reduced product representation for (cr) which we get by factoring c into irreducibles, multiplying by r and then reducing. Taking reduced product representations for (a) and (b), multiplying them together, and reducing, shows that (r) must occur in a reduced product representation for (a) or (b). So $a \in (r)$ or $b \in (r)$. Thus (r) is prime. □

Theorem 4.9. *For a commutative ring R, the following conditions are equivalent.*

(1) *R is a Fletcher unique factorization ring.*

(2) *R is a finite direct product of* UFD*s and* SPIR*s.*

(3) *R is atomic, and any two reduced product representations of a proper principal ideal are isomorphic.*

(4) *R is atomic and any two reduced product representations of a proper principal ideal are homomorphic.*

(5) *R is p-atomic.*

Proof. Fletcher [**29**] has proved that (1) $\Leftrightarrow$ (2), but our proof will be independent of this.

It is easily seen that (1) $\Rightarrow$ (3) and clearly (3) $\Rightarrow$ (4). By Lemma 4.8, (4) $\Rightarrow$ (5) and Theorem 3.6 gives (5) $\Rightarrow$ (2). But it is easily checked that (2) $\Rightarrow$ (1). □

A fundamental result for integral domains is that R a Euclidean domain $\Rightarrow$ R is a PID $\Rightarrow$ R is a UFD. Let us look at the situation where R is a commutative ring. If R is a principal ideal ring (PIR), then it is well known that R is a finite direct product of PIDs and SPIRs and hence is a Fletcher unique factorization ring. (For a nice survey of results on PIRs, see Bouvier [**26**].) Suppose that R is a Euclidean ring as defined by Samuel [**46**]: there is a map $\varphi : R \to W$, $(W, \leq)$ a well-ordered set, such that for $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ with $a = bq + r$ and $\varphi(r) < \varphi(b)$. The usual proof that a Euclidean domain is a PID shows that a Euclidean ring is a PIR. Samuel shows that an SPIR is Euclidean and that a product of Euclidean rings is Euclidean (it is obvious that a direct factor of a Euclidean ring is again Euclidean). Thus, a PIR is Euclidean if and only if its PID factors are Euclidean. Samuel points out that it is necessary to allow the algorithm φ to take on values in a well-ordered set larger than $\mathbf{N}$; for $\mathbf{Z} \times \mathbf{Z}$ has no algorithm $\varphi : \mathbf{Z} \times \mathbf{Z} \to \mathbf{N}$. Indeed, Fletcher [**30**] has shown that a commutative ring R is Euclidean with algorithm $\varphi : R \to \mathbf{N}$ if and only if R is a domain or a finite direct product of SPIRs.

5. Examples. In this section we give a number of examples. These examples use Nagata's method of idealization, which is explained briefly in the next paragraph. For readers wishing more details about this construction, see Huckaba [**35**]. For the idealization $R_1 = R \oplus N$ in Proposition 5.1 and Theorem 5.2 we assume for simplicity that R is an integral domain although parts of these results hold for R any commutative ring.

Let R be an integral domain and N an R-module. Put $R_1 = R \oplus N$, the idealization of R and N, so $(r_1, n_1)+(r_2, n_2) = (r_1+r_2, n_1+n_2)$ and $(r_1, n_1)(r_2, n_2) = (r_1r_2, r_1n_2 + r_2n_1)$. It is easily seen that $Z(R_1) = \{(r, n) \in R_1 \mid r \in Z(N)\}$ and $\operatorname{nil}(R_1) = 0 \oplus N$. Note that $0 \oplus N$ is the unique minimal prime ideal of R_1 (so R_1 is indecomposable) and $(0 \oplus N)^2 = 0$. It is easily checked that R_1 is Noetherian $\Leftrightarrow$ R is Noetherian and N is finitely generated. Note that $(c, n) \in U(R_1) \Leftrightarrow c \in U(R)$. We first find some irreducible elements of R_1.

Proposition 5.1. *Let R be an integral domain, N an R-module, and $R_1 = R \oplus N$.*

(1) *If $0 \neq a \in R$ is irreducible, (a, m) is very strongly irreducible.*

(2) *For $0 \neq a \in R$, the following are equivalent*:

(a) *a is irreducible*,

(b) *$(a, 0)$ is irreducible*,

(c) *$(a, 0)$ is very strongly irreducible.*

(3) *For $0 \neq n \in N$,*

(a) *$(0, n)$ is irreducible $\Leftrightarrow n = am \Rightarrow Rn = Rm \Leftrightarrow Rn$ is a maximal cyclic submodule of N,*

(b) *$(0, n)$ is strongly irreducible $\Leftrightarrow n = am \Rightarrow n = um$ where $u \in U(R)$, and*

(c) *$(0, n)$ is very strongly irreducible $\Leftrightarrow n = am \Rightarrow a \in U(R)$.*

Proof. (1) Suppose that $(a, m) = (b, l)(c, k)$. Then $a = bc$. Now a is irreducible and R is a domain; so say $c \in U(R)$. Then $(c, k) \in U(R_1)$. So (a, m) is very strongly irreducible.

