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THE SEQUENCE x/n AND ITS SUBSEQUENCES

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1. Introduction. We begin by mentioning two problems which seem to have no relation to each other.

Problem 1. *A positive integer n is said to be sparsely totient if*

$$\phi(m) > \phi(n)$$

for all $m > n$, where ϕ is Euler's function. Find the smallest number λ such that, for all sparsely totient numbers n, we have

$$\max_{p|n} p = O_\varepsilon((\log n)^{\lambda+\varepsilon}). \tag{1.1}$$

Here and subsequently, p denotes a prime number and ε an arbitrary positive number.

Now let K be an algebraic number field with degree d; the *size* of an algebraic integer θ in K is the maximum of the set of absolute values of the d conjugates of θ. Let $\alpha_1, \dots, \alpha_n$ be $n \geq 3$ distinct algebraic integers in K and μ a nonzero algebraic integer in K.

Problem 2. *Give a bound for the size of solutions X, Y of the Thue equation*

$$(X - \alpha_1 Y) \cdots (X - \alpha_2 Y) = \mu$$

in algebraic integers X, Y.

Such a bound can be expressed in terms of d and the heights of $\alpha_1, \dots, \alpha_n$, μ and some algebraic integer generating K [**1**, Section 4.2].

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Surprisingly, both problems are linked to the behavior of the sequences of the form

$$x/n \tag{1.2}$$

for given real x, and natural numbers n in a suitable interval.

The authors became aware of Problem 1 in 1983, when D.W. Masser posed the following problem in a letter to G. H.:

For what positive arithmetic functions $f(N)$ is it true that

$$\min_{1\le n\le N} \left\| \frac{x}{n} \right\| < f(N)$$

for *all* real x?

Here $||\dots||$ denotes distance from the nearest integer, while $\{\dots\}$ will denote fractional part. No satisfactory solution has been found. One can show by examples that $f(N)$ has to be *at least* $c_1 N^{-1/2}$. On the other hand, as we shall explain later in this section, we may take

$$f(N) = c_2(\log N)^{-1}.$$

(By $c_1, c_2, \dots$ we denote positive, effectively computable constants, absolute unless otherwise indicated.)

The problem is easier if we restrict the size of x in relation to N, say

$$|x| \le N^{c_3};$$

van der Corput's method of exponential sums comes into play and one can replace $c_2(\log N)^{-1}$ by $c_4 N^{-c_5}$ (c_4, c_5 depend on c_3). See, e.g., Graham and Kolesnik [**18**] for van der Corput's method. It turned out that Masser, in joint work with P. Shiu [**30**] on sparsely totient numbers, needed to solve a Diophantine inequality of the form

$$1 - \frac{x}{16v^2} < \left\{ \frac{x}{p} \right\} < 1 \tag{1.3}$$

with p prime, $2v < p < 3v$, for a given x and v related by

$$v^{c_6} \le x \le v^2.$$

The significance of the constant c_6 in connection with Problem 1 will appear in Section 4. Thus Masser and Shiu were concerned with a *subsequence* of (1.1).

An inequality like (1.3) is best approached via the study of sums

$$\sum_{2v<n<3v} \Lambda(n)e(x/n)$$

where Λ is von Mangoldt's function. These sums can be decomposed into 'bilinear' sums

$$S(x) = \sum_r a_r \sum_s b_s e\left(\frac{x}{rs}\right) \tag{1.4}$$

where a_r, b_s have bounded modulus and r, s run over integers in independent intervals, subject to $2v < rs < 3v$. See, for example, Section 24 in [**10**] ('Vaughan's identity'). Since Vaughan's identity demands special attention to the case

$$\sum_r a_r \sum_s e\left(\frac{x}{rs}\right) \tag{1.5}$$

where the coefficients in the inner sum are 1, this immediately links back to the study of sequences (1.2).

By using results about sums (1.4), (1.5), Harman [**23**] obtained (1.1) with $\lambda = 2 - 8/65$. In a forthcoming paper in Ann. Fac. Sci. Toulouse, the authors reduce the value of λ to 37/20. An outline of the method is presented in Section 4: Vaughan's identity is superseded here by the sieve approach of Harman [**22**].

