ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 4, Fall 1996

ON THE NONEXISTENCE OF POSITIVE SOLUTIONS OF INTEGRO-DIFFERENTIAL INEQUALITIES

FU-HSIANG WONG, CHEH-CHIH YEH AND SHIUEH-LING YU

ABSTRACT. In this paper we show that, under suitable conditions on f and K, the inequalities

$$-\lambda + \theta \int_0^\infty \varepsilon^{\lambda s} K(s)\, ds > 0 \quad \text{for all } \lambda > 0$$

and

$$-\lambda^2 + \theta \int_0^\infty e^{\lambda s} K(s)\, ds > 0 \quad \text{for all } \lambda > 0$$

imply the integro-differential inequalities

$$y'(t) + \int_0^t K(t-s) f(y(s))\, ds \le 0 \quad \text{on } [T, \infty)$$

and

$$y''(t) - \int_0^t K(t-s) f(y(s))\, ds \ge 0 \quad \text{on } [T, \infty)$$

have no positive solution, respectively, where $f(y)/y \ge \theta > 0$ in some interval $(0, y_0)$. We also point out that the function f cannot be a superlinear function, that is, $f(y) \ne y^\beta$ for $\beta \in (1, \infty)$.

1. Introduction. In this paper we consider the nonexistence of positive solutions of the following integro-differential inequalities

$$(\mathrm{E}_1) \qquad y'(t) + \int_0^t K(t-s) f(y(s))\, ds \le 0,$$

Received by the editors on October 28, 1994, and in revised form on December 20, 1994.
1991 *Mathematics Subject Classification.* Primary 34K15, 34C10.
Key words and phrases. Positive solution, integro-differential inequality.

and

$$y''(t) - \int_0^t K(t-s)f(y(s))\,ds \geq 0. \tag{E_2}$$

Such a problem includes several interesting problems arisen in various branches of applications in population dynamics, ecology and a mechanic system with zero external forces, zero kinetic energy; see, for example, Burton [**1**]. This problem has been developed by many authors; see, for example, Burton [**1**], Gopalsamy [**2, 3, 4, 5**], the book of Gyori and Ladas [**6**], Philos and Sficas [**8**], Philos and Sficas [**11**], Lellouche [**9**] and Lewitan [**10**]. To the best of the author's knowledge, the most significant contribution to this problem was made by Ladas, Philos and Sficas [**8, 11**]. The main nonexistence results of Ladas, Philos and Sficas [**8, 11**] are the following two theorems.

Theorem A. *Assume that the following hypotheses hold*:

(H1) $K \in C([0,\infty);[0,\infty))$ *and there is a* $T > 0$ *such that* K *is not identically zero on* $[0, T]$,

(H2) $f \in C(\mathbf{R};\mathbf{R})$ *satisfies* $yf(y) > 0$ *for* $y \neq 0$ *and*

$$\theta := \inf_{y>0} \frac{f(y)}{y} > 0.$$

Then (E_1) *has no positive solution on* $[0,\infty)$ *if the following inequality holds*:

$$-\lambda + \theta \int_0^\infty e^{\lambda s} K(s)\,ds > 0 \quad \text{for all } \lambda > 0. \tag{1}$$

Theorem B. *Assume that* (H1) *and the following inequality hold*:

$$-\lambda^2 + \theta \int_0^\infty e^{\lambda s} K(s)\,ds > 0 \quad \textit{for all } \lambda > 0. \tag{2}$$

Then the integro-differential inequality

$$y''(t) - \int_0^t K(t-s)y(s)\,ds \geq 0 \quad \textit{on } [T,\infty) \tag{E_3}$$

has no bounded and positive solution on $[0, \infty)$.

Obviously, Theorem A is only a minor modification of equation (E_1) with $f(y) = y$, and it cannot be applied to $f(y) := y^\alpha$ for $\alpha \in [0, 1)$ or $\alpha \in (1, \infty)$. The purpose of this paper is to establish some sufficient conditions to guarantee that (E_1) and (E_2) have no positive solutions if $f(y)$ is sublinear. These results generalize Theorems A and B.

An example is also given which explains that (E_1) has a positive solution on $[0, \infty)$ if $f(y)$ is superlinear.