(2) Now (1) gives (a) $\Rightarrow$ (c) and certainly (c) $\Rightarrow$ (b), so it suffices to prove (b) $\Rightarrow$ (a). Suppose that $(a, 0)$ is irreducible and $a = bc$. Then $(a, 0) = (b, 0)(c, 0)$ so $R_1(a, 0) = R_1(b, 0)$ or $R_1(a, 0) = R_1(c, 0)$, hence $Ra = Rb$ or $Ra = Rc$, so a is irreducible.

(3) (a) Suppose that $(0,n)$ is irreducible and $n = am$. Then $(0,n) = (a,0)(0,m)$ so $R_1(0,n) = R_1(0,m)$ and hence $Rn = Rm$. Conversely, suppose that $(0,n) = (a,l)(b,m)$. Now $ab = 0$, so say $b = 0$. Then $n = am$, so by hypothesis $Rn = Rm$. Hence, $R_1(0,n) = R_1(0,m)$. So $(0,n)$ is irreducible. The second implication is obvious. The proofs of (b) and (c) are similar. □

Theorem 5.2. *Let R be an integral domain, N an R-module, and $R_1 = R \oplus N$.*

(1) *If R satisfies* ACCP*, then every ascending chain of principal ideals of R_1 containing a principal ideal of the form $R_1(a,n)$ where $a \neq 0$ stops.*

(2) *R_1 satisfies* ACCP $\Leftrightarrow$ *R satisfies* ACCP *and N satisfies* ACCC (*ascending chain condition on cyclic submodules*).

(3) *R_1 is a* BFR $\Leftrightarrow$ *R is a* BFD *and N is a* BF*-module, i.e., for $0 \neq n \in N$, there exists a natural number $N(n)$ so that $n = a_1 \cdots a_{s-1} n_s \Rightarrow s \leq N(n)$.*

(4) *R_1 is atomic if R satisfies* ACCP *and N satisfies* MCC (*every cyclic submodule of N is contained in a maximal* (*not necessarily proper*) *cyclic submodule*).

(5) *R_1 is présimplifiable $\Leftrightarrow$ N is présimplifiable, i.e., $n = an \Rightarrow n = 0$ or $a \in U(R)$.*

Proof. (1) Suppose $a \neq 0$ and $R_1(a,n) \subset R_1(b,m)$. Then $(a,n) = (b,m)(c,l)$. Now $a = bc$ and c cannot be a unit of R for this gives that $(c,l) \in U(R_1)$. So $Ra \subset Rb$. Thus, if R has ACCP, every ascending chain of principal ideals of R_1 containing a principal ideal of the form $R_1(a,n)$ where $a \neq 0$ stops.

(2) ($\Rightarrow$). R_1 satisfies ACCP $\Rightarrow$ R_1 satisfies ACCP on ideals of the forms $R_1(a_1,0) \subseteq R_1(a_2,0) \subseteq \cdots$ and $R_1(0,n_1) \subseteq R_1(0,n_2) \cdots$. But this gives that R satisfies ACCP and N satisfies ACCC.

($\Leftarrow$). Let $R_1(a_1,n_1) \subseteq R_1(a_2,n_2) \subseteq \cdots$ be an ascending chain. If every $a_i = 0$, the chain gives rise to the chain $Rn_1 \subseteq Rn_2 \subseteq \cdots$ which stops by ACCC and hence the original chain in R_1 stops. If some $a_i \neq 0$, then (1) gives that the chain stops.

(3) $\Rightarrow$. Clear. ($\Leftarrow$). Let $(0,0) \neq (a,n) \in R_1$ be a nonunit and suppose we have a factorization into nonunits $(a,n) = (a_1,n_1)\cdots(a_s,n_s)$. If $a = 0$, $(0,n) = (a_1,n_1)\cdots(a_s,n_s)$ forces say $a_s = 0$ and hence $n = a_1 \cdots a_{s-1} n_s$ so $s \leq N(n)$. If $a \neq 0$, $a = a_1 \cdots a_s$ so $s \leq N(a)$ since R is a BFR.