In a recent paper, E. Bombieri [**7**] used the 'Thue-Siegel principle' to obtain results on Problem 2. We shall not quote the actual results, but rather mention an auxiliary result (Theorem 2 of [**7**]):

Theorem 1. *Let K be a number field of degree d over the rational field $\mathbf{Q}$, let Γ be a finitely generated subgroup of $K^\times$, and let $\xi_1, \dots, \xi_t$ be a set of generators of $\Gamma/\mathrm{tors}\,(\Gamma)$.*

Let $A \in K^\times$, let v be an archimidean absolute value of K, and let $\xi \in \Gamma$ and $\kappa > 0$ be such that

$$0 < |1 - A\xi|_v \le H(A\xi)^{-\kappa}.$$

Let us define $Q = 1$ *if* $t = 0$ *and*

$$Q = (e^{115d/\kappa^2} t)^{t+1} \prod_{i=1}^{t} h(\xi_i)$$

if $t \geq 1$. *Then we have*

$$h(A\xi) \leq \max(Qh(A), [Q]!). \tag{1.6}$$

The presence of the factorial is an unpleasant drawback of (1.6). It originates in the following simple result (Lemma 4 of [**7**]):

Proposition 1. *Let* x_i, $i = 1, \dots, t$, *be rational integers, let* λ_i, $i = 1, \dots, t$ *be positive real numbers with* $\lambda_1 \dots \lambda_t = 1$, *and let* M *and* Q *be positive integers with* $Q > \max \lambda_i^t$. *Then there are a natural number* r *and rational integers* p_i, $i = 1, \dots, t$, *such that*

$$|x_i - rp_i| \leq r\lambda_i Q^{-1/t}, \qquad i = 1, \dots, t \tag{1.7}$$

and

$$(Q-1)!M \leq r \leq Q!M.$$

Proof. Let $\phi_i = x_i/(Q!M)$. By a variant of Dirichlet's theorem [**39**, Theorem 1A], we have

$$|\phi_i q - p_i| \leq \lambda_i Q^{-1/t}$$

for some natural number $q \leq Q$ and integers $p_1, \dots, p_t$. Let $r = Q!M/q$. Then r is an integer and

$$\left| \frac{x_i}{r} - p_i \right| \leq \lambda_i Q^{-1/t}.$$

The proposition follows at once. □

We make two simple observations.

(i) $Q!$ can be replaced by the least common multiple of $1, \dots, Q$, which is, of course,

$$\exp((1+o(1))Q).$$

(ii) The x_i can be real numbers without affecting the proof. Accordingly, taking $t = 1$, $M = 1$, $Q = [(\log N)/2] + 1$, we obtain

$$\min_{1 \le r \le N} \left\| \frac{x}{r} \right\| < 2(\log N)^{-1}$$

for R exceeding an easily computed constant c_7. Hence we can indeed take $f(N) = c_2(\log N)^{-1}$ in Masser's problem.

Bombieri asked one of the authors (R.C.B.) whether he could strengthen the above lemma. For example, one might seek to solve (1.7) in a range

$$R \le r \le 2QR \tag{1.8}$$

where R is a given natural number. (If $c_8 \le Q \le (1/2)\log R$, the range (1.8) is attainable via the argument used in Proposition 1.) However, G. H. had essentially given a counterexample in 1983, in demonstrating that $f(N)$ must be at least $c_1 N^{-1/2}$. See [**3**] for details of such an example; it is still conceivable that one could get a range

$$R \le r \le c_9 Q^2 R$$

in place of (1.8).

To obtain a result that would be serviceable in the context of Problem 2, the authors restricted the size of $\mathbf{x}$ and were led to the following theorem.

Theorem 2. *Let R be a natural number, $\mathbf{x} \in \mathbf{R}^t$ and*

$$|\mathbf{x}| \le R^{c_{10}}.$$

Let Q be a natural number,

$$c_{11} \le Q \le R^{c_{12}} \tag{1.9}$$

and let $\psi_1, \dots, \psi_t$ *be positive numbers with*

$$\psi_i \le c_{13}(\log Q)^{-t}, \qquad i = 1, \dots, t \tag{1.10}$$

$$\psi_1 \cdots \psi_t = Q^{-1}.$$

Then

$$\left\| \frac{x_i}{r} \right\| \le \psi_i, \qquad i = 1, \dots, t,$$

for some integer r *satisfying* (1.8). *Here* c_{12} *depends on* c_{10}*;* c_{11} *and* c_{13} *depend on* c_{10}, t.