2. Main results.

Theorem 1. *Suppose that* (1) *and the following conditions hold*:

(C1) $K \in C([0, \infty); [0, \infty))$ *and there is a* $T > 0$ *such that* K *is not identically zero on* $[0, T]$,

(C2) $f \in C((0, \infty); (0, \infty))$ *and there are two positive constants* $\theta > 0$ *and* $y_0 > 0$ *such that* $f(y)/y \geq \theta$ *in* $(0, y_0)$.

Then there is no solution of (E_1) *which is positive on* $[0, \infty)$.

Proof. Assume to the contrary that there exists a solution $y(t)$ of (E_1) which is positive on $[0, \infty)$. It follows from (C1) that there exist t_0, t_1 such that $0 < t_0 < t_1 < T$, $K(s) > 0$ on $[t_0, t_1]$ and

$$\lambda_1 := \int_{t_0}^{t_1} K(s)\, ds > 0.$$

Since $y(t)$ is decreasing on $[T, \infty)$ and is bounded below by 0,

$$\lim_{t \to \infty} y(t) := y(\infty) \geq 0.$$

It is clear that $y(\infty) = 0$. In fact, if $y(\infty) > 0$, then there exists $t_2 \geq T$ satisfying

$$0 < \frac{y(\infty)}{2} \leq y(t) \leq \frac{3y(\infty)}{2}$$

on $[t_2, \infty)$. Let

$$\inf_{y(\infty)/2 \leq y \leq 3y(\infty)/2} f(y) := m > 0,$$

then

$$\begin{aligned}0 &\geq y'(t) + \int_0^t K(s) f(y(t-s))\, ds \\ &\geq y'(t) + \int_{t_0}^{t-t_2} K(s) f(y(t-s))\, ds \\ &\geq y'(t) + m \int_{t_0}^{t_1} K(s)\, ds \\ &= y'(t) + \lambda_1 m \quad \text{for all } t \geq T_1 := t_2 + t_1 > 0.\end{aligned}$$

Thus,

$$y'(t) \leq -\lambda_1 m \quad \text{on } [T_1, \infty),$$

which implies

$$\lim_{t\to\infty} y(t) = -\infty.$$

This contradiction proves that $y(\infty) = 0$. Hence, there exists $t_3 \geq T$ such that $0 < y(t) < y_0$ on $[t_3, \infty)$ which implies

$$f(y(t)) \geq \theta y(t) \quad \text{on } [t_3, \infty). \tag{3}$$

Since $y(t)$ is decreasing on $[t_3, \infty)$, we see that

$$\begin{aligned}0 &\geq y'(t) + \int_0^t K(s) f(y(t-s))\, ds \\ &\geq y'(t) + \int_{t_0}^{t-t_3} K(s) f(y(t-s))\, ds \\ &\geq y'(t) + \theta \int_{t_0}^{t-t_3} K(s) y(t-s)\, ds \\ &\geq y'(t) + \theta y(t-t_0) \int_{t_0}^{t_1} K(s)\, ds \\ &\geq y'(t) + \lambda_1 \theta y(t) \quad \text{for all } t \geq T_2 := t_3 + t_1 \geq t_3 > 0.\end{aligned}$$

This implies

$$\Gamma_1 := \{\lambda \geq 0 \mid y'(t) + \lambda y(t) \leq 0 \text{ on } [T_\lambda, \infty) \text{ for some } T_\lambda \geq T_2\}$$

contains $\lambda_1\theta > 0$. Moreover, if $\lambda \in \Gamma_1$, then $[0, \lambda] \subset \Gamma_1$. It follows from (E_1) and (3) that

$$\begin{aligned}
0 &\geq y'(t) + \int_0^t K(s) f(y(t-s))\,ds \\
&\geq y'(t) + \int_{t_0}^{t_1} K(s) f(y(t-s))\,ds \\
&\geq y'(t) + \theta \int_{t_0}^{t_1} K(s) y(t-s)\,ds \\
&\geq y'(t) + \theta y(t-t_0) \int_{t_0}^{t_1} K(s)\,ds \quad (t - t_0 \geq t - t_1 \geq T_2) \\
&= y'(t) + \lambda_1 \theta y(t-t_0) \quad \text{for all } t \geq T_2 + t_1 \geq T_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
y(t) &> -y(t + t_0/2) + y(t) \\
&= -\int_t^{t+t_0/2} y'(s)\,ds \\
&\geq \lambda_1 \theta \int_t^{t+t_0/2} y(s-t_0)\,ds \\
&\geq \lambda_1 \theta \frac{t_0}{2} y\left(y - \frac{t_0}{2}\right) \quad \text{for all } t \geq T_2 + t_1 \geq T_2.
\end{aligned}$$

