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DECAYING SOLUTIONS OF ELLIPTIC SYSTEMS IN R^n

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ABSTRACT. We consider nonlinear elliptic systems, with prototype form: $-\Delta\vec{u} = \lambda\vec{f}(x,\vec{u})$ in $\mathbf{R}^n$ and show the existence of positive decaying (componentwise) solutions. Our basic tools are classical estimates of Gidas, Ni, Nirenberg and Egnell coupled with Leray-Schauder degree theory arguments in weighted spaces. We do not assume, in general, that the system is variational, although mountain pass arguments are employed for one such case. This approach enables us to obtain, in particular, the existence of positive solutions also for reducible systems, and the extension of several recent results, some even in the scalar case.

1. Introduction. This paper deals with elliptic nonlinear systems formally given by the equation

$$(1)\qquad -\Delta\vec{u} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_M \end{pmatrix} \vec{f}(x,\vec{u})$$

in $\mathbf{R}^n$, $n \geq 3$, and the related problem $-\Delta\vec{u} = \lambda\vec{f}$, i.e., $\lambda_1 = \cdots = \lambda_M = \lambda$. Here $\vec{u} = (u_1,\dots,u_M)$, and we are interested in the evidence of positive (componentwise) solutions $\vec{u}$ to (1) such that $\vec{u} \to \vec{0}$ at ∞, in the case that $\vec{f}(x,\vec{u})$ is superlinear and subcritical. We do not usually require that (1) admit a variational structure, although some results are obtained under this assumption.

The rough outline of this paper is as follows: We first give conditions under which (1) has solutions for all $\vec{\lambda} = (\lambda_1,\dots,\lambda_M) > \vec{0}$. Our approach here is based on the observation that, under suitable conditions, the fundamental scalar results of Gidas, Ni and Nirenberg [**12**], Gidas and Spruck [**13**] and Egnell [**10**] can be employed to show that all positive solutions of (1) have a norm which is bounded and bounded

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away from zero. An application of degree theory then shows the existence of a nontrivial nonnegative solution. If the system is assumed quasi-irreducible in the sense of Cosner [**9**], the solution must then be positive. These results are obtained under rather severe restrictions on $\vec{f}$, but since they are based on degree theory methods, they continue to hold under perturbations. In Section 2 we employ this fact in order to obtain existence results for (1) for small $\vec{\lambda} > \vec{0}$ under more general conditions on $\vec{f}$. We mention, in particular, that we do not require quasi-irreducibility in this section and indeed obtain the existence of positive decaying solutions also in cases where there exist nonnegative nontrivial solutions. We are not aware of any other results along these lines. These approaches, however, do not deal with cases where all components of $\vec{f}$ involve only cross terms in the components of $\vec{u}$. For completeness we obtain in Section 3 some results for this case under the assumption that the problem is variational. In Section 4 we conclude the paper with some examples and explicit comparisons with earlier results. We also include some results for the scalar case which we believe to be new.

Unlike the case where $\mathbf{R}^n$ is replaced by a bounded domain Ω, there appears to be little known for systems such as (1) or even for superlinear scalar equations not in variational form. The system results in $\mathbf{R}^n$ with which we are familiar involve variational arguments, radial conditions and/or upper-lower solution methods. Furthermore, often it was not required that $\vec{u}$ decay at ∞. We refer in particular to the papers of Gu [**14**], Furusho [**11**], Noussair and Swanson [**22, 23**], Kusano and Swanson [**19**], Allegretto [**1**], Brezis and Lieb [**4**], Kawano [**17**], Kawano and Kusano [**18**], Berestycki and Lions [**4**], P.L. Lions [**21**], and the references therein. These results furnished the basic motivation for this paper.

To avoid technical difficulties, we assume throughout the paper that all functions introduced are smooth in their arguments. Various explicit growth and monotonicity assumptions on $\vec{f}$ will be given below, but we always assume at least that $\vec{f}(x,\vec{\xi}) > \vec{0}$ for $\vec{\xi} > \vec{0}$, $\vec{f}(x,\vec{0}) = \vec{0}$, $\vec{f}$ nondecreasing in $\vec{\xi} \geq \vec{0}$, $\vec{f} \in C^{\alpha}_{\text{loc}}$. Finally, vector inequalities will be understood componentwise, and the various norms of $\vec{u}$ are defined in terms of the component norms in the obvious way. In particular, we define $|\vec{u}| = \sum_{i=1}^{M} u^i$ if $\vec{u} \geq \vec{0}$, $|\vec{u}| = \sum_{i=1}^{n} |u^i|$ otherwise. By E, we denote the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$||v||_E^2 = \int_{\mathbf{R}^n} |\nabla v|^2.$$

We observe that, for $\vec{u} \geq \vec{0}$, the estimate

$$||\,|\vec{u}|\,||_E \leq K||\vec{u}||_E$$

holds.

2. Positive solutions for all $\vec{\lambda} > \vec{0}$. We consider here conditions under which equation (1) has positive solutions for all $\vec{\lambda} > \vec{0}$, and for notational convenience incorporate $\vec{\lambda}$ into $\vec{f}$, i.e., we consider

$$(1') \qquad -\Delta \vec{u} = \vec{f}(x, \vec{u}).$$

Our assumptions on $\vec{f}$ are as follows:

2.1. There exists $\tilde{\mu} > 0$ such that if $\mu > \tilde{\mu}$, then for any $j \in \{1, \dots, n\}$, $i \in \{1, \dots, M\}$ and $\vec{\xi} > \vec{0}$, we have

$$f^i(x, \vec{\xi}) \geq (\not\equiv) f^i(x^{\mu_j}, \vec{\xi}) \quad \text{if} \quad x_j < (x^{\mu_j})_j$$

and

$$f^i(x, \vec{\xi}) \geq (\not\equiv) f^i(x^{-\mu_j}, \vec{\xi}) \quad \text{if} \quad x_j > (x^{-\mu_j})_j.$$

Here x^{μ_j} denotes the reflection of x about the plane $x_j = \mu$ [**12**].