We give the proof for 'large' $\mathbf{x}$ in Section 3; it turns out that a suitable application of Dirichlet's theorem takes care of smaller $\mathbf{x}$, see [**3**]. Bombieri informs us that Theorem 2 can indeed by applied to strengthen Theorem 1, and consequently to give new results on Thue equations; but this is not yet written up.

2. Sequences x/n: Other results and applications. The earliest occurrence of the sequence x/n in the history of mathematics is perhaps in Dirichlet's work on the error term in the divisor function $d(n)$. Let

$$\Delta(x) = \sum_{n \le x} d(n) - x(\log x + 2\gamma - 1).$$

Dirichlet showed that

$$\Delta(x) = -2 \sum_{n \le x^{1/2}} \psi\left(\frac{x}{n}\right) + O(1);$$

see [**18**, p. 40]. Here

$$\psi(x) = \{x\} - 1/2.$$

It follows at once that

$$\Delta(x) = O(x^{1/2}).$$

To go further, one reduces the problem to the study of exponential sums

$$S(hx) = \sum_{m \sim M} e\left(\frac{xh}{m}\right)$$

(h integer). Much effort has gone into improving Dirichlet's exponent by this approach and, in particular, trying to sum nontrivially over h. See [**18**] for further discussion and references. A slight variant of the technique leads to a similar analysis of the error term $R(x)$ in the circle problem,

$$R(x) = \sum_{(a,b);a^2+b^2\le x} 1 - \pi x;$$

see [**18**, p. 42].

The exponential sum (1.4), and in particular (1.5), is also needed when we look for almost primes in short intervals. Here we sieve

$$\mathcal{A} = \{n : x - x^\theta < n \le x\}.$$

It is crucial to have good bounds on average for the remainders R_d defined by

$$R_d = \sum_{\substack{n\in\mathcal{A} \\ n\equiv 0 \pmod d}} 1 - \frac{x^\theta}{d}.$$

Since

$$R_d = \psi\left(\frac{x}{d}\right) - \psi\left(\frac{x-x^\theta}{d}\right),$$

it is apparent that $S(x)$, and indeed $S(hx)$ (averaged over a range of integers h) will be an object of study for this problem. Chen [**8**] showed that one can find a number with at most two prime factors in $\mathcal{A}$ if $\theta = 1/2$ and x is large enough. Later he was able to take $\theta = 0.477$ [**9**]. The value of θ has subsequently been reduced by Halberstam, Heath-Brown and Richert [**20**], Iwaniec and Laborde [**26**], Halberstam and Richert [**21**] and Fouvry [**14**]. The most recent published result is $\theta = 0.44$ given by Wu Jie [**41**]. Besides exponential sums one again needs sieve ideas, this time the Rosser-Iwaniec sieve [**25**] and weighted sieve technique.

Wu also took up another nice problem [**42**] in which the exponential sums (1.4) had been used earlier (by Bantle and Grupp [**6**]). The problem originates with Erdös [**12**], who proved that there is a number $\theta < 1$ with the following property:

Let $\mathcal{B}$ be a sequence of natural numbers

$$b_1 < b_2 < b_3 < \cdots$$

which are coprime in pairs and have convergent reciprocal sum

$$\sum_{i=1}^{\infty} \frac{1}{b_i}.$$

For $x \geq x_0(\theta, \mathcal{B})$, there is an integer in $(x - x^\theta, x]$ which has no divisor in $\mathcal{B}$.

Szemerédi [**40**] was able to take any $\theta > 1/2$; Bantle and Grupp had $\theta > 9/20$; Wu's condition is $\theta > 17/41$. In contrast to Bantle and Grupp, Wu was able to use exponential sums (1.5), instead of (1.4) with 'unknown' a_r, b_s. Sieve ideas are important in this work too—Wu uses a 'fundamental lemma' (Friedlander and Iwaniec [**16**]). The simplest example ($\mathcal{B}$ = squares of the primes) yields a square-free number in $(x - x^\theta, x]$. Here, however, any $\theta > 1/5$ is admissible (Filaseta and Trifonov [**13**]; the technique is combinatorial (no exponential sums)).

