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MOMENT PROBLEM FOR RATIONAL ORTHOGONAL FUNCTIONS ON THE UNIT CIRCLE

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ABSTRACT. The Favard-type theorem for rational functions orthogonal on the unit circle with prescribed poles lying outside the unit circle are studied. We also consider the existence of sequences of orthogonal rational functions whose zeros are everywhere dense in $|z| \leq 1$.

1. Introduction. Let $d\mu$ be a finite positive Borel measure with an infinite set as its support on $[0, 2\pi)$. We define $L^2_{d\mu}$ to be the space of all functions $f(z)$ on the unit circle $T := \{z \in \mathbf{C} : |z| = 1\}$ satisfying $\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\mu(\theta) < \infty$. Then $L^2_{d\mu}$ is a Hilbert space with inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \, d\mu(\theta).$$

Consider a sequence $\{z_n\}$ with $|z_n| < 1$, and let

$$b_n(z) := \frac{z_n - z}{1 - \bar{z}_n z}\eta_n \quad \text{and} \quad \eta_n := \frac{|z_n|}{z_n}, \quad n = 1, \dots ,$$

where for $z_n = 0$ we put $|z_n|/z_n = -1$. Next we define finite Blaschke products recursively as

$$B_0(z) = 1 \quad \text{and} \quad B_n(z) = B_{n-1}(z)b_n(z), \quad n = 1, \dots .$$

The fundamental polynomials $w_n(z)$ are given by

$$w_0(z) := 1 \quad \text{and} \quad w_n(z) := \prod_{i=1}^{n} (1 - \bar{z}_i z), \quad n = 1, \dots .$$

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The space of rational functions of our interest is defined as

$$\mathcal{R}_n = \mathcal{R}[z_1, \dots, z_n] := \left\{ \frac{p(z)}{w_n(z)} : p \in \mathcal{P}_n \right\}, \quad n = 0, 1, \dots .$$

It is easy to verify that $\{B_k\}_{k=0}^n$ forms a basis of $\mathcal{R}_n$, i.e., $\mathcal{R}_n = \operatorname{span}\{B_k(z), k = 0, \dots, n\}$. Finally, for any $r \in \mathcal{R}_n$, we define $r^*(z) := B_n(z)\overline{r(1/\bar{z})}$. Then it is easy to see that $|r^*(z)| = |r(z)|$ for $|z| = 1$ and $r^*(z) \in \mathcal{R}_n$. For each n, we now define the rational version of Szegő polynomials, orthonormal rational functions, $\phi_n(z) = \kappa_n B_n(z) + b_{n-1}B_{n-1}(z) + \cdots$, for $n = 0, 1, 2, \dots$,

$$\begin{aligned} &\phi_n \in \mathcal{R}_n, \quad \kappa_n > 0, \\ &\langle \phi_n, B_k \rangle = 0, \quad k = 0, \dots, n-1, \\ &\langle \phi_n, \phi_n \rangle = 1. \end{aligned}$$

The orthogonal rational functions are of constant interest to both mathematicians and physicists. That is because of the significant relations between the studies in Hankel and Toeplitz operators, continued fractions, moment problem, Carathéodory-Fejer interpolation, Schur's algorithm and function algebras, and solving electrical problems (cf., [**8, 9, 10**–**13, 2**–**7**]). One can get more information for the algebra as well as the analysis properties of $\phi_n(z)$ from some excellent papers (cf., [**10**–**13, 2**–**7**]). One of the properties is that the sequences $\{\phi_n(z)\}$ satisfy the following recurrence relation (cf. [**7**]):

$$\phi_n(z) = \frac{\kappa_n}{\kappa_{n-1}} \left[r_n \frac{z - z_{n-1}}{1 - \bar{z}_n z} \phi_{n-1}(z) + s_n \frac{1 - \bar{z}_{n-1} z}{1 - \bar{z}_n z} \phi_{n-1}^*(z) \right],$$

for $n = 1, 2, \dots$, and $z_0 = 0$, $\phi_0(z) = \kappa_0$. The coefficients r_n and s_n are given by

$$\begin{aligned} r_n &= -\eta_n \frac{(1 - \bar{z}_{n-1} z_n) \overline{\phi_n^*(z_{n-1})}}{(1 - |z_{n-1}|^2)\kappa_n}, \\ s_n &= \frac{(1 - \bar{z}_{n-1} z_n) \phi_n(z_{n-1})}{(1 - |z_{n-1}|^2)\kappa_n}. \end{aligned}$$