2.2. There exists a function $h(x) > 0$ such that

$$\lim_{|\vec{\xi}| \to \infty} \frac{|\vec{f}(x, \vec{\xi})|}{|\vec{\xi}|^\gamma} = h(x)$$

locally uniformly in x, if $\vec{\xi} \geq \vec{0}$.

2.3. There exists a function $C(x) \in L^p \cap L^\infty$ such that $|\vec{f}(x, \vec{\xi})| \leq C(x)|\vec{\xi}|^\gamma$ for $\vec{\xi} \geq 0$ and $C(x) \leq K|x|^{-\beta}$ for some p, β with $p < n/2$ and $\beta + \gamma(n-2) > (n+1)$.

2.4. Problem $(1')$ is quasi-irreducible, [**9**]; i.e., if $\vec{u} \geq \vec{0}$ solves $(1')$, then either $\vec{u} \equiv \vec{0}$ or $\vec{u} > \vec{0}$.

We recall that $1 < \gamma < (n+2)/(n-2)$. The above assumptions are very restrictive on the type of system which can be considered. In the next section we shall obtain solutions, for small $\vec{\lambda}$, for more general systems. Examples of functions $\vec{f}$, for which (2.1)–(2.4) hold, are easy to construct, and we shall do so explicitly later. It is useful to observe here that 2.1 is a condition at infinity. Specifically, if for $|x|$ large, $\vec{f} \geq \vec{0}$ is radial in x and $\vec{f}' \leq 0$ or if f is the product of even R_1 functions decaying monotonically at ∞, then 2.1 holds, and if 2.1 holds for some $\vec{f}$ then it also holds for $\vec{f} + \vec{g}$ where $\vec{g}(x, \vec{\xi}) \geq \vec{0}$, $\vec{g} \equiv \vec{0}$ if $|x|$ is large. Finally, if $\gamma(n-2) > (n+1)$, then we choose $\beta = 0$ in 2.3.

First for positive solutions in E we have the following lemma whose arguments will be useful in the sequel.

Lemma 0. *Let $\vec{u} \geq \vec{0}$ solve* (1)*. Then $|| \, |\vec{u}| \, ||_{L^\infty} \leq a(||\vec{u}||_E)$ for some positive continuous function a.*

Proof. We can reduce the situation to an analogue of scalar results as follows. We set $\vec{u}_K(x) = (u_K^1(x), u_K^2(x), \dots, u_K^M(x))$ where $u_K^i(x) = \min\{u^i(x), K\}$, $i = 1, \dots, M$ and $K = 1, 2, \dots$. For any real number $m \geq 1$, let

$$[\vec{u}_K(x)]^m = ([u_K^1(x)]^m, \dots, [u_K^M(x)]^m).$$

Taking $\vec{\varphi}(x) = [\vec{u}_K(x)]^m$, and denoting by $\langle \quad , \quad \rangle$ the E inner product, we have

$$\langle \vec{u}, [\vec{u}_K(x)]^m \rangle \leq \int_{\mathbf{R}^n} \sum_{j=1}^M |f^j(x, \vec{u})| \cdot [u_K^j]^m \, dx$$

$$\leq \sum_{j=1}^M \int_{\mathbf{R}^n} |f^j(x, \vec{u})| \cdot [u^j]^m \, dx.$$

But

$$\langle \vec{u}, [\vec{u}_K(x)]^m \rangle \geq \sum_{j=1}^M \frac{4m}{(m+1)^2} ||[u_K^j]^{(m+1)/2}||_E^2$$

$$\geq \sum_{j=1}^M C(n, m)[||u_K^j||_{(m+1)n/(n-2)}]^{m+1}.$$

Thus,

$$\begin{aligned}C\| |\vec{u}_k| \|^{m+1}_{(m+1)n/(n-2)} &\leq \sum_{j=1}^{M} C(n,m)[\|u^j_K\|_{(m+1)n/(n-2)}]^{m+1} \\ &\leq \sum_{j=1}^{M} \int_{\mathbf{R}^n} |f^j(x,\vec{u})| \cdot [u^j]^m \, dx \\ &\leq K \int C(x)|\vec{u}|^{m+\gamma},\end{aligned}$$

and the proof then follows the arguments of the scalar case, [**3**], applied to $|\vec{u}|$. ◻

We begin with the following lemma which is a combination of scalar results of Gidas, Ni, Nirenberg [**12**, Lemma 2.1] and Egnell [**10**, Theorem 2] (see also Li and Ni [**20**]).

Lemma 1. *Let $\vec{u} > \vec{0}$ solve $(1')$. Then there exists a $\bar{\mu} > 0$, independent of $\vec{u}$, such that for any $j = 1, \dots, n$, $\vec{u}(x)$ is a decreasing, respectively increasing, function of x_j if $x_j > \bar{\mu}$, respectively $x_j < -\bar{\mu}$.*

Proof. Observe first that $-\Delta(|\vec{u}|) = |\vec{f}(x,\vec{u})|$, and our assumptions on $\vec{f}$ immediately yield $|\vec{u}| \sim |x|^{2-n}$ at ∞ by [**10**, Theorem 2]. It follows that $|\vec{f}(x,\vec{u}(x))| = O(|x|^{-q})$ for some $q > n+1$ and Lemma 2.1 of [**12**] applies to each component of $\vec{u}$. Since our conditions on $\vec{f}$ are uniform in j, it thus suffices to prove the result for $j = 1$ and $x_1 > \bar{\mu}_1$ for some $\bar{\mu}_1$ independent of $\vec{u}$. Set

$$\vec{\mu}_1 = \inf\{\mu \mid \text{if } \alpha \geq \mu \text{ then } \vec{u}(x) > \vec{u}(x^{\alpha_1}) \text{ if } x_1 < \alpha\}.$$

