ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 4, Fall 1996

TWO POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the existence of solutions to differential equations of the form $x'' + g(t, x, x', x'') = f(t)$ with two-point boundary value conditions.

1. Introduction. The purpose of this paper is to establish the existence of solutions to certain *nonlinear* two point boundary value problems. Namely, we consider problems of the form

$$x''(t) + g(t, x, x', x'') = f(t) \tag{1}$$

where $g : [0, 1] \times \mathbf{R}^3 \to \mathbf{R}$ is continuous and $x(t)$ satisfies one of the following boundary conditions:

(D) $\quad x(0) = A, \qquad x(1) = B;$

(N) $\quad x'(0) = A, \qquad x'(1) = B$

(M_1) $\quad x(0) = A, \qquad x'(1) = B$

(M_2) $\quad x'(0) = A, \qquad x(1) = B.$

This and similar types of boundary value problems have recently received considerable attention. (See [**1, 2, 4, 5, 7, 9, 11, 13, 14**], etc.) In most known existence results for the boundary value problem (1), (D), (N), (M_1) and (M_2), the nonlinearity g depends at most on the first derivative x' and hence defines a compact nonlinear operator between some appropriate Banach spaces. In [**11, 13**], the authors studied the similar nonlinear periodic boundary value problems and allowed the nonlinearity g to depend on the highest derivative of $x(t)$. However, our conditions on g and on the boundary in this paper are

Received by the editors on July 18, 1994, and in revised form on December 28, 1994.
1991 *Mathematics Subject Classification.* Primary 34C15, 34C25.
Key words and phrases. Two-point boundary value problem, nonlinear equation, k-set contractive operator.

different from theirs. Our conditions on g in Theorem 3.1 are similar to those in [**15**] and conditions in Theorems 4.1, 4.2, 5.1 and 5.2 are similar to those in [**9**]. It should be noted that we have allowed g in the statement of the problem above to depend on x''. In this case the abstract results used in [**9**] and [**15**] are not applicable to our problems above.

To show the existence of solutions to the considered problems, we will use the continuation theory for k-set contractions [**5, 8**]. Our method in this direction relies on an abstract theorem developed in [**13**] and a priori bounds on solutions. For the convenience of the reader, we state the abstract theorems in Section 2.

2. Abstract existence theorems. In this section we will state some general abstract results that will be used in our study of the equation (1).

Let X and Y be Banach spaces and Ω a bounded open subset of X. Recall that a continuous and bounded map $N : \overline{\Omega} \to Y$ is called k-set-contractive if, for any bounded $A \subset \overline{\Omega}$ we have,

$$\Gamma_Y(N(A)) \leq k\Gamma_X(A)$$

where for a given Banach space Z, $\Gamma_Z(\cdot)$ is a measure of noncompactness given by

$$\Gamma_Z(A) = \sup\{\delta > 0 : \exists \text{ a finite number of subsets } A_i \subset A, A = \cup_i A_i, \operatorname{dim}(A_i) \leq \delta\}.$$

Here, diam (A_i) denotes the maximum distance between the points in the set A_i. Also, for a continuous and bounded map $T : X \to Y$, we define

$$l(T) = \sup\{r > 0 : r\Gamma_X(A) \leq \Gamma_Y(T(A)), A \subset X\}.$$

Now, let $T : X \to Y$ be an invertible and bounded linear map, and let $N : \overline{\Omega} \to Y$ be a k-set-contractive map with $k < l(T)$ and such that, for all $x \in \partial\Omega$ we have $T(x) \neq N(x)$. One can associate with the pair (T, N) a topological degree $D((T, N), \Omega)$, the so-called Sadovskii-Nussbaum degree. See [**5**]. This degree has many important properties. In particular, it has a homotopy invariance property that allows one to prove the following

