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ON CHARACTERIZATION OF STRONGLY EXTREME POINTS IN KÖTHE-BOCHNER SPACES

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ABSTRACT. It is shown that the necessity in the characterization of strongly extreme points in Köthe-Bochner space $E(X)$, given by H. Hudzik and M. Mastyło in [**2**], is true without requiring that E be (LUR) and X be separable. The corollary concerning strongly extreme points in Musielak-Orlicz spaces of Bochner type is presented.

1. Introduction. Denote by $\mathbf{N}$ and $\mathbf{R}$ the sets of natural and real numbers, respectively. Let (T, Σ, μ) denote a measure space with a σ-finite and complete measure μ and $L^0 = L^0(T)$ the space of μ-equivalence classes of Σ-measurable real-valued functions. The notation $f \le g$ for $f, g \in L^0$ will mean that $f(t) \le g(t)$ μ-almost everywhere in T.

A Banach space $(E, \|\cdot\|_E) \subset L^0$ is said to be a Köthe space if

(i) $|f| \le |g|$, $f \in L^0$, $g \in E$ imply $f \in E$ and $\|f\|_E \le \|g\|_E$;

(ii) $\operatorname{supp} E =: \cup\{\operatorname{supp} f : f \in E\} = T$.

Now let us define the type of spaces to be considered in this paper. For a real Banach space $(X, \|\cdot\|_X)$, denote by $\mathcal{M}(T, X)$, or just $\mathcal{M}(X)$, the family of all strongly measurable functions $f : T \to X$ identifying functions which are μ-almost everywhere equal. Let

$$E(X) = \{f \in \mathcal{M}(X) : \|f(\cdot)\|_X \in E\}.$$

Then $E(X)$ becomes a Banach space with the norm

$$\|f\| = \|\|f(\cdot)\|_X\|_E,$$

and it is called a *Köthe-Bochner space*.

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For any Banach space X, we denote by $S(X)$, $B(X)$, the unit sphere, closed unit ball, respectively.

A Banach space X is *locally uniformly rotund* (write (LUR)) if for any $x \in S(X)$ and any sequence (x_n) of elements of $B(X)$ we have

$$\|x_n\|_X \to \|x\|_X \text{ and } \|x_n + x\|_X \to 2\|x\|_X \Rightarrow x_n \to x \text{ strongly in } X.$$

A point $x \in S(X)$ is called a *strongly extreme point of the unit ball* $B(X)$ (write $x \in \delta_{\mathrm{se}} B(X)$) if, for every sequence $(y_n), (z_n) \subset B(X)$, $\lim_{n\to\infty} \|(y_n + z_n) - 2x\|_X = 0$ implies that $\lim_{n\to\infty} \|y_n - x\|_X = 0$.

The following characterization of strongly extreme points of the unit sphere in a Köthe-Bochner space is given in [**2**] (cf. [**2**, Theorems 2 and 3].

Theorem 1. *Let E be a locally uniformly rotund Köthe space over a measure space (T, Σ, μ), and let X be a Banach space.*

a) *If $f \in S(E(X))$ and $f(t)/\|f(t)\|_X \in \delta_{\mathrm{se}} B(X)$ for μ-almost all $t \in \operatorname{supp} f$, then $f \in \delta_{\mathrm{se}} B(E(X))$.*

b) *If, additionally, X is a separable Banach space and $f \in \delta_{\mathrm{se}} B(E(X))$, then $f(t)/\|f(t)\|_X \in \delta_{\mathrm{se}} B(X)$ for μ-almost all $t \in \operatorname{supp} f$.*

2. Main result. Theorem 1 b) is true without requiring that E be (LUR) and X be separable. Adopting some ideas from [**1**], we can prove the following

Theorem 2. *Let E be a Köthe space over a measure space (T, Σ, μ), and let X be a Banach space. If $f \in \delta_{\mathrm{se}} B(E(X))$, then $f(t)/\|f(t)\|_X \in \delta_{\mathrm{se}} B(X)$ for μ-almost all $t \in \operatorname{supp} f$.*

Proof. Let $f \in \delta_{\mathrm{se}} B(E(X))$. Suppose that the theorem is not true, i.e.,

$$\mu\left\{t \in \operatorname{supp} f : \frac{f(t)}{\|f(t)\|_X} \notin \delta_{\mathrm{se}} B(X)\right\} > 0.$$

Denote

$$A_{m,j} = \left\{ x \in S(X) : \left\| \frac{1}{2}(x_1 + x_2) - x \right\|_X < \frac{1}{m} \right.$$
$$\left. \text{for some } x_1, x_2 \in B(X) \Big\backslash \left[x + \frac{1}{j} B(X) \right] \right\}$$

for $m, j \in \mathbf{N}$. The sets $A_{m,j}$ have the following properties:

a) $x \notin \delta_{\mathrm{se}} B(X)$, if and only if there exists j such that for all m we have $x \in A_{m,j}$;

b) $A_{m+1,j} \subset A_{m,j} \subset A_{m,j+1}$;

c) Every $A_{m,j}$ is open in $S(X)$ with respect to the topology induced from X.

