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ASYMPTOTIC BEHAVIOR OF ORTHOGONAL RATIONAL FUNCTIONS CORRESPONDING TO MEASURE WITH DISCRETE PART OFF THE UNIT CIRCLE

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ABSTRACT. For a positive measure μ on the unit circle in the complex plane, m points off the unit circle $z_1, \dots, z_m$ and m positive number A_j, $j = 1, 2, \dots, m$, we investigate the asymptotic behavior of orthogonal rational functions $\psi_n(z)$, $n = 0, 1, 2, \dots$, with prescribed poles lying outside the unit circle corresponding to $d\mu/2\pi + \sum_{j=1}^m A_j \delta_{z_j}$, where δ_z denotes the unit measure supported at point z. We find the relative asymptotics of $\psi_n(z)$ with respect to the orthogonal rational functions corresponding to $d\mu/2\pi$ off the unit circle.

1. Introduction. Let $d\mu$ be a finite positive Borel measure with an infinite set as its support on $[0, 2\pi)$. We define $L^2_{d\mu}$ to be the space of all functions $f(z)$ on the unit circle $T := \{z \in \mathbf{C} : |z| = 1\}$ satisfying $\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\mu(\theta) < \infty$. Then $L^2_{d\mu}$ is a Hilbert space with inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\mu(\theta).$$

We define $\mathcal{P}_n$ to be all polynomials with degree at most n. For any polynomial q_n with degree n, we define $q_n^*(z) = z^n \overline{q_n(1/\bar{z})}$. Consider an arbitrary infinite sequence $\mathbf{S} = \{\alpha_n\}$ with $n \in \mathbf{N}$ and $|\alpha_n| < 1$, and let

$$b_k(z) := \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k}, \qquad k = 1, \dots ,$$

where for $\alpha_k = 0$ we put $|\alpha_k|/\alpha_k = -1$. Next we define finite Blaschke products recursively as

$$B_0(z) = 1 \quad \text{and} \quad B_k(z) = B_{k-1}(z) b_k(z), \qquad k = 1, \dots .$$

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The fundamental polynomials $w_n(z)$ are given by

$$w_0(z) := 1 \quad \text{and} \quad w_k(z) := \prod_{i=1}^{k}(1-\bar{\alpha}_i z), \qquad k = 1, \dots .$$

The space of rational functions of our interest is defined as

$$\mathcal{R}_n = \mathcal{R}[\alpha_1, \dots, \alpha_n] := \left\{ \frac{p(z)}{w_n(z)} : p \in \mathcal{P}_n \right\},$$
$$n = 0, 1, \dots .$$

It is easy to verify that $\{B_k\}_{k=0}^n$ forms the basis of $\mathcal{R}_n$, i.e., $\mathcal{R}_n = \operatorname{span}\{B_k(z), k = 0, \dots, n\}$. Finally, for any $r \in \mathcal{R}_n$, we define $r^*(z) := B_n(z)\overline{r(1/\bar{z})}$. Then it is easy to see that $|r^*(z)| = |r(z)|$ for $|z| = 1$ and $r^*(z) \in \mathcal{R}_n$.

One might as well consider the other basis $C_{n,k}(z) := z^k/w_n(z)$, $k = 0, \dots, n$, for $\mathcal{R}_n$. We denote $\{\varphi_{n,l}(z)\}_{l=0}^n$ the orthogonal rational functions obtained from applying Gram-Schmidt procedure to $\{C_{n,k}(z)\}_{k=0}^n$, and they are uniquely determined by the following conditions:

$$\begin{cases} \varphi_{n,l}(z) = p_{n,l}(z)/w_n(z) & p_{n,l} \in \mathcal{P}_l,\ p_{n,l}^*(0) > 0, \\ \langle \varphi_{n,l}, C_{n,k} \rangle = 0 & k = 0, \dots, l-1, \text{ and} \\ \langle \varphi_{n,l}, \varphi_{n,l} \rangle = 1. & \end{cases}$$

This orthogonal basis has already been studied in [**15**–**17**], not in the form of orthogonal rational functions, but in the setting of orthogonal polynomials with respect to varying measures. Write the numerator of $\varphi_{n,l}(z)$ as $p_{n,l}(z)$, then $\varphi_{n,l}(z) = p_{n,l}(z)/w_n(z)$ and $p_{n,l}(z)$ is the lth orthonormal polynomial with respect to the varying measure $d\mu(\theta)/|w_n(e^{i\theta})|^2$ as defined in [**15**–**18**]. We define $\varphi_n(z) := \varphi_{n,n}(z)$.