Observe that the same arguments as in [**12**] show that $\bar{\mu}_1$ exists and, furthermore, $\bar{\mu}_1 \leq \tilde{\mu}$ with $\tilde{\mu}$ as defined in 2.1. We do not require any system arguments for this result, as found, e.g., in [**25**], since the monotonicity properties of $\vec{f}$ applied to each component equation suffice. ◻

Combining Lemma 1 with the well-known estimates of Gidas and Spruck [**13**], gives

Lemma 2. *There exists a ball B and constant K, independent of $\vec{u}$, such that $|\vec{u}|$ for any positive solution $\vec{u}$ of $(1')$ takes on its maximum in B and $|\vec{u}| < K$.*

Proof. That the maximum of $|\vec{u}|$ must occur in a fixed ball B independent of $\vec{u}$ is immediate from Lemma 1. Suppose there exist sequences $\{\vec{u}_\delta\}$, $\{x_\delta\}$ such that $||\,|\vec{u}_\delta|\,||_{L^\infty} = |\vec{u}_\delta(x_\delta)| \to \infty$. Without loss of generality, $x_\delta \to x$ for some $x \in \overline{B}$. We set $\vec{v}_\delta(y) = \vec{u}_\delta(\lambda_\delta y + x_\delta)/M_\delta$ with $M_\delta = ||\,|\vec{u}_\delta|\,||_{L^\infty}$ and $\lambda_\delta^{2/\gamma-1} \cdot M_\delta = 1$. In view of assumption 2.2, we may then apply directly the procedures of [**13**] to the scalar equation satisfied by $|\vec{v}_\delta|$ and obtain a contradiction. □

We also observe the following elementary estimate indicating the equivalence of the $||\quad||_E$ and L^∞ norms for positive solutions.

Lemma 3. *Let $\vec{u} \geq \vec{0}$ solve $(1')$. Then*

$$||\vec{u}||_E \sim ||\,|\vec{u}|\,||_{L^\infty},$$

in the sense that a bound on one norm implies a bound on the other.

Proof. We recall that Lemma 0 gives $||\,|\vec{u}|\,||_{L^\infty} \leq a(||\vec{u}||_E)$ for some positive continuous function a. On the other hand, we observe

$$\begin{aligned}||\vec{u}||_E^2 &\leq K \int_{\mathbf{R}^n} C(x)|\vec{u}|^{2-\varepsilon}|\vec{u}|^{\gamma-1+\varepsilon}\\ &\leq K||\,|\vec{u}|\,||_{L^\infty}^{\gamma-1+\varepsilon}||\,|\vec{u}|\,||_E^{2-\varepsilon}||C||_{n/[2+\varepsilon(n-2)/2]}\\ &\leq K||\,|\vec{u}|\,||_{L^\infty}^{\gamma-1+\varepsilon}||\vec{u}||_E^{2-\varepsilon}||C||_{n/[2+\varepsilon(n-2)/2]}.\end{aligned}$$

Whence, if ε is sufficiently small, we obtain $||\vec{u}||_E \leq K||\,|\vec{u}|\,||_{L^\infty}^{(\gamma-1+\varepsilon)/\varepsilon}$ and the result.

We observe for future convenience that the same result holds with obvious changes for the more general equation

$$-\Delta\vec{u} = \nu\vec{f}(x, \vec{u}) + t\vec{J}(x)$$

with $0 \leq \nu, t$ bounded and $\vec{J}$ smooth, nonnegative with compact support. Furthermore, if $t = 0$, then $a(0) = 0$ and $a(\xi) > 0$ if $\xi > 0$ where $a(\xi)$ denotes the function given in the above proof. □

As an immediate consequence, we have

Theorem 1. *Let* $F(\vec{u}) = (-\Delta)^{-1}(\vec{f^*}(x,\vec{u}))$ *with* $\vec{f^*}(x,\vec{u}) = \vec{f}(x,\vec{u}^+)$. *Then* $F : E \to E$ *is continuous and compact. Furthermore, there exist* $0 < r_1 < r_2$ *such that* $\deg(I - F, B_{r_2} - \overline{B}_{r_1}, \vec{0}) \neq 0$ *where* B_{r_i} *denotes the ball of radius* r_i *in* E.

Proof. The fact that $F : E \to E$ is continuous and compact is shown in exactly the same way as for the scalar case [**3**]. As for the rest, we recall that it suffices to show that for some r_1, r_2, we have

(i) $\vec{u} \neq \nu F(\vec{u})$ for $0 \le \nu \le 1$, $u \in \partial B_{r_1}$,

(ii) $F(\vec{u}) \neq \vec{u} - t(-\Delta)^{-1}(\vec{z})$ for $t \ge 0$, $\vec{u} \in \partial B_{r_2}$ for some $\vec{0} \le (\not\equiv)\vec{z} \in E$.

