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ALMOST RECONSTRUCTION OF THE 3-DIMENSIONAL BALL FROM $K_{pqrs} \times I$

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ABSTRACT. Let K_{pqrs} denote the 2-complex obtained by attaching two disks to the wedge of two circles by the words $a^p b^q$ and $a^r b^s$, with $ps - qr = \pm 1$. The complex K_{pqrs} is contractible. If the Zeeman conjecture is true, then $K_{pqrs} \times I$ is collapsible. This paper proves that $K_{pqrs} \times I$ collapses to a 2-sphere S^2 plus its interior. The proof exhibits K_{pqrs} as the spine of a 3-ball under a retraction map whose restriction to the boundary S^2 lifts to an embedding into $K_{pqrs} \times I$.

1. Introduction. Let K_{pqrs} denote the 2-complex obtained by attaching two disks to the wedge of two circles by the words $a^p b^q$ and $a^r b^s$, with $ps - qr = \pm 1$. The set K_{pqrs} is contractible. Let I denote a unit interval. In [**3**], Zeeman posed the question: Is $K_{pqrs} \times I$ collapsible? Zeeman proved that the answer is affirmative for K_{1112}, the "topological dunce hat." In [**2**], Lickorish provided an affirmative answer for K_{2334}, but observed that his methodology did not seem to generalize to all K_{pqrs}. In this paper, we display a structure on all K_{pqrs} which seems to correspond to the initial phase of Lickorish's collapse of $K_{2334} \times I$, and which is valid for all K_{pqrs}: The product $K_{pqrs} \times I$ collapses to a 2-*sphere plus its interior*.

Definition. Let S^2 denote a piecewise linear 2-sphere in a contractible 3-complex K^3. A point p of $K^3 - S^2$ lies in the *interior* of S^2 if S^2 is nontrivial in $H_2(K^3 - p; Z_2)$, the second homology group of $K^3 - p$ with Z_2-coefficients.

This definition is commonly used in the special case of a 2-sphere plus its interior in Euclidean 3-space (the three-dimensional PL Schoenflies theorem). In this paper we concern ourselves with constructing a 2-sphere plus its interior in the set $K_{pqrs} \times I$.

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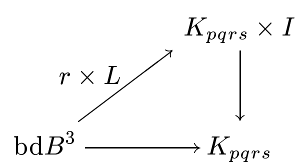
Theorem. *The product $K_{pqrs} \times I$ collapses to a 2-sphere S^2 plus its interior.*

Caution. This 2-sphere plus interior is in general not a 3-manifold. It is only a 3-complex whose homological boundary, $\bmod 2$, is S^2. This phenomenon may be easily viewed in analogy one dimension lower: Let K^2 be a book with four pages. Let X be all of page 1, the bottom 2/3 of page 2, the top 2/3 of page 3 and the middle 1/3 of page 4. Then X is a 1-sphere plus interior which is not a 2-manifold.

Outline of proof. The key to our argument is the fact that K_{pqrs} is a spine of the 3-ball B^3. Thus, B^3 retracts to K_{pqrs}. Details will be provided in Section 2. Consider the restriction r of this retraction map to the 2-sphere $\mathrm{bd}B^3$. We show how r lifts to an *embedding* $r \times L$ of $\mathrm{bd}B^3$ into $K_{pqrs} \times I$. In particular, the map L from $\mathrm{bd}B^3$ to I separates points of $\mathrm{bd}B^3$ with the same r image.



The 2-sphere $r \times L(\mathrm{bd}B^3)$ turns out to be the desired S^2 in $K_{pqrs} \times I$; that is, $K_{pqrs} \times I$ collapses to $S^2 + \mathrm{Int}\, S^2$.

The construction of the lifting map L is tricky. It is done in pieces, by viewing a regular neighborhood of K_{pqrs} in the 3-ball as a handlebody with one 0-handle, two 1-handles, and two 2-handles. This handle structure corresponds to the single vertex, two loops and two disks of K_{pqrs}. The lift L is constructed on the individual pieces provided by this handle structure. Details will be provided in Sections 3–7. The collapse is done in Section 8.