(4) Let $(0,0) \neq (a,n) \in R_1$ be a nonunit. Suppose $a \neq 0$. By (1), (a,n) is a product of irreducibles. Suppose $a = 0$. Then $Rn \subseteq Rm$ where Rm is a maximal cyclic submodule and $n = am$ for some $a \in R^*$. By Proposition 5.1 $(0,m)$ is irreducible. Since $(0,n) = (a,0)(0,m)$ and $(a,0)$ is either a unit or a product of irreducibles, $(0,n)$ is a product of irreducibles.

(5) Clear. □

We now use Proposition 5.1 and Theorem 5.2 to give some examples of bad behavior of factorization in commutative rings with zero divisors. It is well-known that an atomic LCM (or equivalently GCD) domain is a UFD and hence satisfies ACCP. We first show this is not the case for commutative rings.

Example 5.3. A one-dimensional quasi-local ring R_1 that is atomic but does not satisfy ACCP. R_1 is an LCM ring but not a unique factorization ring.

Take $R = \mathbf{Z}_{(2)}$, $N = \mathbf{Z}_2 \oplus \mathbf{Z}_{2^\infty}$, and $R_1 = R \oplus N$. Here $\mathbf{Z}_2 \oplus \mathbf{Z}_{2^\infty}$ does not satisfy ACCC, so R_1 does not satisfy ACCP. Now for any $a \in \mathbf{Z}_{2^\infty}$, $\mathbf{Z}_{(2)}(1,a)$ is a maximal cyclic submodule of $\mathbf{Z}_2 \oplus \mathbf{Z}_{2^\infty}$. For if $\mathbf{Z}_{(2)}(1,a) \subseteq \mathbf{Z}_{(2)}(c,d)$ first $c = 1$ and then $r(1,d) = (1,a)$ forces $r \in U(\mathbf{Z}_{(2)})$ and hence $\mathbf{Z}_{(2)}(1,a) = \mathbf{Z}_{(2)}(c,d)$. Thus, either $\mathbf{Z}_{(2)}(1,a)$ is a maximal cyclic submodule or $\mathbf{Z}_{(2)}(0,a) \subseteq \mathbf{Z}_{(2)}(1,a/2)$ which is maximal. Thus, by Theorem 5.2, R_1 is atomic.

The proper principal ideals of R_1 have the form:

$$(1) \qquad (0,(\alpha,\beta))R_1 = 0 \oplus \mathbf{Z}_{(2)}(\alpha,\beta),$$

$$(2) \qquad (2^n,(0,\alpha))R_1 = 2^n\mathbf{Z}_{(2)} \oplus (0 \oplus \mathbf{Z}_{2^\infty})$$

(for $(u2^{n+m},(0,\beta)) = (2^n,(0,\alpha))(u2^m,(0,(\beta - u2^m\alpha)/2^n))$, $u \in U(\mathbf{Z}_{(2)})$), and

$$(3)\quad (2^n,(1,\alpha))R_1 = \{(u2^n,(1,\beta)) \mid u \in U(\mathbf{Z}_{(2)}), \beta \in \mathbf{Z}_{2^\infty}\} \\ \cup\, 2^{n+1}\mathbf{Z}_{(2)} \oplus (0 \oplus \mathbf{Z}_{2^\infty}).$$

Clearly the intersection of two principal ideals of R_1 is principal; so R_1 is an LCM ring. Note that a complete set of nonassociate atoms is

$$\left\{\left(0,\left(1,\frac{1}{2^n}+\mathbf{Z}\right)\right) \mid n \geq 0\right\} \cup \{(2,(0,0))\} \cup \{(2^n,(1,0)) \mid n \geq 1\}.$$

Example 5.4. A one-dimensional local weakly factorial (i.e., every nonunit is a product of primary elements) LCM ring R_1 that is not a unique factorization ring.

Take $R_1 = \mathbf{Z}_{(2)} \oplus \mathbf{Z}_2$ (idealization), so every ideal of R_1 can be generated by two elements. Here $\langle(0,1)\rangle$ is the unique minimal prime of R_1 and it is principal. Let $n \geq 1$. Here $(2^n,0)R_1 = 2^n\mathbf{Z}_{(2)} \oplus 0$ and $(2^n,0)$ is irreducible $\Leftrightarrow n = 1$. Also, $(2^n,1)R_1 = \{(u2^n,1) \mid u \in U(\mathbf{Z}_{(2)})\} \cup (2^{n+1},0)R_1$ and $(2^n,1)$ is irreducible for each $n \geq 1$. Moreover, since $\sqrt{(2^n,0)R_1} = \sqrt{(2^n,1)R_1}$ is the maximal ideal of R_1, $(2^n,0)$ and $(2^n,1)$ are primary. So R_1 is a one-dimensional local weakly factorial ring that is not a unique factorization ring. Moreover, the intersection of two principal ideals of R_1 is principal: (1) $(2^n,0)R_1 \cap (2^m,0)R_1 = (2^{\max\{n,m\}},0)R_1$, (2) $(2^n,0)R_1 \cap (2^m,1)R_1 = (2^{\max\{n,m+1\}},0)R_1$, and (3) $(2^n,1)R_1 \cap (2^m,1)R_1 = (2^{\max\{n+1,m+1\}},0)R_1$ for $n \neq m$.