Now (i) is immediate, since if $-\Delta\vec{u} = \nu\vec{f^*}(x,u) = \nu\vec{f}(x,\vec{u}^+)$ then $\vec{u} \ge \vec{0}$, $\vec{u} \not\equiv \vec{0}$ whence $\vec{u} > \vec{0}$ by irreducibility. We have

$$-\Delta(|\vec{u}|) \le \nu C(x)(|\vec{u}|)^\gamma \le \nu||\,|\vec{u}|\,||_{L^\infty}^{\gamma-1} C(x)|\vec{u}|.$$

We observe that there exists an eigenfunction $J > 0$, $J \in E$ such that $-\Delta J = \sigma C(x)J$ [**2**], and conclude

$$\sigma \int C(x)J|\vec{u}| \le \nu||\,|\vec{u}|\,||_{L^\infty}^{\gamma-1} \int C(x)J|\vec{u}|$$

whence, by Lemma 3, $[a(||\vec{u}||_E)]^{\gamma-1} \ge ||\,|\vec{u}|\,||_{L^\infty}^{\gamma-1} \ge K\sigma/\nu \ge K\sigma$. As for (ii), we note that it suffices to consider

$$-\Delta\vec{u} = \vec{f^*}(x,\vec{u}) + tJI, \tag{4}$$

where J here denotes the eigenfunction corresponding to the first eigenvalue for the Dirichlet problem for $-\Delta$ in a ball B, extended by $J \equiv 0$ outside B, and I is the identity matrix.

We first note that t must be bounded for a solution to exist. Indeed, if $\vec{u}$ solves (4) with $t > 0$, then $\vec{u} > \vec{0}$ and

$$-\Delta(|\vec{u}|) = \sum f_i(x,\vec{u}) + MtJ = |\vec{f}(x,\vec{u})| + MtJ.$$

Since $|\vec{f}(x,\vec{\xi})|/|\vec{\xi}|^\gamma \to h(x)$ uniformly in B, we conclude that $|\vec{f}(x,\vec{\xi})| \geq (h(x)/2)|\vec{\xi}|^\gamma - |\vec{\xi}|$ in B. It follows that

$$-\Delta(|\vec{u}|) + |\vec{u}| \geq \frac{h(x)}{2}|\vec{u}|^\gamma + MtJ \quad \text{in } B$$

with $|\vec{u}| > d > 0$ in B for some d. On the other hand, let w solve

$$\begin{aligned} -\Delta(w) + w = \varepsilon J \quad &\text{in } B \\ w = 0 \quad &\text{on } \partial B \end{aligned}$$

for $\varepsilon > 0$ small. Then $0 < w < d$ in B, and

$$-\Delta(w) + w \leq \frac{h(x)}{2}w^\gamma + MtJ.$$

We conclude that $w, |\vec{u}|$ form an ordered lower-upper solution pair, and thus obtain the existence of a solution $v > 0$ for

$$\begin{aligned} -\Delta v + v = (h(x)/2)v^\gamma + MtJ \quad &\text{in } B \\ v = 0 \quad &\text{on } \partial B. \end{aligned}$$

Following exactly the arguments in [**7**], we conclude that Mt is bounded. Since J has compact support, the remarks following condition 2.4 imply that Lemma 1 and Lemma 2 hold, and we thus have $||\vec{u}||_E \leq K$ for some K, by Lemma 3. □

Corollary 1. *Problem* $(1')$ *has a positive solution.*

Proof. By the properties of the Leray-Schauder degree we conclude that there exists a $\vec{u}$, nontrivial, such that $-\Delta(\vec{u}) = \vec{f}(x,\vec{u}^+)$. The maximum principle then implies $\vec{u} \geq \vec{0}$ and thus $\vec{u} > \vec{0}$ by irreducibility. □

3. Positive solutions for small $\vec{\lambda}$. Our problem here consists in establishing the existence for small $\vec{\lambda} > \vec{0}$ of positive solutions for (1) without some of the assumptions made in Section 2. In particular, we no longer require quasi-irreducibility and the systems may also have nonnegative nontrivial solutions. We shall also relax condition 2.2. Our approach consists in using the product and homotopy properties of the Leray-Schauder degree and applying these properties to systems which are perturbations of compositions of simpler subsystems which satisfy the conditions of Section 2. More precisely, we consider explicitly the case where the subsystems are scalar equations.

We remark that results for such scalar equations could also be established by variational methods [**3**], but it was essential for us to use degree theory directly since we could not show that the variational solutions were isolated [**15**].

Our new assumption is as follows:

3.1. There exist nonnegative functions h_j, q_j, τ_j such that for each $j = 1, \dots, M$

(1) $h_j(x)\xi_j^{\gamma_j}$ satisfies 2.1–2.4 in the scalar case with $1 < \gamma_j < (n+2)/(n-2)$.

(2) $q_j \in L^{p_1} \cap L^\infty$, $\tau_j \in L^{p_2} \cap L^\infty$ with $p_1 = 2n/[(n+2) - \gamma_j(n-2)]$; $p_2 \in 2n/[(n-2) - \theta_j(n-2)]$ and $1 < \theta_j < \gamma_j$.

(3) for all $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ such that

$$|f^j(x, \vec{\xi}) - h_j(x)\xi_j^{\gamma_j}| \le \varepsilon q_j(x)|\vec{\xi}|^{\gamma_j} + K(\varepsilon)\tau_j(x)|\vec{\xi}|^{\theta_j} \tag{4}$$

for any $\vec{\xi} \ge \vec{0}$.