We believe that the set $S^2 + \mathrm{Int}\, S^2$ produced by the above theorem further collapses to a point. An ambitious approach to this problem is the following:

Conjecture. *Let K denote a contractible* 2*-complex. Let S denote a* 2*-sphere in $K \times I$. If $S + \mathrm{Int}\, S$ is Q-trivial* (*trivial in rational homology*), *then* $S + \mathrm{Int}\, S$ *is collapsible.*

This conjecture would not only prove that $K_{pqrs} \times I$ collapses, but, by using [**1**], it would prove the Poincaré conjecture as well! Note that the conjecture does not imply the Zeeman conjecture. Indeed, we believe that if the Zeeman conjecture is false, then the product of any counterexample to the Zeeman conjecture with an interval does not collapse to a 2-sphere plus its interior.

A less ambitious way to prove that $K_{pqrs} \times I$ collapses to a point would be to use our explicit description of $S^2 + \mathrm{Int}\ S^2$ in $K_{pqrs} \times I$. The goal would be to further collapse $S^2 + \mathrm{Int}\, S^2$ to a 3-ball. Of course, the 3-ball would in turn collapse to a point. This plan, which we call "reconstruction" of the 3-ball from $K_{pqrs} \times I$, seems feasible but involved.

For example, the 2-sphere plus interior obtained from $K_{1223} \times I$ is not a 3-manifold. Does this 2-sphere plus interior collapse to a 3-ball? An affirmative answer could well yield a general methodology which would collapse all $K_{pqrs} \times I$. On the other hand, a negative answer would also be of interest, in that it would restrict the setting in which one could generalize the PL Schoenflies theorem beyond 3-space.

2. Retraction of the 3-ball onto K_{pqrs}. Regard S^3 as the join

$$S^1 * S^1 = S^1 \times S^1 \times I / \{(x, y, 0) \sim (x', y, 0), (x, y, 1) \sim (x, y', 1)\}.$$

The torus $T = S^1 \times S^1 \times 1/2$ determines a Heegaard splitting of S^3, and there is a standard pair a, b of curves bounding disks in the respective halves of the splitting. There are simple closed curves J_1 and J_2 in T representing $a^p b^q$ and $a^r b^s$, respectively, and meeting transversely in a single point x_0. The image K'_{pqrs} of $(J_1 \cup J_2) \times I$ in S^3 becomes K_{pqrs} in the reduced join obtained by collapsing $x_0 \times I$ to a point.

Take a disk D in $T - (J_1 \cup J_2)$. The closure of the complement of $D \times [\varepsilon, 1 - \varepsilon]$ is a 3-ball regular neighborhood N of K_{pqrs} which naturally retracts onto K_{pqrs} in this join structure.

We proceed to make this retraction explicit and to study the restriction r of this retraction to $\mathrm{bd} N = \mathrm{bd} B^3$. This study begins locally around the vertex of K_{pqrs} and proceeds then to the 1- and 2-skeletons. In particular, in the course of defining the retraction and the lift L, we construct abstractly a manifold M homeomorphic to B^3 that plays the role of N above. We encourage the reader to view our construction in terms of the explicit embedding just defined.

3. The link of the vertex of K_{pqrs}. The theorem is trivially true if $p = 0$. Thus, we restrict the discussion to the nontrivial cases, where p, q, r and s are all nonzero. Without loss of generality, we may assume that they are all positive integers.

Regard K_{pqrs} as a CW-complex with a single vertex v. The link of v in K_{pqrs}, denoted lkv, is a graph (a 1-complex) with four vertices. Abstractly we may construct K_{pqrs} from lkv in three steps. First, we embed lkv in the cone over lkv. Second, we attach a book with $p + r$ pages correctly between two of the vertices of lkv and a book with $q + s$ pages correctly between the other two vertices. The boundary of this structure consists of two disjoint closed curves. To these we attach the two open disks.

We now describe lkv, not abstractly, but as a subset of the plane. See Figure 1. This graph has two vertices of degree $p + r$, denoted as a_{out} and a_{in}, which we place in the left half-plane. Similarly, we place the two vertices of degree $q + s$, denoted as b_{out} and b_{in} in the right half-plane.

We add a point to the plane, so that lkv is now viewed as a subset of a 2-sphere, which we denote as $S(lkv)$. The cone over $S(lkv)$ is a three-dimensional ball. By regarding the cone point as v, this places the star of the vertex v in K_{pqrs}, denoted star (v), in a natural way in the 3-ball. This 3-ball is the 0-handle of our handlebody construction.