Theorem 2.1. *Let $T : X \to Y$ be an invertible, bounded linear map and $\Omega \subset X$ bounded, open, and symmetric about $0 \in \Omega$. Let $N : \overline{\Omega} :\to Y$ be k-set-contractive with $k < l(T)$. Then, given $y \in Y$ such that $T(x) + \lambda N(x) \neq \lambda y$, for all $x \in \partial\Omega$ and $\lambda \in (0,1)$, there exists an $x \in \overline{\Omega}$ such that*

$$T(x) + N(x) = y.$$

If we assume that $L : X \to Y$ is a Fredholm operator of index zero and $N : \overline{\Omega} \to Y$ is k-set-contractive with $k < l(L)$, then we can prove another existence theorem similar to Theorem 2.1 above. Using the approach of Mawhin's, it was shown by Hetzer [**8**] that, under the above assumptions, if $Lx \neq Nx$ for all $x \in \partial\Omega$, then one can associate with the pair (L, N) a topological degree $D[(L, N), \Omega]$ which has most of the important properties of the Sadovskii-Nussbaum degree defined previously. Using the corresponding homotopy invariance property for this degree, one can prove the following

Theorem 2.2 [**13**]. *Let $L : X \to Y$ be a Fredholm operator of index zero, and let $y \in Y$ be a fixed point. Suppose that $N : \overline{\Omega} \to Y$ is k-set-contractive with $k < l(L)$ where $\Omega \subset X$ is bounded, open and symmetric about $0 \in \Omega$. Suppose further that*:

a) $Lx \neq \lambda Nx + \lambda y$ for $x \in \partial\Omega$, $\lambda \in (0,1)$.

b) $[QN(x) + Qy, x] \cdot [QN(-x) + Qy, x] < 0$ *for* $x \in \ker(L) \cap \partial\Omega$, *where* $[\ ,\ ]$ *is some bilinear form on* $Y \times X$ *and* Q *is the projection of* Y *onto* $\{\operatorname{coker}(L)\}$. *Then there exists* $x \in \overline{\Omega}$ *such that* $Lx - Nx = y$.

3. The Neumann problem (N)**.** In this section we consider the nonhomogeneous Neumann problem

$$\text{(N)}\qquad \begin{aligned} &x'' + g(t, x, x', x'') = f(t)\\ &x'(0) = A, \qquad x'(1) = B. \end{aligned}$$

This problem is equivalent to the following homogeneous Neumann problem:

$$x'' + g\left(t, x + \frac{1}{2}(B - A)t^2 + At, x' + t(B - A) + A, x'' + (B - A)\right) = f(t) - (B - A).$$

$$\text{(N}'\text{)} \qquad x'(0) = 0, \qquad x'(1) = 0.$$

We write

$$\tilde{g}(t, x, x_1, x_2) = g\bigg(t, x + \frac{1}{2}(B - A)t^2 + At, x_1 + t(B - A), x_2 + (B - A)\bigg)$$

and

$$\tilde{f}(t) = f(t) - (B - A).$$

We put the following conditions on g and f. They are similar to the conditions entertained in [**15**].

H3.1 $g : [0, 1] \times \mathbf{R}^3 \to \mathbf{R}$ is continuous.

H3.2 There exist measurable functions $\mu_+, \mu_- : [0, 1] \to \mathbf{R} \cup \{\pm\infty\}$ such that

$$\mu_+ \le \liminf_{x \to +\infty} g(t, x, x_1, x_2), \qquad t \in [0, 1]$$
$$\mu_- \ge \limsup_{x \to -\infty} g(t, x, x_1, x_2), \qquad t \in [0, 1]$$

uniformly for $(x_1, x_2) \in \mathbf{R}^2$.

H3.3 There exist constants c_1 and c_2 such that

$$g(t, x, x_1, x_2) \ge c_1 \quad \text{for } x \ge 0, (t, x_1, x_2) \in [0, 1] \times \mathbf{R}^2$$

and

$$g(t, x, x_1, x_2) \le c_2 \quad \text{for } x \le 0, (t, x_1, x_2) \in [0, 1] \times \mathbf{R}^2$$

Theorem 3.1. *Let* H3.1–H3.3 *be satisfied, and assume*:

a) *There exists a* $k \in [0, 1)$ *such that*

$$|g(t, x, x_1, p) - g(t, x, x_1, q)| \le k|p - q|$$

for $(t, x, x_1, p), (t, x, x_1, q) \in [0, 1] \times \mathbf{R}^3$.

b) *There exist positive constants* p_0, p_1, p_2 *and* p *such that*

$$|g(t,x,x_1,x_2)| \leq g(t,x,x_1,x_2) + p_0|x| + p_1|x_1| + p_2|x_2| + p$$

for $(t,x,x_1,x_2) \in [0,1] \times \mathbf{R}^3$.