Example 5.5. A ring R_1 which is not atomic but 0 and every regular element of R_1 is a product of irreducible elements.

Take $R = \mathbf{Z}$ and $N = \mathbf{Z}_2 \oplus \mathbf{Q}$, so $R_1 = \mathbf{Z} \oplus (\mathbf{Z}_2 \oplus \mathbf{Q})$. Note that $\mathbf{Z}(1,0)$ is a maximal cyclic submodule of $\mathbf{Z}_2 \oplus \mathbf{Q}$, but no other nonzero cyclic submodule is contained in a maximal cyclic submodule since $\mathbf{Z}(1,a) \subset \mathbf{Z}(1,a/3)$ and $\mathbf{Z}(0,a) \subset \mathbf{Z}(0,a/3)$. Hence R_1 is not atomic. Now $(0,(1,0))$ is irreducible and hence $(0,(0,0)) = (0,(1,0))^2$ is a product of irreducibles. By Theorem 5.2(1), every element of the form $(a,(b,c))$, where $a \neq 0, \neq 1$, is a product of irreducibles.

Example 5.6. A BFR in which 0 does not have a primary decomposition.

Let (R, M) be a local domain with $\dim R > 1$, and let $N = \oplus_{\text{ht } P=1} R/P$. Put $R_1 = R \oplus N$ so $M_1 = M \oplus N$ is the unique maximal ideal of R_1. Now $\cap_{n=1}^{\infty} M_1^n = 0$; so R_1 is a BFR. But $Z(R_1) = \cup_{\text{ht } P=1} P \oplus N$ is not a finite union of primes, so 0 does not have a primary decomposition.

Example 5.7. An irreducible element that is neither prime nor m-irreducible.

Let $R = \mathbf{Z}$, $N = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, and $R_1 = \mathbf{Z} \oplus N$. Then $(0, (0, 1))$ is irreducible since $\langle (0, 1) \rangle$ is a maximal cyclic submodule of N (Proposition 5.1). $R_1(0, (0, 1))$ is certainly not prime. Also, $R_1(0, (0, 1)) \subset R_1(3, (0, 0))$ since $(0, (0, 1)) = (3, (0, 0))(0, (0, 1))$, so $(0, (0, 1))$ is not m-irreducible. In fact, the relation $(0, 1) = 3(0, 1)$ shows that $(0, (0, 1))$ is not even strongly irreducible.

6. Factorization in $R[X]$ and $R[[X]]$. In this section we investigate factorization in $R[X]$ and $R[[X]]$ where R is a commutative ring. If a is an irreducible element of an integral domain D, then a is certainly also irreducible in $D[X]$ and $D[[X]]$, and conversely. We show that an element a of R is irreducible in R if and only if it is irreducible in $R[X]$ or $R[[X]]$. However, in Example 6.1 we give an example of a very strongly irreducible (and hence m-irreducible and strongly irreducible) element of R that is not even strongly irreducible in $R[X]$. Observe that 0 is m-irreducible in K, K a field, but 0 is not m-irreducible in $K[X]$ or $K[[X]]$.

We first note how the three types of associate relations behave with respect to elements of R when considered as elements of $R[X]$ or $R[[X]]$ where R is a commutative ring with zero divisors. Let $a, b \in R$. Then $a \sim_R b \Leftrightarrow a \sim_{R[X]} b \Leftrightarrow a \sim_{R[[X]]} b$ since $aR = bR \Leftrightarrow aR[X] = bR[X] \Leftrightarrow aR[[X]] = bR[[X]]$. Also, $a \approx_R b \Leftrightarrow a \approx_{R[X]} b \Leftrightarrow a \approx_{R[[X]]} b$. Now since a power series is a unit precisely when its constant term is a unit, we see that $a \cong_R b \Leftrightarrow a \cong_{R[[X]]} b$. Note that R is présimplifiable $\Leftrightarrow R[[X]]$ is présimplifiable. (For if $0 \neq f = gf$ where $f = a_n X^n + \cdots$, $g \in R[[X]]$ with $a_n \neq 0$, then $0 \neq a_n = g(0)a_n$. Thus, R présimplifiable gives that $g(0)$ is a unit and hence g is a unit.) Now $a \cong_{R[X]} b \Rightarrow a \cong_R b$, but even $a \cong_R a$ need not imply $a \cong_{R[X]} a$. For if R is présimplifiable, $a \cong_R a$ for each $a \in R$, but for $a \neq 0$, $a \cong_{R[X]} a$ implies $\operatorname{ann}(a) \subset \operatorname{nil}(R)$.