4. Construction of L on the link of the vertex of K_{pqrs}. The boundary of a regular neighborhood $N(lkv)$ of lkv in $S(lkv)$ is a collection J of simple closed curves in one-to-one correspondence with the regions of $S(lkv)$ defined by lkv. We fix such a neighborhood. When we have finished construction of the manifold M with K_{pqrs} as spine, then $\text{bd}M \cap S(lkv)$ will be $S(lkv) - \text{int}\,(N(lkv))$. In what follows, to avoid notational difficulties we view the unit interval I as identical to the interval $[0, \text{MAX}]$, for some sufficiently large MAX.

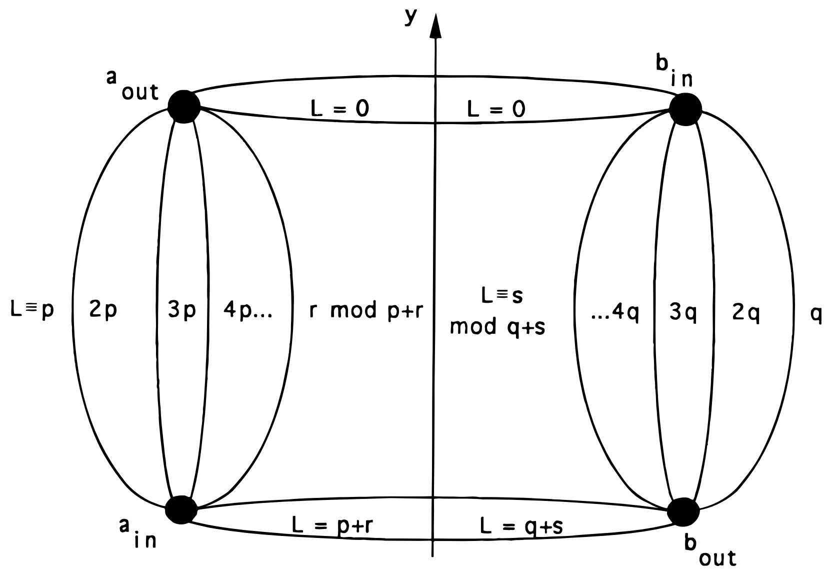


FIGURE 1. The link of the vertex of K_{pqrs}, showing the labelling pattern for the lift L.

Consider a natural retraction ρ carrying $N(lkv)$ to lkv. We begin our construction of the lifting function L by defining L on the finite set $\rho^{-1}(\{a_{\text{out}}, a_{\text{in}}, b_{\text{out}}, b_{\text{in}}\}) \cap J$. This is done in Figure 1.

The set $\rho^{-1}(a_{\text{out}}) \cap J$, denoted A_{out}, consists of $p+r$ points, which will be mapped by L onto the integers $0, 1, 2, \ldots, p+r-1$. The set $\rho^{-1}(a_{\text{in}}) \cap J$, denoted A_{in}, consists of $p+r$ points, which will be mapped by L onto the integers $1, 2, \ldots, p+r$.

In the left half-plane, we begin by lifting a single point of A_{out} to 0, and a single point of A_{in} to $p+r$, as indicated in Figure 1. Specifically, the point of A_{out} which is lifted by the function L to 0 may be described in Cartesian coordinates by adding a small positive number to the x-coordinate of the point a_{out}, and the point of A_{in} which is lifted by L to $p+r$ is described by adding a small positive number to the x-coordinate of the point a_{in}. These points belong to the finite regions bounded by the arc pairs joining a_{out} to b_{in}, and b_{out} to a_{in}. The numbers $1, 2, \dots, p + r - 1$ are then added to Figure 1 by enumerating, modulo $p + r$, around either vertex in the left half-plane. The enumeration proceeds counterclockwise in the pattern $\{0, p, 2p, \dots, (p + r - 1)p\} \bmod p + r$, around a_{out}. Observe that $(p + r - 1)p$ reduces, $\bmod p + r$, to the integer r. Similarly, this enumeration, $\bmod p + r$, is performed clockwise in the same pattern around a_{in}. Because p is relatively prime to $p + r$, this enumeration uses each of the integers $1, 2, \dots, p + r - 1$ exactly once around each vertex.