Another approximation to the classical conjecture that $I_x = (x - x^{1/2}, x]$ contains a prime number, if x is sufficiently large, was given by Ramachandra [**31**]. He found a constant $\theta > 1/2$ such that I_x contains an integer having a prime factor $> x^\theta$, for large x. Ramachandra's first value of θ was 15/26; later he got 5/8 [**32**]. Several authors [**28, 17, 2, 27, 29**] have worked on this problem, and we now know that $\theta = 0.732$ is admissible (Baker and Harman [**4**]). Once again, sieve methods are used in conjunction with exponential sums (1.4) and (1.5); this time, both the Rosser-Iwaniec sieve and the alternative sieve method of Harman [**22**] are helpful.

A well-known conjecture of Erdös states that the binomial coefficient $\binom{2n}{n}$ is not square free for any $n > 4$. Sárközy [**36**] converted this into a problem about exponential sums and was able to prove Erdös's conjecture for sufficiently large n. Recently Granville and Ramaré [**19**] proved Erdös's conjecture for $n \geq 2^{1617}$ via an effective bound for the sum.

$$S(y, y') = \sum_{y < n \leq y'} \Lambda(n) e\left(\frac{x}{n}\right)$$

where $y \leq (1/5)x^{3/5}$ and $y \leq y' \leq 2y$:

$$|S(y, y')| \leq \frac{50}{3} y \left(\frac{x}{y^{(k+3)/2}}\right)^{1/(4(2^k-1))} (\log 16y)^{11/4}$$

for any positive integer k. Vaughan's identity is used and once again generates exponential sums (1.4) and (1.5). A less precise estimate of this kind had already been given in [**11**]. As for $4 < n < 2^{1617}$, this is finished off by Granville and Ramaré via computer verification; only $n = 2^k$ requires effort.

One can, of course, study the sequence x/n in its own right; this is the subject of a paper by Isbell and Schanuel [**24**] and an interesting series of papers by Saffari and Vaughan [**33, 34, 35**]. We quote a couple of results from [**34**].

Suppose that $y = y(x)$ is increasing, $y = o(x)$ and $y \to \infty$ as $x \to \infty$. Suppose further that $0 < \alpha < 1$ and that

$$\theta_{x,y}(\alpha) := y^{-1} \sum_{\substack{n \le y \\ \{x/n\} < \alpha}} 1$$

has a limit as $x \to \infty$. Then the limit is α [**34**, Theorem 2].

Let G be Dickman's function, so that

$$G(u) = 1, \qquad 0 \le u \le 1,$$

G is continuous, and

$$(uG(u))' = -G(u-1), \qquad u > 1.$$

G is monotone decreasing and

$$0 < G(u) \le \Gamma(u+1)^{-1}.$$

We have

$$\limsup_{x \to \infty} \theta_{x,y}(\alpha) \ge G(u)$$

for $0 < \alpha < 1$, $y = (\log x)^u$. Consequently, if $0 < \alpha < G(u)$, $\theta_{x,y}(\alpha)$ does not have a limit as $x \to \infty$ [**34**, Theorem 4].

The following result of Dyer [**11**] answers a question raised in [**34**].

As $x \to \infty$, we have

$$\sup_{A<B<2A} \sup_{\alpha \in [0,1)} \left| \sum_{\substack{\{x/n\} \in [0,\alpha) \\ A \le n \le B}} 1 - (B-A)\alpha \right| = o(A)$$

provided that

$$\exp\left(\frac{(1+\varepsilon)\log x}{\log\log x}\right) \le A = o(x).$$

3. Proof of Theorem 2 for 'large' x. By 'large' $\mathbf{x}$ we shall mean that

$$\max_i \psi_i^{-1}|x_i| \ge c_{13}RQ(\log Q)^t.$$

Our method is adapted from W. Schmidt's beautiful work [**38**] on inequalities

$$||\alpha_i n^2|| < \psi_i, \qquad i = 1, \dots, t,$$

where n is to be found in a given interval $1 \le n \le N$. (See Schäffer [**37**] for recent progress in the case $t = 2$.) We write down a theorem about lattices which implies Theorem 2. The unit ball in $\mathbf{R}^h$ is written $\mathbf{B}_0$ (or, in case $t = 1$, B_0).