The statement a) follows immediately from the negation of the definition of strongly extreme point. b) is a consequence of standard comparison of the sets. To prove c), suppose that $(y_k) \in S(X) \backslash A_{m,j}$, $k \in \mathbf{N}$, and

$$\|y_k - y\|_X \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Obviously, $y \in S(X)$. Moreover, fixing a positive $\varepsilon < 1/m$, a positive integer k_0 can be found such that

$$\|y_k - y\|_X < \varepsilon \quad \text{for } k \geq k_0.$$

Let $x_1, x_2 \in B(X) \backslash [y + (1/j)B(X)]$. Then, for $k \geq k_0$, we have

$$\left\| \frac{1}{2}(x_1 + x_2) - y \right\|_X \geq \left| \left\| \frac{1}{2}(x_1 + x_2) - y_k \right\|_X - \|y_k - y\|_X \right|$$
$$\geq \frac{1}{m} - \varepsilon.$$

Hence, $y \in S(X) \backslash A_{m,j}$, because ε is arbitrary. Therefore, $A_{m,j}$ are open for every $m, j \in \mathbf{N}$.

Define

$$g(t) = \begin{cases} f(t)/\|f(t)\|_X & \text{for } t \in \operatorname{supp} f \\ 0 & \text{for } t \notin \operatorname{supp} f. \end{cases}$$

Then, by a), we can conclude that

$$\begin{aligned} g^{-1}[S(X)\backslash\delta_{\mathrm{se}}B(X)] &= \bigcup_{j=1}^{\infty}\bigcap_{m=1}^{\infty} g^{-1}(A_{m,j}) \\ &= \Big\{t \in \operatorname{supp} f : \frac{f(t)}{\|f(t)\|_X} \notin \delta_{\mathrm{se}}B(X)\Big\}. \end{aligned}$$

Consequently, by our assumption and by b), there are $L \in \mathbf{N}$ and $G \in \Sigma$ of finite measure such that

$$\mu\Big\{ \cap_{m=1}^{\infty} g^{-1}(A_{m,L}) \cap G\Big\} > 0.$$

Choose

$$F \in \Sigma \quad \text{such that } \mu(F) > 0, \qquad F \subset \bigcap_{m=1}^{\infty} g^{-1}(A_{m,L}) \cap G$$

and

$$F \subset \{t \in T : a \leq \|f(t)\|_X\}$$

for some $a > 0$.

Now we will show that, for any $\delta \in (0, 1/L)$, there is a measurable partition $\{F_k\}$ of F such that $\operatorname{diam} f[F_k] < a\delta/3$ for all k, and there exists $t_k \in F_k$ such that $\|f(t_k)\|_X = \inf \|f[F_k]\|_X$ for all k.

Really, since $f[F]$ is separable, there is a measurable partition $\{E_n\}$ of F such that $\operatorname{diam} f[E_n] < a\delta/3$ for all n. For each fixed n, choose a sequence $\{x_j\}$ in E_n such that $\{\|f(x_j)\|_X\}$ is decreasing to $\inf \|f[E_n]\|_X$. Let

$$E_{n,0} = \{t \in E_n : \|f(t)\|_X \geq \|f(x_1)\|_X\}$$

and

$$E_{n,j} = \{t \in E_n : \|f(x_j)\|_X > \|f(t)\|_X \geq \|f(x_{j+1})\|_X\}$$

for $j \in \mathbf{N}$. Then $E_{n,j}$ is measurable for all $n, j \in \mathbf{N}$. Let $\{F_k\}$ be the family consisting of all $E_{n,j}$, $n, j \in \mathbf{N}$. Therefore $\{F_k\}$ satisfies the desirable conditions in an obvious manner.