c)

$$\int_0^1 \mu_-(t)\,dt < \int_0^1 \tilde{f}(t)\,dt < \int_0^1 \mu_+(t)\,dt.$$

Then there exists an $\varepsilon > 0$ *such that when* $\max\{p_0, p_1, p_2\} < \varepsilon$, *the problem* (N) *has a solution.*

Remark. J. Ward [**14**] considered similar conditions for the existence of periodic solutions to a class of semi-linear differential equations with a compact nonlinear perturbation. The interest of our conditions lies in the possibility of proving an existence theorem for the problem (N) without needing any assumption on the growth of $g(t,x,x',x'')$ for $x \geq 0$ (or else for $x \leq 0$). Also, notice that if g is nonnegative then our key condition b) is automatically satisfied.

Proof of Theorem 3.1. Let $X = \{x(t) \in C^2([0,1]) \mid x'(0) = x'(1) = 0\}$ with the norm $||x||_2 = \max\{||x||_0, ||x'||_0, ||x''||_0\}$ where $||h||_0 = \sup_{0\leq t\leq 1}\{|h(t)|\}$. Also, let $Y = C([0,1])$ with the norm $||\cdot||_0$. Of course, X and Y are Banach spaces. Next, let $L : X \to Y$ be given by $x(t) \mapsto x''(t)$. L is a bounded linear map. We will show that L is a Fredholm map of index 0. First we just notice that if $x(t) \in \ker(L)$ then $x(t) = c_1 t + c_2$. However, $x(t) \in X$ and thus $x'(0) = x'(1) = 0$ so that $x(t)$ is in fact a constant function. Conversely, a constant function is clearly in $\ker(L)$. So we see that $\ker(L) = \mathbf{R}$. Next, define a bounded linear functional $Q : Y \to \mathbf{R}$ by

$$Q(y(t)) = \int_0^1 y(t)\,dt.$$

We shall show that $\mathrm{Im}\,(L) = \ker(Q)$. It then follows that coker $= \mathbf{R}$ and that $\mathrm{Index}\,(L) = \dim\ker(L) - \dim \mathrm{coker}\,(L) = 1 - 1 = 0$. Suppose that $y \in \mathrm{Im}\,(L)$ so that $y(t) = x''(t)$ for some $x \in X$. We then have

$$Q(y) = \int_0^1 y(t)\,dt = \int_0^1 x''(t)\,dt = x'(1) - x'(0) = 0.$$

Thus $\operatorname{Im}(L) \subset \ker(Q)$. Conversely, for $y(t) \in \ker(Q)$ (so that $\int_0^1 y(t)\,dt = 0$) we define

$$x(t) = \int_0^t \int_0^\tau y(s)\,ds\,d\tau.$$

It is easy to see that $x(t) \in X$ and $L(x) = x'' = y$. Thus, L is a Fredholm map of index zero as promised. Next, define a (nonlinear) map $N : X \to Y$ by

$$N(x(t)) = -\tilde{g}(t, x(t), x'(t), x''(t)).$$

Now, the problem (N) has a solution $x(t)$ if and only if $Lx - Nx = \tilde{f}$ and $x \in X$.

For the following lemma the reader is asked to recall condition (b) of Theorem 3.1.

Lemma 3.1. *There exist numbers M, $\varepsilon > 0$, such that if condition* (b) *of Theorem* 3.1 *holds and* $\max\{p_0, p_1, p_2\} < \varepsilon$, *then every solution* $x(t)$ *of the problem*

$$Lx - \lambda N x = \lambda \tilde{f}, \qquad \lambda \in (0,1)$$

satisfies $||x||_2 \leq M$.

Proof of Lemma. Let $Lx - \lambda Nx = \lambda\tilde{f}$ for $x(t) \in X$, i.e.,

$$x''(t) + \lambda\tilde{g}(t, x(t), x'(t), x''(t)) = \lambda\tilde{f}. \tag{3.1}$$