The orthogonal rational functions are of constant interest to both mathematicians and physicists. That is, because their significance relates to the studies in Hankel and Toeplitz operators, continued fractions, moment problem, Carathéodory-Fejer interpolation, Schur's algorithm and function algebras, and solving electrical engineering problems (cf. [**7**–**12**, **1**–**6**]).

Suppose $z_1, \dots, z_m$ are m distinct fixed points outside the unit circle. For m positive numbers $A_1, A_2, \dots, A_m$, construct $\nu = \mu/(2\pi) +$

$\sum_{j=1}^{m} A_j \delta_{z_j}$, where δ_z denotes the (Dirac delta) unit measure supported at point z. Then $L^2_{d\nu}$ is a Hilbert space with inner product

$$\langle f, g\rangle_{d\nu} := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\mu(\theta) + \sum_{j=1}^{m} A_j f(z_j)\overline{g(z_j)}.$$

For each n, we now define orthonormal rational functions with respect to $d\nu$, $\{\psi_{n,l}(z)\}_{l=0}^{n}$,

$$\begin{cases} \psi_{n,l} = q_{n,l}(z)/w_n(z) & q_{n,l} \in \mathcal{P}_l,\ q^*_{n,l}(0) > 0, \\ \langle \psi_{n,l}, C_{n,k}\rangle_{d\nu} = 0 & k = 0, \dots, l-1, \text{ and} \\ \langle \psi_{n,l}, \psi_{n,l}\rangle_{d\nu} = 1. & \end{cases}$$

Note that $\psi_n(z)$ depends on A_j, $j = 1, 2, \dots, m$, although this dependence is not explicitly given in our notation. Define $\psi_n(z) = \psi_{n,n}(z)$.

The purpose of this paper is to study the asymptotic behavior of $\psi_n(z)$ off T under fairly mild assumptions on μ. The asymptotics of $\varphi_n(z)$ have been investigated extensively (see, for example, [**15**–**20** and **14**]. To take advantage of this in our study of $\psi_n(z)$, we compare $\psi_n(z)$ with $\varphi_n(z)$ in the form of relative asymptotics, i.e., we investigate $\psi_n(z)/\varphi_n(z)$.

The main results are stated in Section 2, and their proofs are given in Section 4. Section 3 is devoted to the lemmas needed for the proof in Section 4.

2. Main results. Let

$$B(z) := \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z}, \tag{2.1}$$

and $\lambda := |B(0)|/B(0)$.

We write $p_{n,n}(z) = \kappa_n z^n + \cdots$, then $\kappa_n > 0$ since $p^*_{n,n}(0) > 0$. Rewriting $\psi_n(z) = q_n(z)/w_n(z)$, $q_n(z) = \gamma_n z^n + \cdots \in \mathcal{P}_n$, we have $\psi^*_n(z) = \eta_n q^*_n(z)/w_n(z)$. Since $\psi^*_n(0) = q^*_n(0)\eta_n$ and $\bar{\eta}_n \psi^*_n(0) > 0$, then $\gamma_n = q^*_n(0) > 0$.

We first discuss the ratio of the two leading coefficients.

Theorem 2.1. *If $\mu' > 0$ almost everywhere in $[0, 2\pi)$ and $\sum_{i=1}^{\infty}(1 - |\alpha_i|) = \infty$, then*

$$\lim_{n\to\infty} \frac{\gamma_n}{\kappa_n} = \prod_{j=1}^{m} \frac{1}{|z_j|}. \tag{2.2}$$

Remark 1. It is easy to check that

$$\prod_{j=1}^{m} \frac{1}{|z_j|} = \frac{1}{|B(0)|} = \lambda B(\infty),$$

where $B(\infty) := \lim_{z\to\infty} B(z)$.