The goal of the paper is to study the reverse of the above argument, i.e., the Favard-type theorem. Given a system generated by the recurrence relations above, can we find a measure $d\mu \geq 0$ on $[0, 2\pi)$ such that

$\{\phi_n(z)\}$ is orthonormal with respect to $d\mu$? The proof of the Favard theorem for orthogonal rational functions was given in [**4**]. We here want to give a direct proof which uses the same ideas as in [**14**] for the Szegő polynomial case.

In 1980, P. Turán asked in [**15**] whether there is a system of orthogonal polynomials on the unit circle such that the zeros of the polynomials are everywhere dense in $|z| \leq 1$. In [**1**], the authors gave an affirmative answer to this problem. In this paper we consider the same problem for the orthogonal rational functions. We prove that there exist systems of orthonormal rational functions such that the zeros of the orthonormal rational functions are everywhere dense in $|z| \leq 1$.

The main results are given in Section 2, and their proofs are presented in Section 4. Section 3 is used for citing as well as establishing some auxiliary results that are needed in the proofs of our main results.

2. Main theorems. In this section we only state our main theorems, and the proofs will be given in Section 4.

Theorem 2.1. *Assume* $\{a_n\} \in \mathbf{C}$ *and* $|a_n| < 1$. *We define* $\Phi_n(z)$ *in the following way*:

$$n = 0, \qquad z_0 = 0, \qquad \Phi_0(z) = 1, \qquad \Phi_0^*(z) = 1.$$

For $n = 1, 2, \ldots,$

$$\begin{aligned} \psi_n(z) &= \frac{z - z_{n-1}}{1 - \bar{z}_n z} \Phi_{n-1}(z) + a_n \frac{1 - \bar{z}_{n-1} z}{1 - \bar{z}_n z} \Phi_{n-1}^*(z), \\ \Phi_n(z) &= \frac{\psi_n(z)}{\psi_n^*(z_n)}. \end{aligned}$$

Also let

$$e_n := \frac{1}{\overline{\psi_n^*(z_n)}}, \qquad \kappa_0 = 1,$$

and

$$\kappa_n = \sqrt{\frac{1 - |z_n|^2}{\Pi_{m=1}^n |e_m|^2 (1 - |a_m|^2)}}.$$

Then $\{\phi_i(z) = \kappa_i \Phi_i(z)\}_{i=0}^n$ *is orthonormal with respect to*

$$d\mu_n := \frac{P(z, z_n)}{|\phi_n(z)|^2} \frac{d\theta}{2\pi},$$

i.e.,

$$\int_0^{2\pi} \phi_k(z)\overline{\phi_j(z)}\, d\mu_n = \delta_{k,j},$$
$$0 \le k,\ j \le n,\ z = e^{i\theta},$$

where $P(z, z_n) := (1 - |z_n|^2)/|z - z_n|^2$.

Remark. We will prove that $\psi_n^*(z) \neq 0$ in $|z| \le 1$ later (Lemma 3.1), so e_n is well-defined and $\kappa_n > 0$.

Theorem 2.2. *Let* $\sum_{n=1}^{\infty}(1 - |z_n|) = \infty$ *and* $\{\phi_n(z)\}$ *be constructed from* $|a_n| < 1$ *by Theorem* 2.1*. Then there is a unique measure* $d\mu \ge 0$ *on* $[0, 2\pi)$ *and* $\int_0^{2\pi} d\mu = 1$ *so that* $\{\phi_n(z)\}$ *is orthonormal with respect to* $d\mu$*.*

Theorem 2.3. *Let* $\sum_{n=1}^{\infty}(1 - |z_n|) = \infty$*. There exist measures* $d\mu$ *on* $[0, 2\pi)$ *such that the zeros of the orthonormal rational functions with respect to* $d\mu$ *are everywhere dense in* $|z| \le 1$*.*