Integrating this identity, we have

$$\int_0^1 \tilde{g}(t, x(t), x'(t), x''(t))\,dt = \int_0^1 \tilde{f}(t)\,dt.$$

Now, using condition (b) of Theorem 3.1, we have

$$\begin{aligned}|\tilde{g}(t, x(t), x'(t), x''(t)| &\leq \tilde{g}(t, x(t), x'(t), x''(t)) + p_0\left|x(t) + \frac{t^2}{2}(B-A) + At\right| \\ &\quad + p_1|x'(t) + t(B-A) + A| + p_2|x''(t) + B - A| + p \\ &\leq \tilde{g}(t, x(t), x'(t), x''(t)) + p_0|x(t)| + p_1|x'(t)| \\ &\quad + p_2|x''(t)| + p_3\end{aligned}$$

where p_3 is some constant. Integrating again and using (3.1), we get

$$\begin{aligned}\int_0^1 |\tilde{g}(t, x(t), x'(t), x''(t))|\, dt &\le \int_0^1 \tilde{g}(t, x(t), x'(t), x''(t))\, dt \\ &\quad + p_0||x||_0 + p_1||x'||_0 + p_2 \int_0^1 |x''(t)|\, dt + p_3 \\ &= p_0||x||_0 + p_1||x'||_0 \\ &\quad + p_2 \int_0^1 |x''(t)|\, dt + p_3 + \int_0^1 \tilde{f}\, dt \\ &\le p_0||x||_0 + p_1||x'||_0 + p_2 \int_0^1 |x''(t)|\, dt + p_4\end{aligned} \tag{3.3}$$

where p_4 is some constant.

Now, by (3.1), we have

$$\int_0^1 |x''(t)|\, dt \le \int_0^1 |\tilde{g}(t, x(t), x'(t), x''(t)|\, dt + ||\tilde{f}||_0.$$

Combining this inequality with (3.3), we get

$$\int_0^1 |x''(t)|\, dt \le p_0||x||_0 + p_1||x'||_0 + p_2 \int_0^1 |x''(t)|\, dt + p_5$$

for some constant p_5. Now, since $x'(0) = 0$, we have $x'(t) = \int_0^t x''(\tau)\, d\tau$, so that $||x'||_0 \le \int_0^1 |x''(t)|\, dt$, and we get

$$\int_0^1 |x''(t)|\, dt \le p_0||x||_0 + p_1 \int_0^1 |x''(t)|\, dt + p_2 \int_0^1 |x''(t)|\, dt + p_5.$$

Now we may assume that $p_1, p_2 < 1/4$, and so it then follows that

$$\int_0^1 |x''(t)|\, dt \le \frac{p_0}{1 - p_1 - p_2}||x||_0 + \frac{p_5}{1 - p_1 - p_2}. \tag{3.6}$$

Claim. *There is a number, r_0, such that for each solution $x(t)$ to $Lx + \lambda Nx = \lambda\tilde{f}$, $0 < \lambda < 1$, there is a $z \in [0, 1]$ with $|x(z)| \le r_0$. Here z may depend on $x(t)$.*

The proof may be found in [**15**], but for the sake of completeness we give the proof here. Suppose that, for each positive integer n there is a $\lambda_n \in (0,1)$ and a solution x_n of $Lx + \lambda_n Nx = \lambda_n \tilde{f}$ with $x_n(t) \geq n$ for $t \in [0,1]$. Then we would have

$$\int_0^1 Nx_n(t)\,dt = \int_0^1 \tilde{f}(t)\,dt.$$

In other words,

$$\int_0^1 \tilde{g}(t, x_n(t), x_n'(t)', x_n''(t))\,dt = \int_0^1 \tilde{f}(t)\,dt.$$

On the other hand, we also have $\lim_{n\to\infty} \tilde{g}(t, x_n(t), x_n'(t), x_n''(t)) \geq \mu_+(t)$. Now, using this and Fatou's lemma, we get

$$\int_0^1 \tilde{f}(t)\,dt \geq \int_0^1 \mu_+(t)\,dt$$

contradicting condition (c). Thus, there is a number r_1 such that if x is a solution of $Lx + \lambda Nx = \lambda\tilde{f}$, $\lambda \in (0,1)$, then there is a number $s_1 \in [0,1]$ with $x(s_1) \leq r_1$.