For the ratio of the two orthonormal rational functions, we have

Theorem 2.2. *If $\mu' > 0$ almost everywhere in $[0, 2\pi)$ and $\sum_{i=1}^{\infty}(1 - |\alpha_i|) = \infty$, then*

$$\lim_{n\to\infty} \frac{\psi_n(z)}{\varphi_n(z)} = \lambda B(z), \tag{2.3}$$

locally uniformly for $|z| > 1$. Where

$$B(z) = \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z} \quad \textit{and} \quad \lambda = \frac{|B(0)|}{B(0)}.$$

Remark 2. From the known results on the asymptotics of $\varphi_n(z)$, we can use Theorem 2.2 to obtain the corresponding results for $\psi_n(z)$. For example, we can have the ratio asymptotics of $\psi_n(z)$ from that of $\varphi_n(z)$ (cf. [**20**]) and, if we make a stronger assumption on μ (say Szegő's condition), then we can get a better result (cf. [**14**]). We leave the formulation of these results to the reader.

3. Lemmas. The reproducing kernel function K_{n-1} is defined by

$$\begin{aligned} K_{n-1}(z, \zeta) &= \sum_{k=0}^{n-1} \varphi_{n,k}(z)\overline{\varphi_{n,k}(\zeta)} \\ &= \sum_{k=0}^{n-1} \frac{p_{n,k}(z)}{w_n(z)} \overline{\left[\frac{p_{n,k}(\zeta)}{w_n(\zeta)}\right]}, \end{aligned}$$

and by the Christoffel-Darboux formula for orthogonal polynomials (cf. [**13**, p. 3]) we have

$$\sum_{k=0}^{n-1} p_{n,k}(z)\overline{p_{n,k}(\zeta)} = \frac{p^*_{n,n}(z)\overline{p^*_{n,n}(\zeta)} - p_{n,n}(z)\overline{p_{n,n}(\zeta)}}{1-\bar{\zeta}z};$$

thus,

$$\begin{aligned} K_{n-1}(z,\zeta) &= \frac{p^*_{n,n}(z)\overline{p^*_{n,n}(\zeta)} - p_{n,n}(z)\overline{p_{n,n}(\zeta)}}{(1-\bar{\zeta}z)w_n(z)\overline{w_n(\zeta)}} \\ &= \frac{\varphi^*_n(z)\overline{\varphi^*_n(\zeta)} - \varphi_n(z)\overline{\varphi_n(\zeta)}}{(1-\bar{\zeta}z)}. \end{aligned} \tag{3.1}$$

Lemma 3.1. *If $\mu' > 0$ almost everywhere in $[0, 2\pi)$ and $\sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty$, then*

$$\lim_{n\to\infty} \frac{K_{n-1}(z,\zeta)}{\overline{\varphi_n(\zeta)}\varphi_n(z)} = \frac{1}{\bar{\zeta}z - 1},$$

locally uniformly for $|z| > 1$ and $|\zeta| > 1$.

Proof. It is proved in [**17**, Theorem 3] that

$$\lim_{n\to\infty} \frac{\varphi^*_n(z)}{\varphi_n(z)} = 0, \tag{3.2}$$

locally uniformly for $|z| > 1$. The lemma then follows from (3.1). □

Lemma 3.2. *If $\mu' > 0$ almost everywhere in $[0, 2\pi)$ and $\lim_{n\to\infty} \sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty$, then*

$$\lim_{n\to\infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} \frac{(1-\bar{\alpha}_{n+1}z)}{(z-\alpha_{n+1})} = 1, \tag{3.3}$$

locally uniformly for $|z| > 1$. Consequently,

$$\lim_{n\to\infty} |\varphi_n(z)| = \infty, \tag{3.4}$$

locally uniformly for $|z| > 1$.

Proof. Formula (3.3) is from [**19**, Theorem 3.3]. For (3.4), let $r > 1$, then by (3.3) there exists an integer $L > 0$ such that

$$\left|\frac{\varphi_{n+1}(z)}{\varphi_n(z)}\right|\left|\frac{1-\bar{\alpha}_{n+1}z}{z-\alpha_{n+1}}\right| \geq \frac{1}{r},$$

for all $n \geq L$ and $|z| > 1$. So, for $n \geq L$ and $|z| > 1$,

$$\begin{aligned}\left|\frac{\varphi_{n+1}(z)}{\varphi_L(z)}\right| &= \prod_{k=L}^{n}\left|\frac{\varphi_{k+1}(z)(1-\bar{\alpha}_{k+1}z)}{\varphi_k(z)(z-\alpha_{k+1})}\right| \prod_{k=L}^{n}\left|\frac{(z-\alpha_{k+1})}{1-\bar{\alpha}_{k+1}z)}\right| \\ &\geq \left(\frac{1}{r}\right)^{n-L+1} \prod_{k=L}^{n}\left|\frac{(z-\alpha_{k+1})}{(1-\bar{\alpha}_{k+1}z)}\right|.\end{aligned}$$