Theorem 3. *Let Λ be a lattice in $\mathbf{R}^t$ with determinant $d(\Lambda) = Q$,*

$$\Lambda \cap \mathbf{B}_0 = \{\mathbf{0}\}. \tag{3.1}$$

Let R and Q be natural numbers satisfying (1.9). *Let $\mathbf{y} \in \mathbf{R}^t$,*

$$c_{13}RQ(\log Q)^t \le |\mathbf{y}| \le R^{c_{10}+1} \tag{3.2}$$

where $c_{13} = c_{13}(c_{10}, t)$ is sufficiently large. Then

$$r^{-1}\mathbf{y} \in \Lambda + \mathbf{B}_0 \tag{3.3}$$

for some integer r in $[R, c_{14}RQ(\log Q)^{t-1}\lambda^{-1}]$. Here

$$\lambda = \lambda(\Lambda) = \min\{|l| : l \in \Lambda, l \ne \mathbf{0}\}.$$

Naturally, it took some experimentation to arrive at this format—a key point is that 'small' vectors $\mathbf{x}$ in Theorem 2 require a separate treatment. Our proof of Theorem 3, by induction on t, works with far greater efficiency because no orthogonal lattice basis is required to exist.

To deduce Theorem 2 for 'large' $\mathbf{x}$, let Λ be generated by $\psi_j^{-1}\mathbf{e}_j$, $j = 1, \dots, t$, where $\mathbf{e}_1, \mathbf{e}_2, \dots$ is the standard basis. Let $(y_1, \dots, y_t) = (x_1\psi_1^{-1}, \dots, x_t\psi_t^{-1})$, and suppose (3.2) is satisfied. The integer r supplied by Theorem 3 satisfies

$$x_i\psi_i^{-1}r^{-1} \in \psi_i^{-1}\mathbf{Z} + B_0,$$

so

$$x_i/r \in \mathbf{Z} + \psi_i B_0$$

as required. As for the upper bound on r, (1.10) yields

$$r \leq c_{14}RQ(\log Q)^{t-1} \max_j \psi_j < 2RQ.$$

It remains to prove Theorem 3. Implied constants in the rest of the section depend at most on c_{10} and t.

Let us give the induction step from $t-1$ to t. (Having seen it, the reader will be able to fill in the case $t = 1$.) Suppose no suitable r exists. In particular, we cannot solve (3.3) with

$$R \leq r < 2R. \tag{3.4}$$

By Lemma 2 of [**3**] which employs a smooth auxiliary function which is Λ-periodic and vanishes outside $\Lambda + \mathbf{B}_0$, we are led to

$$\sum_{\substack{\mathbf{p}\in\Pi \\ 0<|\mathbf{p}|<t}} \left| \sum_r e\left(\frac{\mathbf{py}}{r}\right) \right| \gg R. \tag{3.5}$$

Here $\mathbf{py}$ is inner product and Π the lattice polar to Λ; the summation condition (3.4) is left implicit.

After dividing up the sum over $\mathbf{p}$ in standard fashion, we find a number B such that

$$B < \left| \sum_r e\left(\frac{\mathbf{py}}{r}\right) \right| \leq 2B \tag{3.6}$$

for all $\mathbf{p}$ in a set $\mathcal{B}$ counted within (3.5), of cardinality

$$|\mathcal{B}| \gg RB^{-1}(\log Q)^{-1};$$

moreover,

$$B \geq RQ^{-1-\varepsilon}.$$

By (1.9), van der Corput's exponential sum estimates, and the lower bound in (3.6), we are driven to conclude that

$$D = \max_{r\in[R,2R)} \left| \frac{d}{dr}\left(\frac{\mathbf{py}}{r}\right)\right|$$