Analogously in the right half-plane, the lift L is constructed by first lifting a single point of $B_{\text{out}} = \rho^{-1}(b_{\text{out}}) \cap J$ to $q+s$ and a single point of $B_{\text{in}} = \rho^{-1}(b_{\text{in}}) \cap J$ to 0. The numbers $1, 2, \dots, q+s-1$ are then added by enumerating, modulo $q+s$, as shown in Figure 1. This enumeration proceeds in the pattern $\{q, 2q, 3q, \dots, s\} \bmod q + s$, counterclockwise around b_{out}, and clockwise in the same pattern around b_{in}.

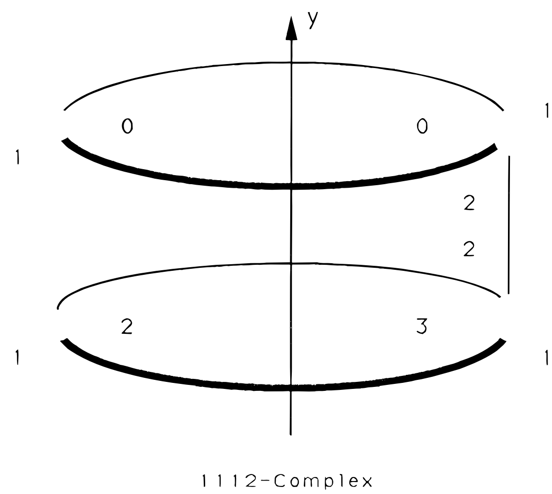
The lifting function L is now extended to the simple closed curves comprising J. The lift L may be defined to be constant on components of J, except for a neighborhood N of the y-axis. On N, we extend L linearly, in order to yield continuity between the left and right halves of this construction.

This yields an embedding of J in the product 2-complex $lkv \times I$: two points of J map to the same point of lkv only if they belong to different member curves, but such points will have distinct lifts, even near the y-axis, as long as p, q, r and s are all greater than 0.

Figure 2 illustrates this structure for the 1112-complex (the dunce hat) and for the 2334-complex.

The set lkv separates K_{pqrs} into two complementary domains, so our problem of extending this lift L is divided into two subproblems. We begin with the complementary domain containing the vertex v.

5. Extending the lift L over the cone point v. The set J consists of $p + q + r + s - 2$ simple closed curves. Select a point w_i in the interior of each curve J_i of J, i.e., in the open region defined by J_i which excludes lkv. Since we have made no assumptions about ρ, we now assume that ρ maps each curve J_i of J radially from the interior point w_i to lkv. Extend the retraction ρ to the retraction map



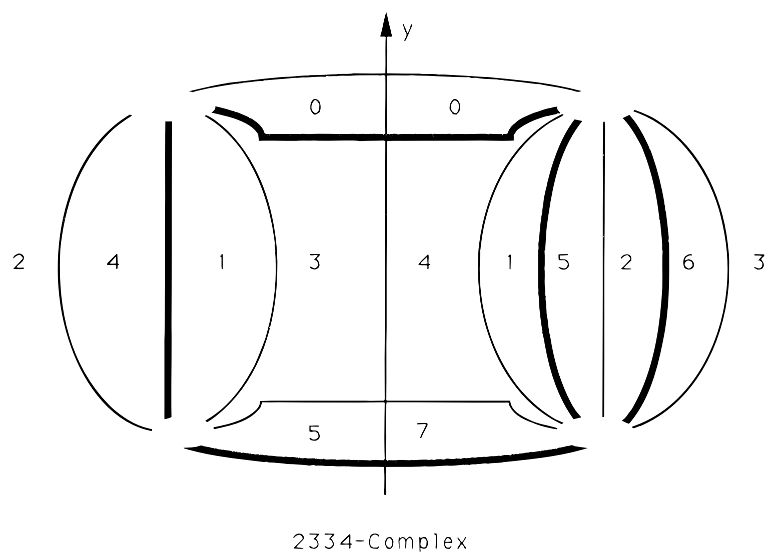


FIGURE 2. The link of the vertex of K_{1112} and K_{2334}, showing the boundary J of a neighborhood of the link, and the labelling for the lift L.

of the 3-ball given by the cone over the sphere S onto star (v) such that $\rho^{-1}(v) \cap S$ consists of the $p+q+r+s-2$ points w_i, with ρ extended linearly between the w_i and the surrounding simple closed curve J_i of J. We will define L on the w_i, then extend linearly to $S - N(lkv)$ using the existing definition on the J_i. Unfortunately, we cannot simply map each w_i to some value of L corresponding to the surrounding curve of J, because there are pairs of curves with the same L-value.