Now formula (3.4) follows from the above inequalities and the facts that $\varphi_L(z) \neq 0$ for $|z| \geq 1$ and $\lim_{n\to\infty}\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$. $\square$

Lemma 3.3. *For all* $n \geq 0$,

$$\frac{\gamma_n}{\kappa_n} \leq 1.$$

Proof. This lemma is the consequence of the extremality of the monic polynomial $\kappa_n^{-1}\varphi_n(z)$ (cf. [**2**, Theorem 11.1.2]), for $z = e^{i\theta}$,

$$\begin{aligned}\frac{1}{\kappa_n^2} &= \min_{p\in P_{n-1}} \frac{1}{2\pi}\int |z^n + p(z)|^2 \frac{d\mu}{|w_n(z)|^2} \\ &\leq \frac{1}{2\pi}\int \left|\frac{q_n(z)}{\gamma_n}\right|^2 \frac{d\mu}{|w_n(z)|^2} \\ &\leq \frac{1}{2\pi}\int \left|\frac{q_n(z)}{\gamma_n w_n(z)}\right|^2 d\mu \\ &= \frac{1}{\gamma_n^2}\int |\psi_n(z)|^2\, d\mu \\ &\leq \frac{1}{\gamma_n^2}\int |\psi_n(z)|^2\, d\nu \\ &= \frac{1}{\gamma_n^2}. \qquad \square\end{aligned}$$

Lemma 3.4. *For points $z_1, \dots, z_m$ outside the unit circle and $z_i \neq z_j$, the following matrix*

$$\mathbf{T}_m := \left(\frac{1}{\bar{z}_j z_k - 1} \right)_{k,j=1}^{m}$$

is positive definite.

Proof. For any $x_1, x_2, \dots, x_m \in \mathbf{C}$, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\bar{x}_j x_i}{\bar{z}_i z_j - 1} = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{i=1}^{m} \frac{x_i}{z \bar{z}_i - 1} \right|^2 d\theta \geq 0,$$

$$z = e^{i\theta}. \qquad \square$$

Lemma 3.5. *For points $z_1, z_2, \dots, z_m$ outside the unit circle and $z_i \neq z_j$, let $B(z)$ be defined as in* (2.1). *Then there exist a unique set of nonzero complex numbers $r_1, r_2, \dots, r_m$ such that*

$$B(z) = \frac{1}{\overline{B(0)}} + \sum_{j=1}^{m} \frac{r_j}{1 - \bar{z}_j z}. \tag{3.5}$$

Proof. The existence of the above partial fraction representation of $B(z)$ is obvious. The uniqueness follows from the linear independence of the set $\{(1 - \bar{z}_j z)^{-1}\}_{j=1}^{m}$. Finally, that none of the r_js is zero follows from comparing the poles on both sides of (3.5). $\square$

4. Proofs of main results.

Now we are ready to prove our theorems.

Proof of Theorem 2.1. We need to show the existence of the limit in (2.2) and calculate the value of the limit.

By Lemma 3.3, every subsequence of $\{\gamma_n/\kappa_n\}_{n=0}^{\infty}$ contains a convergent subsequence. Let $R \geq 0$ be a limit point of this sequence, and let $\Lambda \subseteq \{0, 1, 2, \dots\}$ satisfy

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \gamma_n/\kappa_n = R. \tag{4.1}$$

Note that the orthonormality of $\varphi_n(z)$ yields (on writing $\psi_n(z) = (\gamma_n/\kappa_n)\varphi_n(z) + p(z)/w_n(z)$, with $p \in \mathcal{P}_{n-1}$)

$$\frac{1}{2\pi}\int \psi_n(z)\overline{\varphi_n(z)}d\mu = \frac{\gamma_n}{\kappa_n}.$$