must be $\leq 1/2$, and accordingly [**18**, Lemmata 3.1, 3.5]

$$\sum_r e\left(\frac{\mathbf{py}}{r}\right) \ll D^{-1}$$

for $\mathbf{p} \in \mathcal{B}$. This leads to

$$|\mathbf{py}| \ll R|\mathcal{B}|\log Q$$

and a box principle yields a $\mathbf{p}$ in Π having $|\mathbf{p}| < 2t$,

$$|\mathbf{py}| \ll R\log Q. \tag{3.7}$$

We may suppose without loss of generality that $\mathbf{p}$ is primitive. (Notice that we have again used exponential sums $\sum_r e(x/r)$.) Now

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{py}}{|\mathbf{p}|^2}\mathbf{p}$$

lies in the $(t-1)$-dimensional orthogonal complement $\mathbf{p}^{\perp}$ of $\mathbf{p}$. For a suitably large c_{15}, let

$$\Lambda' = (c_{15}\Lambda) \cap \mathbf{p}^{\perp}, \qquad Q' = \det \Lambda'.$$

This determinant is shown in [**38**] to be

$$Q' = c_{15}^{t-1}Q|\mathbf{p}| \tag{3.8}$$

which in turn (as shown in [**38**]) is

$$\geq c_{15}^{t-1} c_{16}, \tag{3.9}$$

and for suitable choice of c_{15} this is $\geq c_{11}(K, t-1)$.

Suppose, as we may, that $c_{15} \geq 2$. Then

$$\Lambda' \cap \mathbf{B}_0 = 0$$

and

$$\lambda(\Lambda') \geq \lambda(\Lambda). \tag{3.10}$$

We now apply the $(t-1)$-dimensional case of the Proposition, working in $\mathbf{p}^{\perp}$ rather than $\mathbf{R}^{t-1}$. In place of $\mathbf{y}$ and Λ we take $c_{15}\mathbf{z}$, Λ', and in place of R,

$$R' = c_{17} R |\mathbf{p}|^{-1} \log Q \tag{3.11}$$

where $c_{17} > 2t$. We have

$$R' \geq R \log Q, \qquad c_{15}|\mathbf{z}| \leq c_{15}|\mathbf{y}| \leq (R')^{K+1}. \tag{3.12}$$

If c_{17} is suitably chosen, we obtain an inequality we need below,

$$\frac{|\mathbf{py}|}{|\mathbf{p}|R'} < \frac{1}{2}, \tag{3.13}$$

from (3.7) and (3.11). Moreover, $Q' \leq (R')^c$ because

$$Q' \ll Q \leq R^c \leq (R')^c (\log Q)^{-c}$$

from (3.8) and (3.11).

The remaining condition that we have to verify is the appropriate lower bound for $c_{15}|\mathbf{z}|$. We have

$$|\mathbf{z}| \geq |\mathbf{y}| - \frac{|\mathbf{py}|}{|\mathbf{p}|} \geq \frac{1}{2} c_{13}(c_{10}, t) RQ (\log Q)^t$$

from (3.2), (3.7), (3.9). It can be seen that

$$c_{15}|\mathbf{z}| \geq c_{13}(K, t-1)c_{17}R|\mathbf{p}|^{-1}\log Q \cdot c_{15}^{t-1}Q|\mathbf{p}|(\log c_{15}^{t-1}|\mathbf{p}|Q)^{t-1}$$

for a suitable choice of $c_{13}(c_{10}, t)$. The last expression is, of course,

$$c_{13}(K, t-1)R'Q'(\log Q')^{t-1}.$$

Accordingly, there is an integer r,

$$R' \leq r \leq c_{14}(c_{10}, t-1)R'Q'(\log Q')^{t-2}/\lambda(\Lambda') \tag{3.14}$$

such that

$$r^{-1}c_{15}\mathbf{z} \in \Lambda' + \mathbf{B}_0.$$

In particular,

$$r^{-1}\mathbf{z} \in \Lambda + \frac{1}{2}\mathbf{B}_0.$$

We can work back from $\mathbf{z}$ to $\mathbf{y}$, since

$$|r^{-1}(\mathbf{y} - \mathbf{z})| \leq \frac{|\mathbf{p}\mathbf{y}|}{|\mathbf{p}|R'} < \frac{1}{2}$$

from (3.13). Thus,

$$r^{-1}\mathbf{y} \in \Lambda + \mathbf{B}_0.$$

Now, for a suitable choice of $c_{14}(c_{10}, t)$ we have

$$R \leq r \leq c_{14}(c_{10}, t)RQ(\log Q)^{t-1}/\lambda(\Lambda).$$

This is a consequence of (3.14), (3.11), (3.8) and (3.10). The existence of such an integer r contradicts our hypothesis, and we have completed the induction step.