Now we can conclude that, for every k, there exist

$$x_{k,1} x_{k,2} \in B(X) \Big\backslash \left[\frac{f(t_k)}{\|f(t_k)\|_X} + \frac{1}{L} B(X) \right], \tag{1}$$

and

$$\left\| \frac{1}{2}(x_{k,1} + x_{k,2}) - \frac{f(t_k)}{\|f(t_k)\|_X} \right\|_X < \frac{\delta}{3}. \tag{2}$$

Let $\gamma = (a/(3L))\|\chi_F\|_E > 0$. Define

$$f_i = \chi_{T \backslash F} f + \sum_{k=1}^{\infty} \|f(t_k)\|_X x_{k,i} \chi_{F_k},$$

$i = 1, 2$. To finish the proof, it is enough to show that

$$\|f_i - f\| \geq \gamma \quad \text{for } i = 1, 2, \tag{3}$$

and

$$\left\| \frac{1}{2}(f_1 + f_2) - f \right\| < \delta. \tag{4}$$

To prove inequality (3), note that by (1) we conclude

$$\begin{aligned} &\left\| \sum_{k=1}^{\infty} (\|f(t_k)\|_X x_{k,i} - f(t_k)) \chi_{F_k} \right\| \\ &\qquad = \left\| \sum_{k=1}^{\infty} \| \|f(t_k)\|_X x_{k,i} - f(t_k)\|_X \chi_{F_k} \right\|_E \\ &\qquad \geq \frac{1}{L} \left\| \sum_{k=1}^{\infty} \|f(t_k)\|_X \chi_{F_k} \right\|_E \\ &\qquad \geq \frac{a}{L} \left\| \sum_{k=1}^{\infty} \chi_{F_k} \right\|_E \\ &\qquad = \frac{a}{L} \|\chi_F\|_E = 3\gamma \end{aligned}$$

for $i = 1, 2$. Moreover, by the fact that diam $f[F_k] < a\delta/3$, we get

$$\begin{aligned}\Big\|\sum_{k=1}^{\infty} f(t_k)\chi_{F_k} - f\chi_F\Big\| &= \Big\|\sum_{k=1}^{\infty}(f(t_k) - f)\chi_{F_k}\Big\| \\ &\leq \Big\|\sum_{k=1}^{\infty}\|f(t_k) - f(\cdot)\|_X\chi_{F_k}\Big\|_E \\ &\leq \frac{a\delta}{3}\Big\|\sum_{k=1}^{\infty}\chi_{F_k}\Big\|_E \\ &\leq \frac{a}{3L}\|\chi_F\|_E = \gamma.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f_i - f\| &= \Big\|\sum_{k=1}^{\infty}\|f(t_k)\|_X x_{k,i}\chi_{F_k} - f\chi_F\Big\| \\ &\geq \Big\|\sum_{k=1}^{\infty}(\|f(t_k)\|_X x_{k,i} - f(t_k))\chi_{F_k}\Big\| \\ &\quad - \Big\|\sum_{k=1}^{\infty} f(t_k)\chi_{F_k} - f\chi_F\Big\| \\ &\geq 3\gamma - \gamma > \gamma\end{aligned}$$

for $i = 1, 2$. Thus (3) is proved.

It remains to prove the inequality (4). Taking into account the inequality (2), we have

$$\begin{aligned}\Big\|\frac{1}{2}(f_1 + f_2) - f\Big\| &= \Big\|\frac{1}{2}\sum_{i=1}^{2}(f_i - f)\Big\| \\ &= \Big\|\frac{1}{2}\sum_{i=1}^{2}\sum_{k=1}^{\infty}(\|f(t_k)\|_X x_{k,i} - f)\chi_{F_k}\Big\| \\ &= \Big\|\frac{1}{2}\sum_{i=1}^{2}\sum_{k=1}^{\infty}(\|f(t_k)\|_X x_{k,i} \\ &\qquad - f(t_k) + f(t_k) - f)\chi_{F_k}\Big\|\end{aligned}$$

$$
\begin{aligned}
&\le \left\| \frac{1}{2} \sum_{i=1}^{2} \sum_{k=1}^{\infty} (\|f(t_k)\|_X x_{k,i} - f(t_k))\chi_{F_k} \right\| \\
&\quad + \left\| \frac{1}{2} \sum_{i=1}^{2} \sum_{k=1}^{\infty} (f(t_k) - f)\chi_{F_k} \right\| \\
&\le \left\| \sum_{k=1}^{\infty} \left\| \frac{1}{2} \sum_{i=1}^{2} (\|f(t_k)\|_X x_{k,i} - f(t_k)) \right\|_X \chi_{F_k} \right\|_E \\
&\quad + \left\| \sum_{k=1}^{\infty} \|f(t_k) - f(\cdot)\|_X \chi_{F_k} \right\|_E \\
&\le \left\| \sum_{k=1}^{\infty} \frac{\delta}{3} \|f(t_k)\|_X \chi_{F_k} \right\|_E + \left\| \sum_{k=1}^{\infty} \frac{a\delta}{3} \chi_{F_k} \right\|_E \\
&\le \frac{\delta}{3} \| \, \|f(\cdot)\|_X \chi_F \|_E + \frac{\delta}{3} \|a\chi_F\|_E \\
&\le \frac{\delta}{3} \|f\| + \frac{\delta}{3} \| \, \|f(\cdot)\|_X \chi_F \|_E \\
&\le \frac{\delta}{3} + \frac{\delta}{3} < \delta,
\end{aligned}
$$