On the other hand, the orthonormality of ψ_n gives (on writing $\varphi_n(z) = (\kappa_n/\gamma_n)\psi_n(z) + q(z)/w_n(z)$ with $q \in \mathcal{P}_{n-1}$)

$$\begin{aligned}\frac{1}{2\pi}\int \psi_n(z)\overline{\varphi_n(z)}\,d\mu &= \int \psi_n(z)\overline{\varphi_n(z)}\,d\nu \\ &\qquad - \sum_{j=1}^{m} A_j\psi_n(z_j)\overline{\varphi_n(z_j)} \\ &= \frac{\kappa_n}{\gamma_n} - \sum_{j=1}^{m} A_j\psi_n(z_j)\overline{\varphi_n(z_j)}.\end{aligned}$$

So we have

$$\frac{\kappa_n}{\gamma_n} - \frac{\gamma_n}{\kappa_n} = \sum_{j=1}^{m} A_j\psi_n(z_j)\overline{\varphi_n(z_j)}. \tag{4.2}$$

We now consider the limit behavior of the summation on the right side as $n \to \infty$ and $n \in \Lambda$. Note that $w_n(z)(\psi_n(z) - (\gamma_n/\kappa_n)\varphi_n(z)) \in \mathcal{P}_{n-1}$, so according to the reproducing property of the kernel function (cf. [**21**]) and orthogonality of $\varphi_n(z)$ and $\psi_n(z)$, with $\zeta = e^{i\theta}$,

$$\begin{aligned}&w_n(z)\left[\psi_n(z) - \frac{\gamma_n}{\kappa_n}\varphi_n(z)\right] \\ &\quad = \frac{1}{2\pi}\int w_n(\zeta)\left(\psi_n(\zeta) - \frac{\gamma_n}{\kappa_n}\varphi_n(\zeta)\right)\sum_{k=0}^{n-1} p_{n,k}(z)\overline{p_{n,k}(\zeta)}\frac{d\mu(\theta)}{|w_n(\zeta)|^2} \\ &\quad = \frac{1}{2\pi}\int w_n(\zeta)\psi_n(\zeta)\sum_{k=0}^{n-1} p_{n,k}(z)\overline{p_{n,k}(\zeta)}\frac{d\mu(\theta)}{|w_n(\zeta)|^2} \\ &\quad = \frac{w_n(z)}{2\pi}\int \psi_n(\zeta)K_{n-1}(z,\zeta)\,d\mu(\theta) \\ &\quad = -w_n(z)\sum_{j=1}^{m} A_j\psi_n(z_j)K_{n-1}(z,z_j),\end{aligned}$$

and so

$$\begin{aligned}\frac{\psi_n(z)}{\varphi_n(z)} &= \frac{\gamma_n}{\kappa_n} \\ &\quad - \sum_{j=1}^{m} \left\{ A_j \psi_n(z_j) \overline{\varphi_n(z_j)} \frac{(\bar{z}_j z - 1) K_{n-1}(z, z_j)}{\overline{\varphi_n(z_j)} \varphi_n(z)} \right\} \\ &\qquad \cdot \frac{1}{\overline{z_j} z - 1}.\end{aligned} \tag{4.3}$$

By Lemma 3.1, we can write

$$\begin{aligned}A_j \psi_n(z_j) \overline{\varphi_n(z_j)} \frac{(\overline{z_j} z_k - 1) K_{n-1}(z_k, z_j)}{\overline{\varphi_n(z_j)} \varphi_n(z_k)} &= A_j \psi_n(z_j) \overline{\varphi_n(z_j)} (1 + o(1)) \\ &=: X_j (1 + o(1)),\end{aligned}$$

as $n \to \infty$, uniformly for $j, k = 1, 2, \dots, m$. On the other hand, since $A_j |\psi_n(z_j)|^2 \le \int |\psi_n|^2 \, d\nu = 1$, $j = 1, 2, \dots, m$, we have

$$\lim_{n \to \infty} \frac{\psi_n(z_j)}{\varphi_n(z_j)} = 0,$$

by Lemma 3.2, and the limit is locally uniform for the choice of $A_j > 0$, $j = 1, 2, \dots, m$. So, letting $z = z_k$, $k = 1, 2, \dots, m$ in (4.3) and using the above limit relations, we can obtain