Now, we claim that

$$\sup \Gamma_1 \leq \eta := -\frac{2}{t_0} \ln\left(\lambda_1 \theta \frac{t_0}{2}\right).$$

Assume to the contrary that $\sup \Gamma_1 > \eta$. Hence, $\eta \in \Gamma_1$ and there exists a $T^* \geq T_2 + t_1$ such that

$$y'(t) + \eta y(t) \leq 0 \quad \text{on } [T^*, \infty).$$

This implies

$$\begin{aligned}
\lambda_1 \theta \frac{t_0}{2} y\left(t - \frac{t_0}{2}\right) &< y(t) \\
&\leq \exp\left(-\eta \frac{t_0}{2}\right) y\left(t - \frac{t_0}{2}\right) \\
&= \lambda_1 \theta \frac{t_0}{2} y\left(t - \frac{t_0}{2}\right) \quad \text{for all } t \geq T^* \geq T_2 + t_1,
\end{aligned}$$

which gives a contradiction. Thus, $0 < \lambda_1^* := \sup \Gamma_1$ and there exists $T_2^* \geq T_2$ such that

$$y'(t) + \lambda_1^* y(t) \leq 0 \quad \text{on } [T_2^*, \infty).$$

Furthermore, it follows from (3) that, for any t, s with $t \in [T_2^*, \infty)$ and $s \in [0, t - T_2^*]$,

$$\theta y(t) \exp(\lambda_1^* s) \leq \theta y(t-s) \leq f(y(t-s)).$$

This and (E_1) imply, for all $t \in [T_2^*, \infty)$,

$$\begin{aligned} 0 &\geq y'(t) + \int_0^t K(s) f(y(t-s))\, ds \\ &\geq y'(t) + \int_0^{t-T_2^*} K(s) f(y(t-s))\, ds \\ &\geq y'(t) + \theta \int_0^{t-T_2^*} K(s) y(t-s)\, ds \\ &\geq y'(t) + \left(\theta \int_0^{t-T_2^*} K(s) \exp(\lambda_1^* s)\, ds\right) y(t). \end{aligned} \tag{4}$$

Finally, we claim that

$$\theta \int_0^{t-T_2^*} K(s) \exp(\lambda_1^* s)\, ds \leq \lambda_1^* \quad \text{for all } t \geq T_2^*. \tag{5}$$

Suppose to the contrary that there exists $T_3 > T_2^*$ such that

$$\lambda_2 := \theta \int_0^{T_3 - T_2^*} K(s) \exp(\lambda_1^* s)\, ds > \lambda_1^*.$$

This and (4) imply that $\lambda_2 \in \Gamma_1$, which contradicts the definition of λ_1^*. This contradiction proves that (5) holds. Hence,

$$-\lambda_1^* + \theta \int_0^{\infty} K(s) \exp(\lambda_1^* s)\, ds \leq 0.$$

This contradicts (1). This contradiction completes the proof. □

Theorem 2. *Suppose that* (C1), (C2) *and the following inequality hold*:

$$-\lambda^2 + \theta \int_0^\infty e^{\lambda s} K(s)\, ds > 0 \quad \text{for } \lambda > 0. \tag{6}$$

Then there is no solution of (E_2) *which is bounded and positive on* $[0, \infty)$.