3. Lemmas.

Lemma 3.1. *Suppose that* $\{a_n\} \subset \mathbf{C}$ *and* $|a_n| < 1$*. Let* $\phi_n(z)$ *be constructed from Theorem* 2.1*, then*

(i) *all zeros of* $\phi_n(z)$ *are in* $|z| < 1$*,*

(ii)

$$\frac{1 - |z_n|^2}{1 - |z_{n-1}|^2} \frac{\kappa_{n-1}^2}{\kappa_n^2} = |e_n|^2(1 - |a_n|^2),$$

(iii)

$$\frac{\phi_n^*(z_{n-1})}{\kappa_n} = -\bar{e}_n \frac{1 - |z_{n-1}|^2}{1 - \bar{z}_n z_{n-1}} \eta_n,$$

(iv)

$$\bar{e}_n\phi_n(z) = \frac{1-|z_n|^2}{1-|z_{n-1}|^2}\frac{\kappa_{n-1}}{\kappa_n}\frac{(z-z_{n-1})}{(1-\bar{z}_n z)}\phi_{n-1}(z) - \frac{a_n e_n}{\eta_n}\phi_n^*(z).$$

Proof. (i) We only need to prove (i) holds for $\psi_n(z)$. We use induction to prove that all zeros of $\psi_n^*(z)$ are in $|z|>1$.

Case $n=1$. Notice that $z_0=0$ and $\psi_1(z)=(z+a_1)/(1-\bar{z}_1 z)$. So the zero of $\psi_1(z)$ is in $|z|<1$, hence the zero of $\psi_1^*(z)$ is in $|z|>1$.

Suppose (i) holds for $n=k$, then $|\Phi_k(z)/\Phi_k^*(z)|\le 1$ for $|z|\le 1$ since $\psi_k^*(z)\neq 0$ on $|z|\le 1$ and $|\Phi_k(z)/\Phi_k^*(z)|=1$ for $|z|=1$. Hence,

$$\begin{aligned}\psi_{k+1}^*(z) &= \left\{-\frac{\eta_{k+1}(1-\bar{z}_k z)}{(1-\bar{z}_{k+1}z)}\Phi_k^*(z) - \bar{a}_{k+1}\frac{\eta_{k+1}(z-z_k)}{(1-\bar{z}_{k+1}z)}\Phi_k(z)\right\}\\ &= -\Phi_k^*(z)\eta_{k+1}\frac{(1-\bar{z}_k z)}{(1-\bar{z}_{k+1}z)}\\ &\quad \left\{1+a_{k+1}\frac{(z-z_k)}{1-\bar{z}_k z}\frac{\Phi_k(z)}{\Phi_k^*(z)}\right\}\neq 0, \quad |z|\le 1,\end{aligned}$$

since $|a_{k+1}|<1$.

(ii) The formula follows from the definition of κ_n.

(iii) Notice that

$$\frac{\phi_n^*(z)}{\kappa_n} = -\eta_n\bar{e}_n\left\{\frac{1-\bar{z}_{n-1}z}{1-\bar{z}_n z}\frac{\phi_{n-1}^*(z)}{\kappa_{n-1}} + \bar{a}_n\frac{z-z_{n-1}}{1-\bar{z}_n z}\frac{\phi_{n-1}(z)}{\kappa_{n-1}}\right\}.$$

Letting $z=z_{n-1}$, then

$$\begin{aligned}\frac{\phi_n^*(z_{n-1})}{\kappa_n} &= -\bar{e}_n\eta_n\frac{1-|z_{n-1}|^2}{1-\bar{z}_n z_{n-1}}\frac{\phi_{n-1}^*(z_{n-1})}{\kappa_{n-1}}\\ &= -\bar{e}_n\eta_n\frac{1-|z_{n-1}|^2}{1-\bar{z}_n z_{n-1}}.\end{aligned}$$