To circumvent this difficulty, we may choose any total ordering $J_1, J_2, \dots, J_{p+q+r+s-2}$ of the simple closed curves comprising J and then define $L(w_i) = i$. The linear extension of such an L over $S - N(lkv)$ need not yield a one-to-one map $\rho \times L$ on $S - N(lkv)$. We now show how to pick an ordering for which it will.

The lifting map L already specifies four partial orders on the J_i, one for each of the four vertices of lkv, as follows. Fixing one of these vertices, say x, we define a partial order on J by $J_i < J_j$ if $\rho^{-1}(x)$ meets J_i and J_j, say at y_i and y_j, and if $L(y_i) < L(y_j)$. We call a total ordering extending all of these partial orderings *compatible*. Any compatible ordering will serve our purpose.

We address the issue of compatible orderings by considering a directed graph G related to the labelings and associated partial orders defined around the four vertices of lkv. The $p+q+r+s-2$ points of G are the member curves of J (equivalently, the regions of $S(lkv) - lkv$). An arc of G extends from J_i to J_j if and only if J_i and J_j lie in adjacent regions of $S(lkv) - lkv$ and $J_i < J_j$ in one of the four partial orderings. It is standard that a compatible ordering exists if and only if the directed graph G contains no cycles. In our case G has an obvious structure as the nondisjoint union of two directed paths. Referring to Figure 1, the first path, P_1, arises from the $p+r+1$ regions in the left half-plane, labelled in directed order 0 through $p+r$. The second path, P_2, arises from the $q+s+1$ regions in the right half-plane, labelled 0 through $q+s$. P_1 and P_2 share endpoints. They intersect in one other point if $p = r = 1$ and otherwise in two other points, labelled p and r along P_1, and labelled q and s along P_2. In the first case there is clearly no cycle. In the second case a cycle arises if and only if $p < r$ and $q > s$ or $p > r$ and $q < s$. However, the K_{pqrs} condition that $ps - qr = \pm 1$ guarantees $p < r$ if and only if $q < s$.

We conclude that compatible orderings exist, so consider one such, and define $L(w_i) = i$. Extend L over $S(lkv) - N(lkv)$ linearly, using the existing definition on J.

We now claim that $\rho \times L$ is a one-to-one map on $S(lkv) - \text{int}\,(N(lkv))$. Suppose that $\rho \times L(x) = \rho \times L(y)$. If $x \neq y$, then x and y lie in distinct components of $S - lkv$ whose boundaries share an edge, corresponding, say, to J_i and J_j, with $i < j$. Therefore, $\rho(x) = \rho(y)$ implies there exist f in $[0, 1]$ and z_i and z_j on J_i and J_j, respectively, such that $\rho(z_i) = \rho(z_j)$, $x = (1 - f)w_i + fz_i$ and $y = (1 - f)w_j + fz_j$. Thus, $L(x) = (1 - f)i + fL(z_i) = L(y) = (1 - f)j + fL(z_j)$. But by the ordering of the curves of J, and the definition of L on J, we must have $L(z_i) < L(z_j)$, a contradiction.

6. Attaching the 1-handles. Consider a disk neighborhood D of the point a_{out} in the 2-sphere $S(lkv)$. See Figure 3. We assume that D is contained in $N(lkv)$ and that bd$D$ meets J in a short interval surrounding each point of $J \cap \rho^{-1}(a_{\text{out}})$.

This disk neighborhood is now abstractly viewed as the bottom disk $D \times 0$ of the product $D \times I$, where $I = [0, 1]$, a 1-handle, here casually ignoring our previous extension of I to $[0, \text{MAX}]$. Abstractly, again, we may view the 1-handle as retracting to a book B_{p+r} on $p + r$ pages, such that bd$B_{p+r} \cap D = D \cap lkv$; i.e., each page attaches along one arc of lkv incident at a_{out}. Our problem is to decide how to attach $D \times 1$ to a neighborhood of a_{in}.