Lemma 4. (a) *For each $j = 1, \dots, M$, there exists an annulus $A_j \subset E$ such that*

$$\deg(u_j - (-\Delta)^{-1}(h_j(u_j^+)^{\gamma_j}), A_j, 0) \ne 0.$$

(b) *Let $S(\vec{u}) = (-\Delta)^{-1}[(h_1(u_1^+)^{\gamma_1}, \dots, h_M(u_M^+)^{\gamma_M})^T]$ and $A = A_1 \times \cdots \times A_M$. We then have*

$$\deg(\vec{u} - S(\vec{u}), A, \vec{0}) \ne 0.$$

Proof. Part (a) is immediate from the results in Section 2 specialized to the scalar case. Part (b) follows from the product properties of the Leray-Schauder degree, [**25**, p. 573]. □

Theorem 2. *If* 3.1 *holds, then* (1) *has a positive solution for some* $\vec{\lambda} > \vec{0}$, $|\vec{\lambda}|$ *small.*

Proof. Let T denote the set of solutions to $\vec{u} = S(\vec{u})$ in A. We observe immediately that T is a compact set in E, and let $N \subset A$ denote an open neighborhood of T chosen as follows. Set $J(u_j) = \int_B [u_j^+]^{2n/(n-2)}$ for $j = 1, \dots, M$, and some fixed ball $B \subset \mathbf{R}^n$. Since T is compact, we observe that, for each j, $J(u_j)$ assumes its (positive) infimum on T. We choose N small enough so that if $\vec{u} \in N$ then $J(u_j) > \alpha > 0$ for some constant α and any $j = 1, \dots, M$. Now $\deg(\vec{u} - S(\vec{u}), N, \vec{0}) \neq 0$, and equation (4) may be rewritten as

$$(5) \quad \left| \alpha^{\gamma_j} f^j\left(x, \frac{\vec{\xi}}{\alpha}\right) - h_j(x)\xi_j^{\gamma_j} \right| \le \varepsilon q_j(x)|\vec{\xi}|^{\gamma_j} + K(\varepsilon)\alpha^{\gamma_j - \theta_j}\tau_j(x)|\vec{\xi}|^{\theta_j}$$

for any $\alpha > 0$.

We set $P(\vec{u}) = (-\Delta)^{-1}[(\Lambda \cdot \vec{f}(x, \vec{u}^+/\alpha) - \mathrm{Diag}\,(h_j(x)u_j^{+\gamma_j})]$, where $\Lambda = \mathrm{Diag}\,(\alpha^{\gamma_j})$, and observe that (4) and (5) imply that $P : E \to E$ is continuous, compact, and, for any given small $\eta > 0$, by choosing first $\varepsilon > 0$ small enough and then α small, we conclude from (5) that $||P(\vec{u})||_E \le \eta$ for $\vec{u} \in \overline{N}$ by assumption 3.1. Next observe that $||\vec{u} - S(\vec{u})||_E$ is bounded away from zero on ∂N by the properties of S and the definition of N. In particular, if ε and α are chosen small enough, then for $u \in \partial N$ and $0 \le t \le 1$, $||\vec{u} - S(\vec{u}) - tP(\vec{u})||_E \ge ||\vec{u} - S(\vec{u})||_E - ||P(\vec{u})||_E > 0$ and $\deg(\vec{u} - S(\vec{u}) - tP(\vec{u}), N, \vec{0})$ is well defined. Since $\deg(\vec{u} - S(\vec{u}), N, \vec{0}) \neq 0$, then also

$$\deg(\vec{u} - S(\vec{u}) - P(\vec{u}), N, \vec{0}) \neq 0$$

i.e., there exists a solution $\vec{u}$ to

$$-\Delta\vec{u} = \Lambda\vec{f}\left(x, \frac{\vec{u}^+}{\alpha}\right).$$

Observe that $\vec{u} \ge 0$ and $\vec{u}$ is clearly nontrivial. Indeed, since $\vec{u} \in N$, then $J(u_i) > 0$ and hence $u_i^+ \neq 0$ and, consequently, $u_i > 0$ by the (scalar) maximum principle. Finally, setting $\vec{v} = (\vec{u}/\alpha)$ gives the result. $\square$

Note that since $\vec{u} \in E$, then $\vec{u} \to \vec{0}$ at ∞ as shown in the proof of Lemma 1.

Corollary 2. *If* $\gamma_1 = \cdots = \gamma_M = \gamma$, *then* (1) *has a positive solution with* $\lambda_1 = \cdots = \lambda_M = \lambda$ *for* $\lambda > 0$ *small. That is, there exists a function* $\vec{u} > \vec{0}$ *such that* $-\Delta\vec{u} = \lambda\vec{f}(x, \vec{u})$.

4. Some variational cases. We consider here a class of systems which do not satisfy the assumptions made in the earlier sections. We are in particular interested in the case where each f_i involves precisely cross products of the various components of $\vec{u}$. We no longer require the various monotonicity assumptions made earlier but now do ask that the problem have a variational structure, i.e., that there exists a function F such that $\nabla_{\vec{u}} F(x, \vec{\xi}) = \vec{f}(x, \vec{\xi}^+)$. This is also a somewhat severe condition. In compense, we can deal in exactly the same way with a more general case where $-\Delta\vec{u}$ is replaced by $-\sum D_i(a_{ij}(x)D_j\vec{u})$, but will not do this for consistency. The added variational assumption implies that the problem may be treated by mountain pass arguments. Since most of the steps are very similar to the ones in the scalar case, e.g., [**3**], our presentation will be brief.

We now make the following assumptions on $\vec{f}$:

$$\left|\sum f^i(x, \vec{\xi})\right| \leq g(x)|\vec{\xi}|^\gamma \quad \text{with} \quad g \in L^{p_0} \cap L^\infty \tag{4.1}$$

and

$$p_0 = 2n/[2n - (\gamma + 1)(n - 2)], \qquad \vec{\xi} \geq \vec{0}.$$

4.2. There exists a function F such that $\nabla_{\vec{u}} F(x, \vec{\xi}) = \vec{f}(x, \vec{\xi}^+)$. For notational simplicity we write $\vec{f}(x, \vec{t})$ in place of $\vec{f}(x, \vec{t}^+)$ henceforth. Since we seek positive solutions, this will not cause difficulty.