$$\frac{\gamma_n}{\kappa_n} \mathbf{1} = \mathbf{T}_m [\mathbf{X}(1 + o(1))] + o(\mathbf{1}),$$

where $\mathbf{1} := (1, 1, \dots, 1)^t$, $\mathbf{T}_m$ is defined as in Lemma 3.4, $\mathbf{X} := (X_1, X_2, \dots, X_m)^t$, and the first $o(1)$ is independent of A_js and the second $o(\mathbf{1})$ is locally uniform for $A_j > 0$, $j = 1, 2, \dots, m$. So, by Lemma 4,

$$\mathbf{X}(1 + o(1)) = \frac{\gamma_n}{\kappa_n} \mathbf{T}_m^{-1} \mathbf{1} + o(\mathbf{1}). \tag{4.4}$$

But, letting $z = z_k$, $k = 1, 2, \dots, m$ in (3.5) will yield

$$\frac{1}{\overline{B(0)}} = \sum_{j=1}^{m} \frac{r_j}{\overline{z_j} z_k - 1}, \qquad k = 1, 2, \dots, m,$$

i.e.,

$$\mathbf{T}_m(r_1, r_2, \dots, r_m)^t = \frac{1}{\overline{B(0)}}\mathbf{1},$$

and so

$$\mathbf{T}_m^{-1}\mathbf{1} = \overline{B(0)}(r_1, r_2, \dots, r_m)^t.$$

Thus, by (4.4)

$$\mathbf{X}(1 + o(1)) = \frac{\gamma_n}{\kappa_n}\overline{B(0)}(r_1, r_2, \dots, r_m)^t + o(\mathbf{1}),$$

and so we have, by use of (4.1),

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} X_j = \lim_{\substack{n\to\infty \\ n\in\Lambda}} A_j\psi_n(z_j)\overline{\varphi_n(z_j)} = R\overline{B(0)}r_j, \quad j = 1, 2, \dots, m. \tag{4.5}$$

Now, letting $n \to \infty$ and $n \in \Lambda$ in (4.2), we see that

$$\frac{1}{R} - R = R\overline{B(0)}\sum_{j=1}^{m} r_j = R\overline{B(0)}\left(B(0) - \frac{1}{\overline{B(0)}}\right).$$

Hence, $R = |B(0)|^{-1}$. Since R is an arbitrary limit point of $\{\gamma_n/\kappa_n\}_{k=0}^{\infty}$, we see that the limit $\lim_{n\to\infty} \gamma_n/\kappa_n$ exists and is equal to $|B(0)|^{-1}$. The proof of the theorem is complete. □

Proof of Theorem 2.2. From (4.3) and the Christoffel-Darboux formula, we can write

$$\frac{\psi_n(z)}{\varphi_n(z)} = \frac{\gamma_n}{\kappa_n} - \sum_{j=1}^{m} A_j\psi_n(z_j)\overline{\varphi_n(z_j)} \left[\left(\overline{\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}}\right)\frac{\varphi_n^*(z)}{\varphi_n(z)} - 1\right]\frac{1}{\overline{z_j}z - 1}. \tag{4.6}$$

So, together with (3.5), this gives for $|z| > 1$

$$\begin{aligned}\left|\frac{\psi_n(z)}{\varphi_n(z)} - \lambda B(z)\right| \le{}& \left|\frac{\gamma_n}{\kappa_n} - \frac{\lambda}{B(0)}\right| \\ &+ \sum_{j=1}^{m}\left\{|A_j\psi_n(z_j)\overline{\varphi_n(z_j)} - \lambda r_j| \right. \\ &\left. + \left|A_j\psi_n(z_j)\overline{\varphi_n(z_j)}\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right|\right\}\frac{1}{|z_j| - 1},\end{aligned}$$

where we have used the fact that $|\varphi_n^*(z)/\varphi_n(z)| \leq 1$ for $|z| \geq 1$. Now, using (4.5) (by Theorem 2.1, Λ there can be taken as $\{1, 2, 3, \dots\}$, $R = |B(0)|^{-1}$), and so the limit values in (4.5) are λr_j, $j = 1, 2, \dots, m$ and (3.2) we have

$$\lim_{n\to\infty} \left| \frac{\psi_n(z)}{\varphi_n(z)} - \lambda B(z) \right| = 0,$$

locally uniformly for $|z| > 1$. $\square$

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