i.e., (4) is satisfied.

Inequalities (3) and (4) imply that $f \notin \delta_{\mathrm{se}} B(E(X))$. This contradiction completes the proof of the theorem. □

3. Corollaries. Now we apply our results to the case of Musielak-Orlicz space of Bochner type. To do it, we agree on some terminology.

A function $\varphi : \mathbf{R} \times T \to [0, \infty]$ is said to be a *Musielak-Orlicz function* if

a) $\varphi(u, \cdot)$ is measurable for each $u \in \mathbf{R}$,

b) $\varphi(0, t) = 0$ and $\varphi(\cdot, t)$ is convex, even, lower semi-continuous, not identically equal to zero, continuous at zero for μ-almost all $t \in T$.

By the *Musielak-Orlicz space* L^φ we mean

$$L^\varphi = \left\{ f \in L^0 : I_\varphi(cf) = \int_T \varphi(cf(t), t)\, d\mu < \infty \text{ for some } c > 0 \right\},$$

equipped with so-called *Luxemburg norm* defined as follows

$$\|f\|_\varphi = \inf\{\varepsilon > 0 : I_\varphi(f/\varepsilon) \le 1\}.$$

The Musielak-Orlicz space $L^\varphi(X)$ of Bochner type we define as the family of functions $f \in \mathcal{M}(T,X)$ such that $\|f(\cdot)\| \in L^\varphi_X$, i.e., $E(X)$ with $E = L^\varphi$.

We say that the Musielak-Orlicz function φ *satisfies the* Δ_2*-condition* if there exist a real number $b > 0$ and a nonnegative integrable function $a(\cdot)$ such that

$$\varphi(2u,t) \le b\varphi(u,t) + a(t)$$

for μ-almost all $t \in T$ and every $u \in \mathbf{R}$.

For more details, we refer to [**4**].

The following corollary is an immediate consequence of Theorem 2.

Corollary 2. *Let* φ *be a Musielak-Orlicz function.*

a) *If* φ *satisfies the* Δ_2*-condition,* $\varphi(\cdot,t)$ *is strictly convex for* μ*-almost all* $t \in T$*,* $f \in S(L^\varphi(X))$*, and* $f(t)/\|f(t)\|_X \in \delta_{\text{se}}B(X)$ *for* μ*-almost all* $t \in \operatorname{supp} f$*, then* $f \in \delta_{\text{se}}B(L^\varphi(X))$*.*

b) *If* $f \in \delta_{\text{se}}B(L^\varphi(X))$*, then* $f(t)/\|f(t)\|_X \in \delta_{\text{se}}B(X)$ *for* μ*-almost all* $t \in \operatorname{supp} f$*.*

Proof. By a Kamińska's result, cf. [**3**], the Musielak-Orlicz space is (LUR) if and only if $\varphi(\cdot,t)$ is strictly convex μ-almost everywhere in T, and φ satisfies the Δ_2 condition. Applying Theorem 1a), we get Corollary 2a).

Corollary 2b) is an immediate consequence of Theorem 2. □

Open question. Is Theorem 1a) true without requiring that E be (LUR)?

REFERENCES

1. Z. Hu and B.L. Lin, *Strongly exposed points in Lebesgue Bochner function spaces*, Proc. Amer. Math. Soc. **120** (1994), 1159–1165.

2. H. Hudzik and M. Mastylo, *Strongly extreme points in Köthe-Bochner spaces*, Rocky Mountain J. Math. **23** (1993), 899–909.

3. A. Kamińska, *On some convexity properties of Musielak-Orlicz spaces*, Rend. Circ. Mat. Palermo, Serie II, Suppl. **5** (1984), 63–72.

4. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. **1034** (1983), 1–222.

5. M.A. Smith, *Strongly extreme points in* $L^p(\mu, X)$, Rocky Mountain J. Math. **16** (1986), 1–5.

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