The answer is provided for us by extending the lifting function L, already defined on $(\text{bd}D \cap J) \times 0$, to $(\text{bd}D \cap J) \times I$ as follows. We continuously increase all values of the lifting function L by t, $0 \leq t \leq 1$, as we move up the 1-handle, so that on $D \times 1$, all values of the lift L have been increased by 1. Thus, we may attach $D \times 1$ to a disk neighborhood D' of a_{in} such that the L values on $(\text{bd}D \cap J) \times 1$ match the L values on bdD'! See Figure 3. We use the same procedure to attach a second 1-handle to a neighborhood of b_{in} and b_{out} at disk neighborhoods D_1 and D'_1.

Adding two 1-handles to a 0-handle yields a solid torus with two handles, denoted T^3. Assuming, as we may, that all relevant retractions are compatible, we may combine the retractions to the books B_{p+r} and B_{q+s} with the retraction of the 3-ball to star$\,(v)$, yielding a retraction ρ of T^3 onto a two-dimensional spine Y. Does Y have the

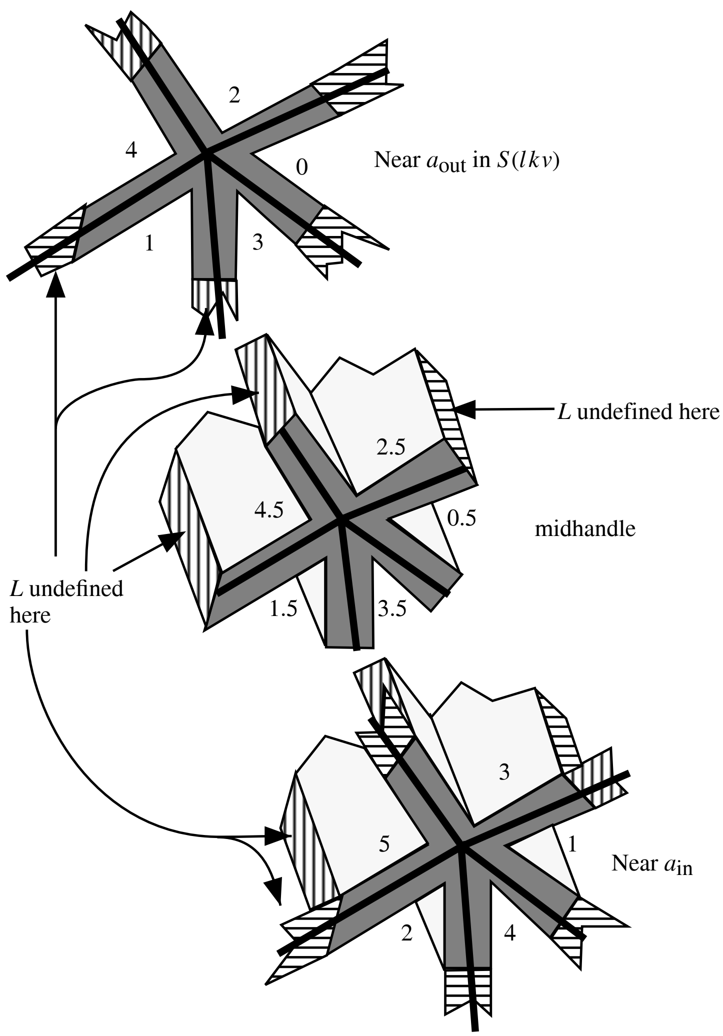


FIGURE 3. Disk neighborhoods of a_{in} and a_{out}, and a cross-section of the connecting 1-handle, with values of the lift L. L is undefined on the interior of the hatched regions.

desired structure? We now verify that our attaching procedure actually produces a spine Y which is a subset of the complex K_{pqrs} in T^3, a "twice punctured K_{pqrs}," that is to say, a neighborhood of the two loops "a" and "b" in K_{pqrs}.

The boundary bdry Y of Y consists of two parts: the part of lkv not inside the disk neighborhoods around its four vertices, and the boundaries of the books B_{p+r} and B_{q+s} not inside those disk neighborhoods. The resulting one complex bdry Y is a disjoint union of closed cycles. For convenience, we now suppose that $lkv \cap Y$ inherits an orientation from counterclockwise orientations of the loops a and b, so that edges are directed from a_{in} towards a_{out} and from b_{in} towards b_{out}. Also direct the edges from a_{in} towards b_{out} and from b_{in} towards a_{out}. Finally direct the book edges in the opposite ways: from a_{out} towards a_{in}, and from b_{out} towards b_{in}. It now suffices to show two things about bdry Y: (1) there is a directed cycle using p book edges from B_{p+r}, $p-1$ edges from a_{in} towards a_{out} in $lkv \cap Y$, one edge from a_{in} towards b_{out} in $lkv \cap Y$, q book edges from B_{q+s}, $p-1$ edges from b_{in} towards b_{out} in $lkv \cap Y$, and one edge from b_{in} towards a_{out} in $lkv \cap Y$; (2) there is a directed cycle disjoint from that of (1) using r book edges from B_{p+r} and s book edges from B_{q+s}.