(For $b \in \operatorname{ann}(a)$ gives $a = (1 - bX)a$ and thus $1 - bX \in U(R[X])$ gives $b \in \operatorname{nil}(R)$. Thus, $a \cong_{R[X]} a$ for each $a \in R \Leftrightarrow Z(R) = \operatorname{nil}(R)$.) Thus, $R[X]$ présimplifiable $\Rightarrow R$ is présimplifiable, but $R[X]$ is présimplifiable $\Leftrightarrow 0$ is primary. Clearly the remarks of this paragraph may be extended to arbitrary families of indeterminates.

Example 6.1. Let $R = \mathbf{Z}_{(2)} \oplus \mathbf{Z}_4$ (idealization), so R is a one-dimensional local ring. Now $a = (0,1)$ is very strongly irreducible and prime in R. Hence a is prime and therefore irreducible in $R[X]$. But a is not strongly irreducible in $R[X]$. Let $f = (1,0) + (2,0)X$. Note that $af^2 = (0,1)((1,0)+(4,0)X+(4,0)X^2) = (0,1) = a$, so $a = (af)f$. Now certainly $f \not\approx a$. Suppose $af \approx a$, so $af = au$ for some unit $u \in R[X]$. Then $u = r_0 + r_1X + \cdots + r_sX^s$ where r_0 is a unit of R and $r_1, \ldots, r_s$ are nilpotent and hence in $0 \oplus \mathbf{Z}_4$. Thus $ar_i = 0$ for $i = 1, \ldots, s$. Hence $af = au = ar_0 \in R$, a contradiction. Here $a \cong_R a$, but $a \not\cong_{R[X]} a$. Also, $a \sim af$, but $a \not\approx af$.

We next show that if $a \in R$ is irreducible as an element of R, then a is also irreducible as an element of $R[X]$. Thus the weakest form of irreducibility is the only one preserved by adjunction of an indeterminate. This gives some evidence that the weakest form of being irreducible is indeed the correct generalization of "irreducible" to commutative rings with zero divisors. Of course, if a is prime in R, a is also prime as an element of $R[X]$, and conversely.

Theorem 6.2. *Let $a \in R$. Then a is irreducible in R if and only if a is irreducible in $R[\{X_\alpha\}]$.*

Proof. ($\Leftarrow$). Suppose $a = bc$ where $b, c \in R$. Since a is irreducible in $R[\{X_\alpha\}]$, say $aR[\{X_\alpha\}] = bR[\{X_\alpha]$. But then $aR = aR[\{X_\alpha\}] \cap R = bR[\{X_\alpha\}] \cap R = bR$. ($\Rightarrow$). Since any factorization of a involves only finitely many indeterminates, we may assume that $\{X_\alpha\}$ is finite. Then by induction it is enough to show that a is irreducible in $R[X]$.

If aR is idempotent, then a irreducible gives that aR is a prime ideal (Corollary 2.7). Thus $aR[X]$ is a prime ideal and hence a is irreducible in $R[X]$. So we may assume that aR is not idempotent.

Suppose $a = fg$ in $R[X]$ where $f = b_0 + b_1X + \cdots + b_sX^s$ and

$g = c_0 + c_1X + \cdots + c_tX^t$. Now $a = b_0c_0$ and a is irreducible, so say $a \sim b_0$. Let $b_0 = r_0a$; so $a = r_0ac_0$. Note that $a \not\sim c_0$; for if $a \sim c_0$, then we would have $a = b_0c_0 \sim a^2$, contradicting our assumption that (a) is not idempotent. It suffices to show that $a \mid b_j$ for each j, for then $a \sim b_0 + b_1X + \cdots + b_sX^s = f$. Suppose that we have already shown that $a \mid b_j$ for $j = 0, \ldots, i-1$; say $b_j = r_ja$. Equating the coefficients of X^i in the product $a = fg$ gives $0 = c_0b_i + c_1b_{i-1} + \cdots + c_ib_0$. So

$$\begin{aligned} a = c_0r_0a &= c_0r_0a + c_0b_i + c_1b_{i-1} + \cdots + c_ib_0 \\ &= c_0r_0a + c_0b_i + c_1r_{i-1}r_0ac_0 + \cdots + c_ir_0^2ac_0 \\ &= c_0(r_0a + b_i + c_1r_{i-1}r_0a + \cdots + c_ir_0^2a). \end{aligned}$$