4.3. There exists a constant $\theta > 0$ such that $\theta u_j f_j(x, \vec{u}) \geq F(x, \vec{u})$ for $\vec{u} \geq \vec{0}$, $x \in \mathbf{R}^n$ and $j = 1, \ldots, M$, where F denotes the potential associated with $\vec{f}$ normalized by $F(x, \vec{0}) = 0$ and $\theta/M < 1/2$.

4.4. There exists $0 \le w \in C_0^\infty$ such that, for all large β,

$$\int_{\mathbf{R}^n} F(x, \beta w\vec{1})\, dx \ge \beta^{\theta_1} C(w) - K(w).$$

Here $C(w)$ and $K(w)$ represent constants dependent on w, $\theta_1 > 2$ and $\vec{1} = (1, \dots, 1)$.

We first remark that 4.3 implies irreducibility. Indeed, assume that $-\Delta\vec{u} = \lambda\vec{f}(x, \vec{u})$ with $\vec{u} \ge \vec{0}$, nontrivial. If, for some i, we have $u^i \not\equiv 0$, then for some x_0 we have $f^i(x_0, \vec{u}(x_0)) \ne 0$ and $u^i(x_0) > 0$. It follows that

$$\begin{aligned}\theta u^j(x_0) f^j(x_0, \vec{u}(x_0)) &\ge F(x_0, \vec{u}(x_0)) \\ &= \int_0^1 \sum_k f^k(x_0, t\vec{u}(x_0)) \cdot u^k(x_0)\, dt > 0\end{aligned}$$

since $f^j \ge 0$. It follows that $u^j \ge 0$, $\not\equiv 0$ and hence $u^j > 0$ by the (scalar) maximum principle. Condition (4.3) could thus be replaced by the assumption that the problem is irreducible and $F(x, \vec{u}) \le \theta \sum f^j(x, u) u^j$, with $\theta < 1/2$. Note that this condition will hold, for example, in cases where the components of $\vec{f}$ consist of cross products of various components of $\vec{u}$. In such cases M and n are, however, related. For example, if $F(x, \vec{u}) = p(x)(u^1)^{\alpha_1} \cdots (u^M)^{\alpha_M}$ for $\vec{u} \ge \vec{0}$, $F(x, \vec{u}) = 0$ otherwise; then, to ensure smooth differentiability and irreducibility, we choose $\alpha_i \ge 1$ for each i, while $\alpha_1 + \dots + \alpha_M < 2n/(n-2)$ in order to establish suitable maps on E, and it follows that $M < 2n/(n-2)$. For further discussion of this argument, we refer to [**23**]. Observe that no such restriction was present in our earlier results, and that, even here, we can consider the case $M = 2$ for any n.

We let

$$G(\vec{u}) = \int_{\mathbf{R}^n} F(x, \vec{u})\, dx$$

and

$$J(\vec{u}) = (1/2)||\vec{u}||_E^2 - \lambda G(\vec{u}) \quad \text{for } \vec{u} \in E. \tag{6}$$

We recall that the arguments in [**24**] show how to treat the scalar case in bounded domains, while those of [**3**] extend some of the results of

[**24**] to scalar problems in unbounded domains. Based on these results we observe the following analogies for the system case.

Lemma 5. *G' is continuous and compact from E to E, with*

$$G'(\vec{u})(\vec{\varphi}) = \int_{\mathbf{R}^n} \vec{f}(x, \vec{u}) \cdot \vec{\varphi}\, dx.$$

We now present our existence result.

Theorem 3. *If* (4.1) *and* 4.2–4.4 *hold, then for any parameter* $\lambda > 0$, *system* (1) *has at least one positive decaying solution with* $\lambda_1 = \cdots = \lambda_M = \lambda$.

Proof. The proof of the theorem is a verification of the conditions of the mountain pass theorem of Ambrosetti and Rabinowitz (see, for example, [**24**]). The fact that $F(x, \vec{0}) = 0$ implies that $J(\vec{0}) = 0$. Next we observe that

$$\begin{aligned} J(\vec{u}) &= (1/2)||\vec{u}||_E^2 - \lambda \int_{\mathbf{R}^n} F(x, \vec{u})\, dx \\ &\geq (1/2)||\vec{u}||_E^2 - \lambda C \int_{\mathbf{R}^n} g|\vec{u}|^{\gamma+1}\, dx \\ &\geq (1/2)||\vec{u}||_E^2 - \lambda C ||g||_{L^{p_0}} ||\vec{u}||_E^{\gamma+1} \\ &= (1/2 - \lambda C||g||_{L^{p_0}} ||\vec{u}||_E^{\gamma-1})||\vec{u}||_E^2. \end{aligned}$$

Choosing $\rho > 0$ sufficiently small, we can find $\alpha > 0$ such that $||\vec{u}||_E = \rho$ implies $J(\vec{u}) \geq \alpha$.

Let w denote the function in condition 4.4, and observe that for some constant K,

$$J(\beta w\vec{1}) \leq K\beta^2 - \lambda C(w)\beta^{\theta_1} + \lambda K(w).$$

From this inequality, we conclude that $J(\beta\vec{w}) < 0$ for β large enough. Finally, Lemma 5 shows that $J \in C^1(E, \mathbf{R})$, and J satisfies the (P.S.) condition as a result of the compactness of $G'(\cdot)$, [**24**], and the fact that

$$J'(\vec{u})(\vec{\varphi}) = \langle \vec{u}, \vec{\varphi}\rangle_E - \lambda \int_{\mathbf{R}^n} \vec{f}(x, \vec{u}) \cdot \vec{\varphi}\, dx \quad \text{for } \vec{\varphi} \in E.$$