Similarly, by using μ_- and Fatou's lemma, we can show that there must be a number $r_2 > 0$ such that, for any solution x there is a corresponding value $s_2 \in [0,1]$ with $x(s_2) \geq -r_2$. By continuity we conclude that, for any solution x, there is some $z_x \in [0,1]$ with $|x(z_x)| \leq r_0$, $r_0 = \max\{r_1, r_2\}$. Now, for any given solution $x(t)$, we have

$$x(t) = x(z_x) + \int_{z_x}^t x'(\tau)\,d\tau$$

so that

$$||x||_0 \leq r_0 + ||x'||_0 \leq r_0 + \int_0^1 |x''(t)|\,dt. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\int_0^1 |x''(t)|\,dt \leq \frac{p_0}{1-p_1-p_2}\left(r_0 + \int_0^1 |x''(t)|\,dt\right) + \frac{p_5}{1-p_1-p_2}.$$

Now there exists an $\varepsilon > 0$ such that, when $p_0, p_1, p_2 < \varepsilon$, then $\int_0^1 |x''(t)|\, dt \le p_6$ for some fixed constant p_6. Therefore, we also have that, when $p_0, p_1, p_2 < \varepsilon$,

$$||x'||_0 \le \int_0^1 |x''(t)|\, dt \le p_6 \tag{3.8}$$

and

$$||x||_0 \le r_0 + ||x'||_0 \le p_6 + r_0. \tag{3.9}$$

Now, given such a solution of $x'' + \lambda\tilde{g}(t, x, x', x'') = \lambda\tilde{f}(t)$, we have, using (3.8), (3.9) and condition (a),

$$\begin{aligned} |x''(t)| &\le |\tilde{g}(t, x, x', x'')| + |\tilde{f}(t)| \\ &\le |\tilde{g}(t, x, x', x'') - \tilde{g}(t, x, x', 0)| \\ &\quad + |\tilde{g}(t, x, x', 0)| + |\tilde{f}(t)| \\ &\le k|x''(t)| + |\tilde{g}(t, x, x', 0)| + |\tilde{f}(t)| \\ &\le k|x''(t)| + p_7. \end{aligned}$$

Here p_7 is some constant. From this we see that $||x''||_0 \le p_7/(1-k)$ so that if we let $M = \max\{p_6, p_7 + r_0, p_7/(1-k)\}$, then

$$||x||_2 \le M$$

for some sufficiently small ε and $p_0, p_1, p_2 < \varepsilon$.

Lemma 3.2.

$$l(L) \ge 1.$$

Proof. Let $A \subset Z$ be a bounded subset, and let $a = \Gamma_Y(L(A)) > 0$. Given $\varepsilon > 0$, according to the definition, there is a finite number of subsets A_i of A such that $\operatorname{diam}_0(L(A_i) \le a + \varepsilon$. Since X is compactly embedded into $C^1([0,1])$ with norm $||\cdot||_1$, and since A_i are bounded in X, it follows that there is a finite number of subsets A_{ij} of A_i such that $\operatorname{diam}_1(A_{ij}) < \varepsilon$ and, hence, $\operatorname{diam}_2(A_{ij}) \le a + \varepsilon$, where

$\operatorname{diam}_0(\cdot), \operatorname{diam}_1(\cdot)$ and $\operatorname{diam}_2(\cdot)$ are defined with respect to the norms $|| \ ||_0$, $|| \ ||_1$ and $|| \ ||_2$, respectively. This proves

$$\Gamma_X(A) \le a = \Gamma_Y(L(A)).$$

That is, $l(L) \ge 1$. $\square$

Lemma 3.3. *$N : X \to Y$ is a k-set-contractive map with k as in condition* (a).

Proof. Let $A \subset X$ be a bounded subset, and let $a = \Gamma_X(A)$. Then, for any $\varepsilon > 0$, there is a finite family of subsets $\{A_i\}$ with $A = \cup_i A_i$ and $\operatorname{diam}_2(A_i) \le a + \varepsilon$. Now it follows from the fact that $\tilde{g}$ is uniformly continuous on any compact subset of $\mathbf{R} \times \mathbf{R}^3$, and from the fact that A and A_i are precompact in $C^1([0,1])$ with norm $|| \cdot ||_1$, that there is a finite family of subsets $\{A_{ij}\}$ of A_i such that $A_i = \cup_j A_{ij}$ with