Now a is irreducible and $a \not\sim c_0$, so $a \sim r_0a + b_i + c_1r_{i-1}r_0a + \cdots + c_ir_0^2a$. But then $a \mid b_i$. □

We next see how irreducibility extends from R to $R[[X]]$. We state our theorem for the power series ring $R[[\{X_\alpha\}]]_1 = \cup\{R[[X_{\alpha_1}, \ldots, X_{\alpha_n}]] \mid \{\alpha_1, \ldots, \alpha_n\} \subseteq \Lambda\}$ where $\{X_\alpha\}_{\alpha\in\Lambda}$ is a set of power series indeterminates over R. Similar results may be given for the power series rings $R[[\{X_\alpha\}]]_2$ and $R[[\{X_\alpha\}]]_3$ which are defined in Gilmer [**32**].

Theorem 6.3. *Let $a \in R$. Then a is irreducible, respectively strongly irreducible, very strongly irreducible, or prime, in R if and only if a is irreducible, respectively strongly irreducible, very strongly irreducible or prime, in $R[[\{X_\alpha\}]]_1$.*

But a is m-irreducible in $R[[\{X_\alpha\}]]_1$ if and only if a is m-irreducible in R and (a) is not idempotent.

Proof. Again, it is enough to prove the theorem for $R[[X]]$. The case where a is prime is well-known, and the proof given for a irreducible in Theorem 6.2 works mutatis mutandis for $R[[X]]$.

Suppose that a is strongly irreducible as an element of R. Again, we can assume that (a) is not idempotent. The proof for $R[X]$ shows, in the notation of the previous theorem, but with $f = b_0 + b_1X + \cdots$ and $g = c_0 + c_1X + \cdots$ power series, that we can take $a \approx b_0$. So $b_0 = ar_0$ where $r_0 \in U(R)$. But then $f = a(r_0 + r_1X + \cdots)$ where $r_0 + r_1X + \cdots$ is a unit in $R[[X]]$ and hence $f \approx a$. The converse is easily proved.

Next suppose that a is very strongly irreducible as an element of R. Here (a) cannot be idempotent unless $a = 0$, a trivial case. So suppose $a \neq 0$. As before, let $a = fg$ where $f = b_0 + b_1X + \cdots$ and $g = c_0 + c_1X + \cdots$ are power series. Now $a = b_0c_0$ and a is very strongly irreducible, so say c_0 is a unit. But then $g = c_0 + c_1X + \cdots$ is a unit, so a is very strongly irreducible in $R[[X]]$. The converse is easily proved.

If a is m-irreducible in $R[[X]]$, certainly a is m-irreducible in R. Suppose that a is m-irreducible in R. If (a) is idempotent, then $aR = eR$ where $e^2 = e$. Then $aR[[X]] = eR[[X]]] \subset (e + (1 - eX))R[[X]]] \neq R[[X]]$ so a is not m-irreducible in $R[[X]]$. So suppose that a is m-irreducible in R and (a) is not idempotent. If $aR[[X]] \subseteq fR[[X]] \neq R[[X]]$ where $f = b_0 + b_1X + \cdots$, then $a = fg$ for some $g = c_0 + c_1X + \cdots$. Now $a = b_0c_0$ gives $aR \subseteq b_0R \neq R$ and hence $aR = b_0R$. Since $aR = b_0Rc_0R$ and aR is not idempotent, we must have $c_0R = R$. But then c_0 is a unit in R, so g is a unit in $R[[X]]$ and hence $aR[[X]] = fR[[X]]$. □

We next consider the question of when X is irreducible in $R[X]$ or $R[[X]]$. Note that since X is regular, X is irreducible if and only if it is very strongly irreducible.

Theorem 6.4. *Let R be a commutative ring. Then X is a finite product of irreducible elements of $R[X]$ if and only if R is a finite direct product of indecomposable rings. In this case, the number of irreducible factors in any factorization of X into irreducibles is equal to the number of factors in any decomposition of R into a direct product of indecomposable rings. In particular, X is irreducible if and only if R is indecomposable. The previous statements of this theorem also hold if $R[X]$ is replaced by $R[[X]]$.*

Proof. We only prove the polynomial ring version; the proof of the power series ring case is similar.

If $R \cong R_1 \times \cdots \times R_m$, then $R[X] \cong R_1[X] \times \cdots \times R_m[X]$ where $X \to (X, \dots, X)$. Since an irreducible element of a direct product is a unit in all but one coordinate, we see that if X is a product of n irreducibles, then $m \leq n$. Thus, if X is a finite product of irreducibles, R is a finite direct product of indecomposable rings.