Proof. Assume to the contrary that there exists a solution $y(t)$ of (E_2) which is bounded and positive on $[0, \infty)$. It follows from the lemma of Philos and Sficas [**11**] that $y(t)$ is decreasing on $[T, \infty)$. Let t_0, t_1 and λ_1 be defined as in the proof of Theorem 1. Since $y(t)$ is decreasing on $[T, \infty)$ and is bounded below by 0, we see that

$$\lim_{t\to\infty} y(t) := y(\infty) \geq 0.$$

We claim that $y(\infty) = 0$. In fact, if $y(\infty) > 0$, then

$$\begin{aligned} 0 &\leq y''(t) - \int_0^t K(s) f(y(t-s))\, ds \\ &\leq y''(t) - \int_{t_0}^{t-t_2} K(s) f(y(t-s))\, ds \\ &\leq y''(t) - m \int_{t_0}^t K(s)\, ds \\ &= y''(t) - \lambda_1 m \quad \text{for all } t \geq T_1 := t_2 + t_1 \geq 0, \end{aligned}$$

where m, T_1 and t_2 are defined as in the proof of Theorem 1. Hence,

$$\lim_{t\to\infty} y'(t) = \infty,$$

which gives a contradiction. Thus, $y(\infty) = 0$ and hence (3) holds on

$[t_3, \infty)$ for some $t_3 \geq T$. Since $y(t)$ is decreasing, we see that

$$\begin{aligned} 0 &\leq y''(t) - \int_0^t K(s) f(y(t-s))\, ds \\ &\leq y''(t) - \int_{t_0}^{t-t_3} K(s) f(y(t-s))\, ds \\ &\leq y''(t) - \theta \int_{t_0}^{t-t_3} K(s) y(t-s)\, ds \\ &\leq y''(t) - \theta y(t-t_0) \int_{t_0}^{t_1} K(s)\, ds \\ &\leq y''(t) - \lambda_1 \theta y(t) \quad \text{for all } t \geq T_2 := t_3 + t_1 \geq t_3 > 0. \end{aligned}$$

This implies the set

$$\begin{aligned} \Gamma_2 := \{\lambda \geq 0 \mid y''(t) - \lambda^2 y(t) \geq 0 \\ \text{on } [T_\lambda, \infty) \text{ for some } T_\lambda \geq T_2\} \end{aligned}$$

contains $(\lambda_1 \theta)^{1/2} > 0$. Clearly, if $\lambda \in \Gamma_2$, then $[0, \lambda] \subset \Gamma_2$. Moreover, using the same technique of Philos and Sficas [**11**], we see that $\Gamma_2 \subset \Gamma_1$. Thus, $\lambda_2^* := \sup \Gamma_2 \in \Gamma_1$ and there exists $T_2^* \geq T_2$ such that $y'(t) + \lambda_2^* y(t) \leq 0$ on $[T_2^*, \infty)$. Moreover, as in the proof of Theorem 1, we see that, for any t, s with $t \in [T_2^*, \infty)$ and $s \in [0, t - T_2^*]$,

$$\theta y(t) \exp(\lambda_2^* s) \leq \theta y(t-s) \leq f(y(t-s)).$$

This and (E_2) imply for all $t \in [T_2^*, \infty)$ that

$$\begin{aligned} 0 &\leq y''(t) - \int_0^t K(s) f(y(t-s))\, ds \\ &\leq y''(t) - \int_0^{t-T_2^*} K(s) f(y(t-s))\, ds \\ &\leq y''(t) - \theta \int_0^{t-T_2^*} K(s) y(t-s)\, ds \\ &\leq y''(t) - \left(\theta \int_0^{t-T_2^*} K(s) \exp(\lambda_2^* s)\, ds\right) y(t). \end{aligned} \tag{7}$$

Finally, we claim that

$$\left\{\theta \int_0^{t-T_2^*} K(s)\exp(\lambda_2^* s)\,ds\right\}^{1/2} \le \lambda_2^* \quad \text{for all } t \ge T_2^*. \tag{8}$$

Suppose to the contrary that there exists $T_3 > T_2^*$ such that

$$\lambda_3 := \left\{\theta \int_0^{T_3-T_2^*} K(s)\exp(\lambda_2^* s)\,ds\right\}^{1/2} > \lambda_2^*.$$