$$|\tilde{g}(t, x, x', u'') - \tilde{g}(t, u, u', u'')| < \varepsilon$$

for any $x, u \in A_{ij}$. Therefore, for $x, u \in A_{ij}$, we have

$$\begin{aligned} ||Nx - Nu||_0 &= \sup_{0 \le t \le 1} |\tilde{g}(t, x(t), x'(t), x''(t)) - \tilde{g}(t, u(t), u'(t), u''(t))| \\ &\le \sup_{0 \le t \le 1} |\tilde{g}(t, x(t), x'(t), x''(t)) - \tilde{g}(t, x(t), x'(t), u''(t))| \\ &\quad + \sup_{0 \le t \le 1} |\tilde{g}(t, x(t), x'(t), u''(t)) - \tilde{g}(t, u(t), u'(t), u''(t))| \\ &\le k||x'' - u''||_0 + \varepsilon \le ka + (k+1)\varepsilon. \end{aligned}$$

That is,

$$\Gamma_Y(N(A)) \le k\Gamma_X(A).$$

We are now in a position to finish the proof of Theorem 3.1. We apply Theorem 2.2 for $\Omega = \{x \in X \mid ||x||_0 < r\}$. It is easy to see now that all of the necessary conditions in Theorem 2.2 hold except for condition (b). We will now show that condition (b) also holds. Define a bounded bilinear form $[\cdot, \cdot]$ on $Y \times X$ by $[y, x] = \int_0^1 y(t)x(t)\,dt$. Also, define $Q : Y \to \operatorname{coker}(L)$ by $y \mapsto \int_0^1 y(t)\,dt$. Notice that, for

$x \in \ker(L) \cap \partial\Omega$, we must have $x = r$ or $x = -r$ so that, for such an x,

$$[QN(x)+Q\tilde{f}, x] \cdot [QN(-x)+Qy, x] = r^2 \int_0^1 (\tilde{g}(t, r, 0, 0) - \tilde{f}(t))\,dt \cdot \int_0^1 (\tilde{g}(t, -r, 0, 0) - \tilde{f}(t))\,dt.$$

By condition (c), there is a number $M_1 > 0$ such that, if $r > M_1$, then

$$\int_0^1 (\tilde{g}(t, r, 0, 0) - \tilde{f}(t))\,dt \cdot \int_0^1 (\tilde{g}(t, -r, 0, 0) - \tilde{f}(t))\,dt < 0.$$

Thus, if we pick $r > \max\{M, M_1\}$, then all of the conditions required in Theorem 2.2 hold. It now follows by Theorem 3.2 that there is a function $x(t) \in X$, such that

$$Lx - Nx = \tilde{f}.$$

This finishes the proof of Theorem 3.1. □

Example. Let $g(t, x, x_1, x_2) = h_1(t, x_1, x_2)e^{-x^2} + h_2(t, x_1)e^x$, here $h_1 \geq 0$ and $h_2 > 0$ are bounded continuous functions and h_1 satisfies $\sup |(\partial/\partial x_2)h_1(t, x, x_1, x_2)| < 1$. (For example, $h_1(t, x, x_1, x_2) = (1/4)\sin^2(x_2)e^{-x^2}$.) By Theorem 3.1, the Neumann boundary value problem

$$x'' + h_1(t, x_1, x_2)e^{-x^2} + h_2(t, x_1)e^x = f(t)$$
$$x'(0) = A, \qquad x'(1) = B$$

has solutions provided $\int_0^1 f(t)\,dt > B - A$.

4. The mixed problems (M$_1$) and (M$_2$).

In this section we consider the mixed problems

$$(\mathrm{M}_1) \qquad x'' + g(t, x, x', x'') = 0, \qquad x(0) = A, \qquad x'(1) = B$$

and

$$(\mathrm{M}_2) \qquad x'' + g(t, x, x', x'') = 0, \qquad x'(0) = A, \qquad x(1) = B.$$

These are equivalent to

$$(\mathrm{M}_1') \qquad \begin{gathered} x'' + g(t, x(t) + tB + A, x'(t) + B, x''(t)) = 0, \\ x(0) = x'(1) = 0 \end{gathered}$$

and

$$(\mathrm{M}_2') \qquad \begin{gathered} x'' + g(t, x(t) + tA + B - A, x'(t) + A, x''(t)) = 0, \\ x'(0) = x(1) = 0. \end{gathered}$$

Theorem 4.1. *Let $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ be a continuous map, and assume that*

(a) *There is a $k \in [0,1)$ such that $|g(t,x,x_1,p) - g(t,x,x_1,q)| \leq k|p-q|$ for $p, q \in \mathbf{R}$.*

(b) *There are constants L_i, $i = 1,2,3,4$, such that $L_4 < L_3 \leq B \leq L_1 < L_2$ and*

$$g(t,x,x_1,x_2) \leq 0 \quad \textit{for } (t,x_1) \in [0,1] \times [L_1, L_2]$$

and

$$g(t,x,x_1,x_2) \geq 0 \quad \textit{for } (t,x_1) \in [0,1] \times [L_4, L_3].$$

Then there is at least one solution $x \in C^2([0,1])$ to the problem (M_1).