Indeed, it suffices to show that if $|J(\vec{u}_m)| \leq C$ and $|J'(\vec{u}_m)| \to 0$ as $m \to \infty$ then $\{\vec{u}_m\}$ is a bounded sequence, but this is immediate since, for m large,

$$\begin{aligned} C + ||\vec{u}_m||_E & \\ &\geq J(\vec{u}_m) - \frac{\theta}{M} J'(\vec{u}_m)(\vec{u}_m) \\ &= \left(\frac{1}{2} - \frac{\theta}{M}\right) ||\vec{u}_m||_E^2 - \int_{\mathbf{R}^n} \left[F(x, \vec{u}_m) - \frac{\theta}{M} \sum_i f^i(x, \vec{u}_m) u_m^i\right] \\ &\geq \left(\frac{1}{2} - \frac{\theta}{M}\right) ||\vec{u}_m||_E^2. \end{aligned}$$

We conclude that $J(\cdot)$ has a critical point, say $\vec{u}$. From our assumptions, it is seen that $\vec{u} \geq \vec{0}$ almost everywhere, $\vec{u} \not\equiv 0$, since $\vec{f}(x, \vec{u}) = \vec{f}(x, \vec{u}^+)$. To finally show that $u^i > 0$ for $i = 1, \dots, M$, we observe that our problem is irreducible by the remark following 4.3. □

5. Remarks and examples. We first observe that somewhat similar results hold for the equation

$$-\Delta \vec{u} + \vec{u} = \lambda \vec{f}(x, \vec{u}). \tag{7}$$

Indeed, as is well known, the linear term is of help in the considerations and exponential decay of $\vec{u}$ is obtained. In this case the space E is replaced by $H^{1,2}$ and conditions 2.1, 2.2, and 2.4 are kept, while in conditions 2.3 and 3.1, the estimates on which the various compactness and continuity arguments are made can be replaced by the results of Berger and Schechter [**5**]. We recall that a solution in $H^{1,2}$ must decay exponentially by another result of Egnell [**10**, Theorem 5] (see also [**12**]). Analogous remarks apply for the variational situation considered in Section 4.

To be specific, we illustrate the above remarks by considering the analogue of Theorem 1 for this case. We first recall the following definitions, [**5**].

$$\begin{aligned} M_{\alpha,1}(w) &= \sup_{x \in \mathbf{R}^n} \int_{|y|<1} |w(x-y)|\, |y|^{\alpha-n}\, dy \\ M_{\alpha,1}(w, \Omega) &= M_{\alpha,1}(w|_\Omega) \end{aligned}$$

with $w|_\Omega(x) = w(x)$ if $x \in \Omega$ and equals zero otherwise. We have

Theorem 1′. *Assume conditions* 2.1, 2.2 *and* 2.4 *hold, but assume that* $C(x)$ *in* 2.3 *is in* L^∞ *and*

$$M_{\alpha,1}(C, \mathbf{R}^n - B_R) \to 0 \quad \text{as } R \to \infty$$

for some $0 < \alpha < n - ((\gamma+1)/2)(n-2)$ *where* B_R *denotes the ball of radius* R. *Then* (7) *has a positive solution* $\vec{u}$ *decaying exponentially at* ∞.

Proof. Monotonicity and boundedness in L^∞ of any positive solution $\vec{u}$ follow exactly as in Theorem 1 by means of the results in [**12**]. Since $M_{\alpha,1}(C, \mathbf{R}^n) < \infty$ and $0 < \alpha < 2$, then Lemma 3 also holds by the results of Berger and Schechter [**5**]. Finally, $M_{\alpha,1}(C, \mathbf{R}^n - B_R) \to 0$ implies continuity and compactness in Theorem 1 (see, e.g., [**3**]) and the result follows by degree theory.

Note that

$$\int_{|y|<1} C(x-y)|y|^{\alpha-n} \leq K||C||_{L^{p_0}(B_1(x))}$$

for some constant K if $p_0 > n/\alpha$. It thus suffices to assume that $||C||_{L^{p_0}(B_1(x))} \to 0$ as $|x| \to \infty$ for some $p_0 > 2n/[2n - ((\gamma+1)/2)(n-2)]$. □

We observe that this theorem extends results of Chaljub-Simon and Volkmann [**8**], even in the scalar case since we do not require exponential decay of $C(x)$.

We conclude with the following illustrative examples.

Example 1. Assume that (1′) is

$$-\Delta u_1 = p(x)(u_1 + u_2)^\gamma$$
$$-\Delta u_2 = q(x)(u_1 + u_2)^\gamma$$

with $0 < p, q$ radial near infinity and $p', q' \leq 0$ there. If $1 < \gamma < (n+2)/(n-2)$, then 2.1 and 2.2 hold, and since irreducibility is obvious, 2.4 follows. Finally, if we assume that $p, q \in L^{n/2-\varepsilon}$, $p, q \leq K|x|^{-\beta}$ with $\beta + \gamma(n-2) > (n+1)$, then 2.3 holds and we conclude the existence of positive decaying solutions.

Observe that p, q need not be radial at ∞ for this result to hold. For a simple example, let $0 < K_0 \leq h(t) \leq K_1$ be a smooth function with bounded derivative. Then the result holds if $p(x) = \exp(-|x|^2)h(x_1)$ in the above system.