4. Prime factors of sparsely totient numbers. We shall sketch a proof of the following result: $P_j(n)$ denotes the jth largest prime factor of n.

Theorem 4. *Let n be a sparsely totient number. Then*

$$P_1(n) < c_{16}(\log n)^{37/20}. \tag{4.1}$$

The key to the improvement of [**23**] is work of Fouvry and Iwaniec [**15**] on exponential sums

$$\sum_{m}\sum_{m_1}\sum_{m_2} a_m b_{m_1,m_2} e(Am^{\alpha} m_1^{\alpha_1} m_2^{\alpha_2}) \tag{4.2}$$

where $e(\theta) = e^{2\pi i\theta}$. The sums we need here are of the particular form

$$\sum_{h}\sum_{s}\sum_{t} a_s b_t c_h e\left(\frac{hx}{st}\right). \tag{4.3}$$

It is well known that there are devices for estimating this sum more efficiently than (4.3); see, e.g., Iwaniec and Laborde [**26**], Baker [**2**], Fouvry and Iwaniec [**15**], Wu [**41**], Liu [**29**] and Baker and Harman [**5**]. These devices would, perhaps surprisingly, make no difference to the final result if we employed them here.

It is interesting to note that for $j \geq 2$ and $n \geq n_0(j,\varepsilon)$, a sparsely totient number n satisfies

$$P_j(n) \leq \left(\frac{j}{j-1} + \varepsilon\right) \log n; \tag{4.4}$$

see [**23**]. We shall make use of (4.4) in proving (4.1).

Proposition 2. *For all x, v sufficiently large and*

$$v^{37/20} \leq x \leq v^2,$$

there are

$$\gg \frac{x}{v \log x}$$

solutions in primes p to

$$1 - \frac{x}{16v^2} < \left\{\frac{x}{p}\right\} < 1, \quad \text{with} \quad 2v < p < 3v. \tag{4.5}$$

Proposition 2 is proved by the sieve method developed by Harman [**22**] and Baker, Harman and Rivat [**5**]. We are able to use the same numerical work as in [**5**]; this saves a great deal of space. Sums (4.3) arise, as one would expect, in bounding the remainder terms of the sieve.

The deduction of Theorem 4 from Proposition 2 follows [**23**]. Suppose that n is a sparsely totient number and

$$P_1(n) \geq c_{16}(\log n)^{37/20},$$

so that n is large. From [**30**], we know that

$$P_1(n) < (\log n)^2.$$

Let $p_1 = P_1(n)$ and write $m = n/p_1$. We apply the Proposition with $x = p_1$, $v = \log n$. It follows that there are

$$\gg p_1/(v \log p_1)$$

solutions to (4.5). From (4.4), there are at most three primes between $2v$ and $3v$ which divide n. We deduce that (4.5) has a solution with $p \nmid n$. Let

$$r = [p_1/p] + 1.$$

Evidently $mrp > n$. We now use (4.5) to show that $\phi(mrp) < \phi(n)$. We have

$$\phi(mrp) \leq r\phi(m)p\left(1 - \frac{1}{p}\right) \leq \frac{rp}{p_1}\frac{(1-1/p)}{(1-1/p_1)}\phi(n). \tag{4.6}$$