Next suppose that R is indecomposable. We show that X is very strongly irreducible. Suppose that $X = fg$ where $f = a_0 + a_1X + \cdots + a_sX^s$ and $g = b_0 + b_1X + \cdots + b_tX^t$. Then $0 = a_0b_0$ and $1 = a_0b_1 + a_1b_0$. Now both a_0 and b_0 cannot be 0, say $a_0 \neq 0$. Then $a_0 = a_0(a_0b_1 + a_1b_0) = a_0^2 + a_0b_0a_1 = a_0^2b_1$. Thus $(a_0)^2 = (a_0)$, so since R is indecomposable, a_0 is a unit. Hence $0 = a_0b_0$ gives $b_0 = 0$. So $X = (a_0 + a_1X + \cdots + a_sX^s)X(b_1 + b_2X + \cdots + b_tX^{t-1})$. Cancelling X gives $1 = (a_0 + a_1X + \cdots + a_sX^s)(b_1 + b_2X + \cdots + b_tX^{t-1})$, so f is a unit. Thus, X is very strongly irreducible.

Thus, if $R = R_1 \times \cdots \times R_m$ where each R_i is indecomposable, $(X, \ldots, X) = X_1 \cdots X_m$ in $R_1[X] \times \cdots \times R_m[X]$ where $X_i = (1, \ldots, 1, X, 1, \ldots, 1)$ (with X in the ith coordinate) is irreducible. Moreover, since any irreducible element of $R_1[X] \times \cdots \times R_m[X]$ is a unit in all but one coordinate with that coordinate being irreducible, we see that any factorization of X into irreducibles has exactly m irreducible factors. The result follows since the number of indecomposable factors in any decomposition of R into indecomposable rings is an invariant of R. □

We next briefly investigate how certain factorization properties defined on R extend to $R[X]$ or $R[[X]]$. First suppose that D is an integral domain. It is well known and easily proved that D satisfies ACCP $\Leftrightarrow D[X]$ satisfies ACCP $\Leftrightarrow D[[X]]$ satisfies ACCP. Certainly if $D[X]$ is atomic, so is D. Anderson, Anderson and Zafrullah [**7**] raised the question of whether D atomic implies that $D[X]$ is atomic. Soon afterwards Roitman [**45**] gave an example of an atomic domain D for which $D[X]$ is not atomic. (Thus, none of the forms of atomicity are inherited by $R[X]$.) The relationship between $D[[X]]$ and D being atomic seems not to have been investigated. Of course, D is a UFD $\Leftrightarrow D[X]$ is a UFD. If $D[[X]]$ is a UFD, so is D, but Samuel has given an example of a Noetherian UFD D with $D[[X]]$ not a UFD. Besides bounded factorization domains (BFDs) which were briefly mentioned in Section 3, the previously mentioned paper [**7**] considered *finite factorization domains* (FFDs). Recall that D is an FFD if one of the following three equivalent conditions holds: (1) every nonzero nonunit of D has only a finite number of factorizations up to order and associates, (2) every nonzero nonunit of D has only a finite number of nonassociate divisors, and (3) D is atomic and each nonzero element of D has at most a finite number of nonassociate irreducible divisors. It was shown that D is a BFD $\Leftrightarrow$ $D[X]$ is a BFD $\Leftrightarrow$ $D[[X]]$ is a BFD and that D is an FFD $\Leftrightarrow$ $D[X]$ is an FFD. Clearly, D an FFD $\Rightarrow$ D is a BFD $\Rightarrow$ D satisfies ACCP. While any Noetherian domain is always a BFD, it need not be an FFD. But a Krull domain is an FFD.

We have already defined (Definition 3.8) a bounded factorization ring (BFR) in Section 3. For commutative rings with zero divisors, the three equivalent conditions given in the previous paragraph to define the FFDs are no longer equivalent. This leads to three different definitions.

Definition 6.5. Let R be a commutative ring. R is called a *finite factorization* ring (FFR) if every nonzero nonunit of R has only a finite number of factorizations up to order and associates; R is called a *weak finite factorization ring* (WFFR) if every nonzero nonunit of R has only a finite number of nonassociate divisors; and R is called an *atomic idf -ring* if R is atomic and each nonzero element of R has at most a finite number of nonassociate irreducible divisors.