Remark. Similar conditions were used in [**9**].

Proof. Let

$$X = \{x(t) \in C^2([0,1]) \mid x'(0) = x(1) = 0\}$$

with norm $||\cdot||_2$, and let $Y = C([0,1])$ with norm $||\cdot||_0$, X and Y are Banach spaces. If we let $T : X \to Y$ be defined by $x(t) \mapsto x''(T)$, then $\ker(T) = \{0\}$ and $\operatorname{coker}(T) = \{0\}$ so that T is an invertible linear bounded operator. In a manner similar to the method used in the proof of Theorem 3.1, one can prove that $l(T) \geq 1$. Now write

$$\tilde{g}(t,x,x_1,x_2) = g(t, x + tB + A, x_1 + B, x_2),$$

and let $N : X \to Y$ be defined by

$$x(t) \mapsto \tilde{g}(t, x(t), x'(t), x''(t)).$$

As in the proof of Theorem 3.1, N can easily be shown to be a k-set-contractive map. We will use Theorem 2.1, and so we need to show that there is an open, bounded subset $\Omega \subset X$ which is symmetric about $0 \in \Omega$, and such that $Lx + \lambda Nx \neq 0$ has no solution on $\partial\Omega$ with $\lambda \in (0,1)$. We will show that, for solution $x(t)$,

$$L_3 - B \leq x'(t) \leq L_1 - B, \qquad t \in [0,1].$$

That is, $L_3 \leq x'(t) + B \leq L_1$ for $t \in [0,1]$. Suppose, by way of contradiction, that there is a $t_0 \in [0,1)$ such that $x'(t) + B > L_1$. Since $x'(t)$ is continuous and $x'(1) + B = B < L_1$, we may assume that $L_1 < x'(t_0) + B < L_2$. Also, by the continuity of $x'(t)$, there is a $\delta > 0$ such that

$$L_1 < x'(t) + B < L_2$$

for $t \in (t_0 - \delta, t_0 + \delta)$. In fact, since $x'(1) = 0$ there must be a $t_0' \in (t_0, t_0 + \delta)$ such that

$$x'(t_0') < x'(t_0). \tag{4.1}$$

But, when $t \in (t_0, t_0 + \delta)$, we have $L_1 < x'(t) + B < L_2$, and so it follows that

$$g(t, x(t) + tB + A, x'(t) + B, x''(t)) \leq 0,$$

and hence, $x''(t) \geq 0$. This means that $x'(t_0') \geq x'(t_0)$ which contradicts (4.1). Therefore, we must have $x'(t) + B \leq L_1$ for $t \in [0,1]$.

Similarly, one can show that $x'(t) + B \geq L_3$. Hence,

$$||x'||_0 \leq C_1 \tag{4.2}$$

Now, $x(0) = 0$ and $x(t) = \int_0^\tau x'(\tau)\, d\tau$, so in fact

$$||x||_0 \leq C_1. \tag{4.3}$$

Finally,

$$\begin{aligned} |x''(t)| &\leq |g(t, x(t) + tB + A, x'(t) + B, x''(t))| \\ &\leq k|x''(t)| + |g(t, x(t) + tB + A, x'(t) + B, 0)| \end{aligned}$$

so that

$$||x''||_0 \le C_2. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we see that $||x||_2 \le M < +\infty$.