To illustrate our general procedure, the part of the punctured disk boundary corresponding to $a^p b^q$ and lying in $lkv \cap Y$ appears in bold in Figure 4 for the special cases of K_{1112} and K_{2334}. The other partial edges in Figure 4 represent the punctured disk $a^r b^s$. The general procedure requires some notation. Each edge in Figure 1 is notated as two distinct half-edges. A half-edge can be notated by the vertex from which it emanates and the two values of L which surround the half-edge. We use the same labels for the partial half edges in $lkv \cap Y$. We may now describe the cycle in bdry Y corresponding to the punctured disk $a^p b^q$ in a way which may be tracked in Figure 1. Begin with half-edge $\{a_{\text{out}}, 0, r\}$. Proceed around the 1-handle to $\{a_{\text{in}}, 1, r+1\}$. Return along lkv to half-edge $\{a_{\text{out}}, 1, r+1\}$. Proceed around the 1-handle to $\{a_{\text{in}}, 2, r+2\}$, and so one. This procedure ends with the half-edge $\{a_{\text{in}}, p, p+r\}$. It requires exactly p trips around the 1-handle denoted "a." We now proceed along lkv to $\{b_{\text{out}}, q, q+s\}$ and work our way down through the other 1-handle, denoted "b." After q such reducing procedures, we have returned to the starting half-edge and completed the loop corresponding to the punctured disk $a^p b^q$.

An identical argument, starting with half edge $\{a_{\text{out}}, 0, p\}$ yields the loop corresponding to the punctured disk $a^r b^s$. Thus, we do have the desired 2-complex, a "twice punctured K_{pqrs}," as the spine of our handlebody T^3.

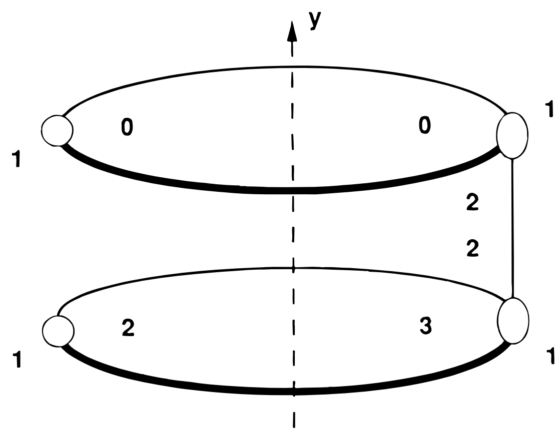
When we finish the embedding of K_{pqrs} in the 3-manifold M, $\text{bd}M \cap T^3$ will be just the subset of $\text{bd}T^3$ on which L is now defined.

7. Attaching the 2-handles. The solid 2-handled torus T^3 may be viewed as a subset of Euclidean 3-space. The spine Y constructed above intersects $\text{bd}T^3$ in two simple closed curves. Do these curves bound disjoint disks in the exterior of T^3?