Now

$$r - \frac{p_1}{p} = 1 - \left\{\frac{p_1}{p}\right\} < \frac{p_1}{16v^2} < \frac{9p_1}{16p^2}$$

from (4.5). Hence

$$\frac{rp}{p_1} < 1 + \frac{9}{16p}. \tag{4.7}$$

Combining (4.6) and (4.7),

$$\begin{aligned}\phi(mrp) &\leq \phi(n)\left(1 - \frac{1}{p} + O\left(\frac{1}{p_1}\right)\right)\left(1 + \frac{9}{16p}\right)\\ &\leq \phi(n)\left(1 - \frac{7}{16p} + O\left(\frac{1}{p^{37/20}}\right)\right).\end{aligned}$$

Since p is large, we have

$$\phi(mrp) < \phi(n)$$

which is absurd. Theorem 4 is proved. □

We now give a brief outline of the proof of Proposition 2. Let ε be a sufficiently small positive number and $\eta = \varepsilon^2$. Constants implied by '$\ll$,' '$\gg$' and '$O_\varepsilon(\ )$' will depend at most on ε. Constants implied by 'O' will be absolute. We use the abbreviation '$m \sim M$' for

$$M < m \leq 2M.$$

We write

$$\alpha = 3/20, \qquad \delta = x/(16v^2).$$

Let $\mathcal{B}$ be the set of integers in $(2v, 3v)$, and let $\mathcal{A}$ be the set of k in $\mathcal{B}$ for which

$$1 - \delta < \left\{ \frac{x}{k} \right\} < 1.$$

For $\mathcal{E} = \mathcal{A}$ or $\mathcal{B}$, we write

$$\mathcal{E}_d = \{k \in \mathcal{E} : d|k\},$$
$$S(\mathcal{E}, z) = |\{k \in \mathcal{E} : p|k \Rightarrow p \geq z\}|.$$

Then the number of primes in $\mathcal{A}$ is $S(\mathcal{A}, (3v)^{1/2})$. We prove that

$$S(\mathcal{A}, (3v)^{1/2}) > \frac{\delta v}{4 \log v}$$

which establishes Proposition 2.

We begin with the asymptotic formulae

$$\sum_{s \sim M} a_s |\mathcal{A}_s| = \delta v \sum_{s \sim M} \frac{a_s}{s} + O_\varepsilon(\delta v^{1-3\eta}) \tag{4.8}$$

$(M \leq v^{1-3\alpha-\varepsilon})$ and

$$\sum_{\substack{st \in \mathcal{A} \\ s \sim M, t \sim N}} a_s b_t = \delta v \sum_{\substack{st \in \mathcal{B} \\ s \sim M, t \sim N}} \frac{a_s b_t}{st} + O_\varepsilon(\delta v^{1-3\eta}) \tag{4.9}$$

(M in any of the intervals

$$[v^{\alpha+\varepsilon}, v^{1-5\alpha-\varepsilon}], \quad [v^{3\alpha+\varepsilon}, v^{1-3\alpha-\varepsilon}], \quad [v^{5\alpha+\varepsilon}, v^{1-\alpha-\varepsilon}]).$$

Here a_s, $s \le 2M$, and $b_t t \sim N$ are complex numbers with $|a_s|$, $|b_t| \ll v^\eta$. We prove (4.8) by combining the argument of Lemma 2 of [**22**] with bounds for exponential sums taken from [**15, 5**]; the procedure for (4.9) is analogous.

Proceeding as in [**22, 5**], via a fundamental lemma, we reach the asymptotic formula

$$\sum_{m\sim M} a_m S(\mathcal{A}_m, v^{1/10-2\varepsilon}) = \delta \sum_{m\sim M} a_m S(\mathcal{B}_m, v^{1/10-2\varepsilon})(1 + O(g(\nu)) + O_\varepsilon((\log v)^{-1})) + O_\varepsilon(\delta v^{1-2\eta}).$$

Here $M \le v^{11/20-\varepsilon}$, $0 \le a_m \ll v^\eta$, $a_m = 0$ unless all prime divisors of m are at least $v^{1/10-2\varepsilon}$; $\nu = 100\varepsilon$, and

$$g(x) = \exp\left(1 - \frac{1}{x}\log\left(\frac{1}{x}\right)\right).$$

We may now carry out the decomposition of $S(\mathcal{A}, (3v)^{1/2})$ in exactly the same way as [**5**, Section 5] with v in the role of x, and push the argument to a conclusion by following that paper with very little adaptation.

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