Let $\Omega = \{x \in X \mid ||x||_2 < M+1\}$. Then

$$Lx + \lambda Nx \ne 0$$

for $x \in \partial\Omega$, $\lambda \in (0,1)$. Applying Theorem 2.1, we finish the proof of Theorem 4.1. □

Theorem 4.2. *Let* $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ *be continuous, and assume that*

a) $|g(t,x,x_1,p) - g(t,x,x_1,q)| \le k|p-q|$ *for a constant* $k \in [0,1)$ *and* $(t,x,x',x_1,p),(t,x,x_1,q) \in [0,1] \times \mathbf{R}^3$.

b) *There are constants* L_1, L_2, L_3 *and* L_4 *such that* $L_4 < L_3 \le A \le L_1 < L_2$ *and*

$$g(t,x,x_1,x_2) \ge 0 \quad \textit{for } (t,x_1) \in [0,1] \times [L_1,L_2]$$

and

$$g(t,x,x_1,x_2) \le 0 \quad \textit{for } (t,x_1) \in [0,1] \times [L_4,L_3].$$

Then the boundary value problem (M_2) *has at least one solution in* $C([0,1)]$.

Proof. The proof is entirely similar to that of Theorem 4.1. □

5. The Dirichlet problem (D). In this section we consider the Dirichlet problem,

$$x'' = g(t,x(t),x'(t),x''(t)), \qquad x(0) = A, \qquad x(1) = B. \tag{D}$$

This is equivalent to

$$x''(t) = g(t, x(t) + t(B-A) + A, x'(t) + B - A, x''(t)) \tag{D$'$}$$

with the homogeneous boundary conditions $x(0) = 0$, $x(1) = 0$.

Theorem 5.1. *Let* $g : [0,1] \times \mathbf{R}^3 \to \mathbf{R}$ *be continuous, and suppose that*

a) $|g(t,x,x-1,p) - g(t,x,x_1,q)| \leq k|p-q|$, $0 \leq k < 1$.

b) *There exist constants* L_i, $i = 1,\dots,8$, *such that* $L_2 > L_1 \geq c$, $L_4 > L_3 \geq c$, $L_5 < L_6 \leq c$, $L_7 < L_8 \leq c$, *where* $c = B - A$, *and such that*

$$g(t,x,x_1,x_2) \geq 0$$

for

$$(t,x,x_1,x_2) \in [0,1] \times \mathbf{R} \times ([L_1,L_2] \cup [L_5,L_6]) \times \mathbf{R}$$

and

$$g(t,x,x_1,x_2) \leq 0$$

for

$$(t,x,x_1,x_2) \in [0,1] \times \mathbf{R} \times ([L_3,L_4] \cup [L_7,L_8]).$$

Then the Dirichlet problem (D) *has at least one solution in* $C^2([0,1])$.

Proof. Let

$$X = \{x \in C^2([0,1]) \mid x(0) = x(1) = 0\}$$

and $Y = C^1([0,1])$. Define $T : X \to Y$ by $x(t) \mapsto x''(t)$. Then T is an invertible bounded linear map and $l(T) \geq 1$. Define $N : X \to Y$ by

$$x(t) \mapsto g(t, x(t) + t(B-A) + A, x'(t) + B - A, x''(t)).$$

N is easily seen to be a k-set-contractive map. Now there exists an M such that, if $x(t) \in C^2([0,1])$ is a solution to $Tx = \lambda N(x)$ for $\lambda \in (0,1)$, then $||x||_2 \leq M$. For this, see Theorem 4.1 of [**9**]. We can now use Theorem 2.1 to prove the theorem. For completeness, we state one last theorem whose proof uses methods exactly similar to that of Theorem 5.1 above.

Theorem 5.2. *Let* $g : [0,1] \times |R^3 \to \mathbf{R}$ *be continuous, and assume that*

a) $|g(t,x,x_1,p) - g(t,x,x_1,q)| \leq K|p-q|$;

b) *there is a constant* $M \geq 0$ *such that* $xg(t,x,0) \geq 0$ *for* $|x| > M$;

c) *for the constants* L_i, $i = 1, \dots, 8$ *of Theorem* 5.1,

$$g(t, x, x_1, x_2) \geq 0,$$

for $(t, x, x_1, x_2) \in [0,1] \times [-M, M] \times ([L_1, L_2] \cup [L_5, L_6])$ *and*

$$g(t, x, x_1, x_2) \leq 0$$

for $(t, x, x_1, x_2) \in [0,1] \times [-M, M] \times ([L_3, L_4] \cup [L_7, L_8])$, *then the Dirichlet problem* (D) *has a solution in* $C^2([0,1])$.

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