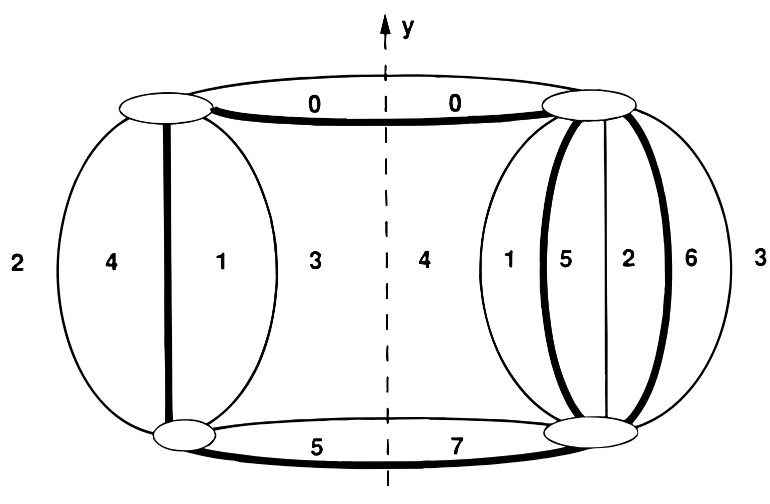
The answer is yes, if care is used in how the 1-handles are placed in 3-space. We choose to avoid this difficulty entirely. Instead, observe that the curves are bicollared in the orientable 2-manifold $\text{bd}T^3$. In fact, we may take for their bicollars the union of $\text{bd}T^3 \cap N(lkv)$ and of $\text{bd}D \cap N(lkv) \times I$ and $\text{bd}D_1 \cap N(lkv) \times I$, i.e., precisely the closure of the part of $\text{bd}T^3$ on which L is undefined. See Figure 3. We may now complete the construction by abstractly attaching two 2-handles to these two bicollars. This yields a 3-manifold M^3 whose spine is K_{pqrs} via the obvious extension of the retraction ρ.

Denote by X and Y the boundaries of the top and bottom 1-cells of one of the 2-handles. The lifting function L is already defined on X and Y. We must show that L extends over those 1-cells so that $\rho \times L$ is an embedding there. For points x of X and y of Y corresponding under ρ, we have $L(x) \neq L(y)$, whence by continuity, we may assume that $L(x) > L(y)$ for all corresponding pairs. We now extend L over the 1-cells by picking any points z_1 inside X and z_2 inside Y corresponding under ρ, selecting arbitrary values $L(z_1) > L(z_2)$, and extending linearly between the selected points and the boundaries. Repeating this extension for the other 2-handle, we now have $\rho \times L$, an embedding on all of $\text{bd}M^3$.

The 3-manifold M^3 is in fact homeomorphic to the 3-ball, but we do not need to prove this. All we need is that $\text{bd}M^3$ is a 2-sphere, which follows from the fact that M^3 is contractible, because its spine is K_{pqrs}. This completes the construction of the desired 2-sphere $S = \rho \times L(\text{bd}M^3)$ in the product $K_{pqrs} \times I$.



1112-Complex



2334-Complex

FIGURE 4. $lkv \cap Y$ for K_{1112} and K_{2334}. The cycle of bdry Y corresponding to $a^p b^q$ meets $lkv \cap Y$ in the bold partial edges; the other cycle contains the other partial edges.

8. Collapsing $K_{pqrs} \times I$ to $S + \operatorname{Int} S$. Consider the *vertical hull* V of S in $K_{pqrs} \times I$, that is, the convex hull of S taken on each vertical line $p \times I$ individually. We assert that $V = S + \operatorname{Int} S$. To see this, first observe that $S + \operatorname{Int} S$ is contained in V. This holds in general for every 2-sphere in a contractible product $K^2 \times I$. The fact that V is contained in $S + \operatorname{Int} S$, however, is not true for any 2-sphere, and must be argued based on our specific construction of S. Let U denote the open 2-skeleton of K_{pqrs}, consisting of the two open disks $a^p b^q$ and $a^r b^s$. Then $V \cdot (U \times I) = (S + \operatorname{Int} S) \cdot (U \times I)$, since for any point p in U, the interval $p \times I$ intersects S in exactly two points. Furthermore, the closure of $V \cdot (U \times I)$ is V. This is verified by examination. Therefore, V is contained in $S + \operatorname{Int} S$.

The collapse of $K_{pqrs} \times I$ to the set $V = S + \operatorname{Int} S$ is done vertically, up from the bottom and down from the top on each vertical line $p \times I$. This proves the theorem.

9. Conclusion. We believe that the above theorem can be extended to read:

Theorem. *Let K denote any acyclic 2-dimensional complex without separating points. If K is the spine of a punctured 3-manifold, then $K \times I$ collapses to a 2-sphere plus its interior.*

Note that without the "acyclic" hypothesis, projective 2-space as a spine of punctured projective 3-space provides a counterexample. Without the "no separating points" hypothesis, one constructs a counterexample by first letting K^* be a spine of a punctured dodecahedral homology sphere.

Then the wedge of two copies of K^*, considered as a spine of the boundary connected sum of two copies of the punctured dodecahedral homology sphere, provides a counterexample K.

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