Example 2. To illustrate the perturbation arguments, let $p(x), q(x)$ be as above. We have existence for

$$\begin{aligned} -\Delta u_1 &= p(x)[u_1^\gamma + \varepsilon u_1^{\gamma_1} u_2^{\gamma_2}] \\ -\Delta u_2 &= q(x)[u_2^\gamma + \varepsilon u_1^{\gamma_1} u_2^{\gamma_2}] \end{aligned}$$

if ε is small enough and $1 < \gamma_1 + \gamma_2 < \gamma$. Setting $\vec{u} = \alpha \vec{v}$ then gives

$$\begin{aligned} -\Delta v_1 &= \lambda p(x)[v_1^\gamma + v_1^{\gamma_1} v_2^{\gamma_2}] \\ -\Delta v_2 &= \lambda q(x)[v_2^\gamma + v_1^{\gamma_1} v_2^{\gamma_2}] \end{aligned}$$

if we choose $\alpha^{\gamma-(\gamma_1+\gamma_2)} = \varepsilon$ and $\lambda = \alpha^{\gamma-1}$. Note that this system also has the nonnegative nontrivial solution $(v_1', 0)$ with $-\Delta v_1' = \lambda p(x)(v_1')^\gamma$. We are not aware of other results for such systems. For another example, assume that $p(x)$ is as above but that now $0 < q(x)$, $z(x) \in L^{p_0} \cap L^\infty$, with $p_0 = 2n/[(n+2) - (\gamma_1 + \gamma_2)(n-2)]$. We then have existence, by the same method, of a positive solution for small $\lambda > 0$ of

$$\begin{aligned} -\Delta v_1 &= \lambda[p(x)v_1^\gamma + q(x)v_1^{\gamma_1} v_2^{\gamma_2}] \\ -\Delta v_2 &= \lambda[p(x)v_2^\gamma + z(x)v_1^{\gamma_1} v_2^{\gamma_2}]. \end{aligned}$$

Notice that we do not require monotonicity nor decay of q, z at ∞.

Example 3. Consider the system

$$\left.\begin{aligned} -\Delta u &= \lambda[\alpha_3 h(x) u^{\alpha_1} v^{\alpha_2} + \beta_3 k(x) u^{\beta_1} v^{\beta_2}] \\ -\Delta v &= \lambda[\alpha_2 h(x) u^{\alpha_3} v^{\alpha_4} + \beta_2 k(x) u^{\beta_3} v^{\beta_4}] \end{aligned}\right. \quad \text{in } \mathbf{R}^n.$$

Assume that $\alpha_i \geq 0$ for $i = 1, \dots, 4$, $\alpha_4 + 1 = \alpha_2$, $\alpha_1 + 1 = \alpha_3$, $1 < \alpha_1 + \alpha_2 < (n+2)/(n-2)$ and $0 < h(x) = O(|x|^{-a})$, $a > 0$ with $\max\{(n+2-2a)/(n-2), 1\} < \alpha_1 + \alpha_2$, and k, β_i satisfy the same conditions. Observe that this system is variational and $h \in L^{p_0}(\mathbf{R}^n)$, $p_0 = 2n/(2n - (\alpha_1 + \alpha_2 + 1)(n-2))$ since $ap_0 > n$, and that a similar remark holds for k. By Theorem 3, this system has a positive decaying solution $(u(x), v(x))_\lambda$ for any parameter $\lambda > 0$. Systems of this type have been considered in [**23**] under a variety of technical conditions. The authors obtained the existence of positive decaying solution only for some $\lambda = \tilde{\lambda}$ and not for any λ, and for $n \leq 5$.

Example 4. Consider the scalar equation

$$-\Delta u + 2\lambda \sum b_j(x) D_j u = p(x) u^\gamma$$

with γ, p as in the earlier examples. Observe that, while this problem is scalar, it is not variational if $\lambda \neq 0$ and the usual approaches based on mountain pass arguments fail. Our methods, however, can be used to show the existence of a solution for small $\lambda > 0$. Indeed, we may rewrite the equation as $u - S(u) + \lambda T(u) = 0$ with $S(u) = (-\Delta)^{-1}(p(x)(u^+)^\gamma)$, $T(u) = (-\Delta)^{-1}(\sum b_j D_j u)$. The procedures of Section 2, in particular Lemma 4 applied to the scalar case, show that there exists $A \subset E$ such that $\deg(u - S(u), A, 0) \neq 0$. Observe that T is linear, and, thus, if we show it is compact $E \to E$, then it will also be continuous. Assume that this is done for the moment, and let T, N be exactly as in Theorem 2. If λ is small enough, then $u - S(u) - t\lambda T(u) \neq 0$ on ∂N for $0 \leq t \leq 1$, and we conclude that $\deg(u - S(u) - \lambda T(u), N, 0) \neq 0$. To conclude, we need to give sufficient conditions for T to be compact and, for this, let $u_n \to u$ weakly in E and assume first that $\vec{b}$ has compact support. Setting $-\Delta v_n = \sum b_j D_j u_n$ and integrating by parts gives

$$\begin{aligned} ||v_n - v_m||_E^2 = &- \int \sum b_j D_j (v_n - v_m)(u_n - u_m) \\ &- \int \operatorname{div}(\vec{b})(v_n - v_m)(u_n - u_m). \end{aligned}$$

Since we assume that $\vec{b} \in C_0^1(\mathbf{R}^n)$, then that T is compact in this case follows from the observation that, given any ball $B \subset R^n$, then E is continuously embedded in $H^1(B)$ [**3**] and $H^1(B)$ is compactly embedded in $L^q(B)$ for $q < 2n/(n-2)$. In the more general case, let $g_j^R = b_j \varphi(x/R)$ where φ is a smooth, nonnegative radial function,

$\varphi(t) \equiv 1$ if $t \leq 1$, $\equiv 0$ if $t \geq 2$. Setting $T_R(u) = (-\Delta)^{-1}(\sum g_j^R D_j u)$ gives

$$||T_R(u) - T(u)||_E \leq K||\vec{b}(1 - \varphi(x/R)||_n \cdot ||u||_E.$$

Since T_R is compact, the result follows if we assume that $\vec{b} \in L^n$.

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