This and (7) imply $\lambda_3 \in \Gamma_2$, which contradicts the definition of λ_2^*. Thus (8) holds, and hence

$$-(\lambda_2^*)^2 + \theta \int_0^\infty K(s)\exp(\lambda_2^* s)\,ds \le 0.$$

This contradicts condition (6); thus we complete the proof. □

Remark C. The condition (C2) in Theorems 1 and 2 is indispensable. For example, let $f(y) := y^3$ for $y > 0$ and $K(s) := e^{-s}$ for $s \ge 0$. It is clear that f does not satisfy (H2) and (C2). Moreover, for any given $\theta \ge 1$, we see that

$$-\lambda + \theta \int_0^\infty e^{\lambda s} K(s)\,ds > 0 \quad \text{for all } \lambda > 0.$$

Let $y(t) := e^{-t}$ for all $t \ge 0$. A simple calculation shows that

$$y'(t) + \int_0^t K(t-s) f(y(s))\,ds = -\frac{1}{2}(e^{-t} + e^{-3t}) \quad \text{on } [0,\infty).$$

Thus, (E1) has a positive solution on $[0,\infty)$.

Remark D. There are many functions f satisfying (C2) but which do not satisfy the hypothesis "$\inf_{y>0} f(y)/y \ge \theta > 0$" in Theorem A; see, for example, $f(y) := \ln(y+1)$, $f(y) := y^\alpha + cy^\beta$, $f(y) := 1 - \exp(-y)$, $f(y) := \exp(y) + cy^\beta$, or $f(y) := |\sin y|$ for $y > 0$, where $\alpha \in [0,1]$ and $c, \beta \in [0,\infty)$. Therefore, Theorems 1 and 2 generalize Theorems A and B, respectively.

Corollary 3. *Let* $\alpha \in [0,1]$ *and* $n = 1$, *(*$n = 2$ *respectively). If* (C1) *and the following inequality hold*:

$$-\lambda^n + \int_0^\infty e^{\lambda s} K(s)\, ds > 0 \quad \text{for all } \lambda > 0, \tag{9}$$

then the following integro-differential inequalities

$$(-1)^{n+1} y^{(n)}(t) + \int_0^t K(t-s) y^\alpha(s)\, ds \le 0 \quad \text{on } [T,\infty), \tag{E$_4$}$$

and

$$(-1)^{n+1} y^{(n)}(t) + \int_0^t K(t-s) \exp(y(s))\, ds \le 0 \quad \text{on } [T,\infty), \tag{E$_5$}$$

have no solution which is positive (*bounded and positive, respectively*) *on* $[0,\infty)$.

REFERENCES

1. T.A. Burton, *Volterra integral and differential equations*, Academic Press, New York, 1983.

2. K. Gopalsamy, *Stability, instability, oscillation and nonoscillation in scalar integro-differential system*, Bull. Austral. Math. Soc. **28** (1983), 223–246.

3. ———, *Oscillation in a delay-logistic equation*, Quart. Appl. Math. **44** (1986), 447–461.

4. ———, *Oscillation in systems of integro-differential equations*, J. Math. Anal. Appl. **113** (1986), 78–87.

5. ———, *Oscillation of integro-differential equations*, J. Math. Anal. Appl. **126** (1987), 100–109.

6. I. Gyori and G. Ladas, *Oscillation theory of delay differential equations with applications*, Oxford University Press, Cary, 1991.

7. G. Ladas, Y.G. Sficas and I.P. Stavroulakis, *Necessary and sufficient conditions for oscillations*, Amer. Math. Monthly **90** (1983), 637–640.

8. G. Ladas, Ch.G. Philos and Y.G. Sficas, *Oscillations of integro-differential equations*, Differential Integral Equations **4** (1991), 1113–1120.

9. G. Lellouche, *A frequency criterion for oscillatory solutions*, SIAM J. Control **8** (1970), 202–206.

10. B. Lewitan, *On an integral equation with an almost periodic solution*, Bull. Amer. Math. Soc. **43** (1937), 677–679.

11. Ch.G. Philos and Y.G. Sficas, *On the existence of positive solutions of integro-differential equations*, Appl. Anal. **36** (1990), 189–210.

Department of Mathematics and Science, National Taipei Teacher's College, Taipei 10659, Taiwan, Republic of China

Department of Mathematics, National Central University, Chung-Li, 32054 Taiwan, Republic of China

St. John's & St. Mary's Institute of Technology, Tamsui, Taipei, Taiwan, Republic of China