Clearly, if R is an FFR, then R is a WFFR and if R is a WFFR then R is an atomic idf-ring. However, $R = \mathbf{Z}_2 \times \mathbf{Z}_2$ being finite is certainly a WFFR, but $(0,1) = (0,1)^n$ is an infinite set of nonassociate factorizations of $(0,1)$. Next let $R = \mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)}$. Then R has only four nonassociate irreducibles: (0,1), (0,2), (1,0) and (2,0). However, $(1,0) = (1,0)(0,2^n)$ for each $n \geq 1$. Thus, R is an atomic idf-ring that is not a WFFR. Note that neither $\mathbf{Z}_2 \times \mathbf{Z}_2$ nor $\mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)}$ is a BFR or is présimplifiable.

Proposition 6.6. *For a commutative ring R, the following conditions are equivalent.*

(1) *R is an* FFR.

(2) *R is a* BFR *and* WFFR.

(3) *R is présimplifiable and a* WFFR.

(4) *R is a* BFR *and an atomic* idf-*ring.*

(5) *R is présimplifiable and an atomic* idf-*ring.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate, so it suffices to prove (5) $\Rightarrow$ (1). However the proof of Theorem 5.1 [**7**] (which shows that an atomic idf-domain is an FFD) may be easily adapted to show that (5) $\Rightarrow$ (1). Just use the hypothesis of présimplifiable rather than cancellation to yield a contradiction. □

Since an FFR is présimplifiable, all the forms of associate coincide. However, in the definitions of WFFRs and atomic idf-rings, we could get variations of these definitions by replacing "nonassociate" by "non strongly associate" or "non very strongly associate." In the definition of an atomic idf-ring we could replace the atomic hypothesis by strongly atomic, very strongly atomic, etc.

It is an open question whether R satisfies ACCP implies $R[X]$ or $R[[X]]$ satisfies ACCP. See Heinzer and Lantz [**34**] for a discussion of this question. They do show that if a zero-dimensional ring R satisfies ACCP then so does $R[X]$. For the related question of when $R[X]$ inherits the ascending chain condition on annihilator ideals, see Camillo and Guralnick [**27**] and Kerr [**38**].

$\mathbf{Z}_4$ is a unique factorization ring and a Fletcher unique factorization ring, but neither $\mathbf{Z}_4[X]$ nor $\mathbf{Z}_4[[X]]$ is a unique factorization ring or a Fletcher unique factorization ring. This example also shows that R p-atomic does not imply that $R[X]$ or $R[[X]]$ is p-atomic. Using Theorem 4.4 we see that $R[X]$ is a unique factorization ring if and only if R is a UFD, and, using Theorem 4.9, we see that $R[X]$ is a Fletcher unique factorization ring if and only if R is a finite direct product of UFDs. Also, $R[[X]]$ a unique factorization ring forces R to be a UFD while $R[[X]]$ a Fletcher unique factorization ring forces R to be a finite direct product of UFDs.

Let R be a local ring with 0 not primary. By Theorem 3.9 R is a BFR since it is présimplifiable, but $R[X]$ is not a BFR because it is not présimplifiable. However, $R[[X]]$ is a BFR. In fact, for R Noetherian, R is a BFR $\Leftrightarrow$ $R[[X]]$ is a BFR. More generally, for any commutative ring R, R is présimplifiable and satisfies ACCP $\Leftrightarrow$ $R[[X]]$ is présimplifiable and satisfies ACCP. We conjecture that R is a BFR $\Leftrightarrow$ $R[[X]]$ is a BFR and that $R[X]$ is a BFR $\Leftrightarrow$ R is a BFR and 0 is primary.

We next show that R an FFR need not imply that $R[X]$ or $R[[X]]$ is an FFR. Let (R, M) be a quasi-local ring with $M^2 = 0$. Then certainly R is an FFR. Now for $m, n \in M - \{0\}$, $mX^2 = mX(n + X)$ in $R[X]$ and $R[[X]]$. Since $n_1 + X$ and $n_2 + X$ are associates $\Leftrightarrow n_1 = n_2$, we see that if M is not finite, then $R[X]$ or $R[[X]]$ is not an FFR. The same example shows that for any commutative ring R, if $R[X]$ or $R[[X]]$ is an FFR, then for $0 \neq m \in R$, $\operatorname{ann}(m)$ must be finite.

We can also consider when the various factorization properties ascend to or descend from the rings $R(X)$ and $R\langle X\rangle$. For a discussion of the rings $R(X)$ and $R(X)$, the reader is referred to [**6**].

Note added in proof. The question mentioned in Section 6 of whether R has ACCP implies $R[X]$ has ACCP has been answered in the negative by W. Heinzer and D. Lantz, *ACCP in polynomial rings: a counterexample*, Proc. Amer. Math. Soc. **121** (1994), 975–977.

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