(iv) Since

$$\phi_n(z) = e_n \frac{\kappa_n}{\kappa_{n-1}} \frac{z - z_{n-1}}{1 - \bar{z}_n z} \phi_{n-1}(z) + e_n a_n \frac{\kappa_n}{\kappa_{n-1}} \frac{1 - \bar{z}_{n-1} z}{1 - \bar{z}_n z} \phi^*_{n-1}(z)$$

and

$$\phi^*_n(z) = -\bar{e}_n \eta_n \frac{\kappa_n}{\kappa_{n-1}} \frac{1 - \bar{z}_{n-1} z}{1 - \bar{z}_n z} \phi^*_{n-1}(z) + \bar{e}_n \eta_n \bar{a}_n \frac{\kappa_n}{\kappa_{n-1}} \frac{z - z_{n-1}}{1 - \bar{z}_n z} \phi_{n-1}(z),$$

eliminating $\phi^*_{n-1}(z)$, we have

$$\eta_n \bar{e}_n \phi_n(z) + a_n e_n \phi^*_n(z) = \eta_n |e_n|^2 \frac{\kappa_n}{\kappa_{n-1}} \frac{z - z_{n-1}}{1 - \bar{z}_n z} \phi_{n-1}(z)(1 - |a_n|^2).$$

Then, from (ii), we obtain

$$\eta_n \bar{e}_n \phi_n(z) + a_n e_n \phi^*_n(z) = \eta_n \frac{\kappa_{n-1}}{\kappa_n} \frac{1 - |z_n|^2}{1 - |z_{n-1}|^2} \frac{z - z_{n-1}}{1 - \bar{z}_n z} \phi_{n-1}(z). \qquad \square$$

Lemma 3.2. *There is a* $Q_{n-2}(z) \in \mathcal{R}_{n-2}$ *such that*

$$\frac{1 - \bar{z}_{n-1} z}{z - z_n} B_n(z) = \frac{\bar{z}_n - \bar{z}_{n-1}}{\kappa_n (1 - |z_n|^2)} \phi_n(z) - \frac{\bar{z}_n (1 - \bar{z}_{n-1} z_n) \overline{\phi^*_n(z_{n-1})}}{|z_n| \kappa_n (1 - |z_n|^2)} B_{n-1}(z) + Q_{n-2}(z).$$

Proof. For $z_n = z_{n-1}$ the formula is immediately verified (note that $\overline{\phi^*_n(z_{n-1})} = \overline{\phi^*_n(z_n)} = \kappa_n$). So we assume that $z_n \neq z_{n-1}$. We first observe that we may write

$$\frac{1 - \bar{z}_{n-1} z}{z - z_n} B_n(z) = \frac{\bar{z}_n - \bar{z}_{n-1}}{1 - |z_n|^2} B_n(z) - \eta_n \frac{1 - \bar{z}_{n-1} z_n}{1 - |z_n|^2} B_{n-1}(z). \tag{3.1}$$

On the other hand,

$$\phi_n(z) = \kappa_n B_n(z) + c_{n-1}B_{n-1} + \cdots + c_0 B_0 \tag{3.2}$$
$$= \kappa_n B_n(z) + c_{n-1}B_{n-1}(z) + L_{n-2}(z),$$

where $L_{n-2} \in \mathcal{R}_{n-2}$, and

$$\phi_n^*(z) = \kappa_n + \bar{c}_{n-1}b_n(z) + b_n(z)b_{n-1}(z)L_{n-2}^*(z).$$

Note that $b_{n-1}(z_{n-1}) = 0$, therefore

$$\phi_n^*(z_{n-1}) = \kappa_n + \bar{c}_{n-1}b_n(z_{n-1}).$$

Hence,

$$\bar{c}_{n-1} = [\phi_n^*(z_{n-1}) - \kappa_n]\bar{\eta}_n \frac{1 - z_{n-1}\bar{z}_n}{z_n - z_{n-1}}. \tag{3.3}$$

From (3.1) and (3.2), we have

$$\frac{1 - \bar{z}_{n-1}z}{z - z_n}B_n(z) = \frac{\bar{z}_n - \bar{z}_{n-1}}{\kappa_n(1 - |z_n|^2)} [\phi_n(z) - c_{n-1}B_{n-1} - L_{n-2}(z)] - \eta_n \frac{1 - \bar{z}_{n-1}z_n}{1 - |z_n|^2}B_{n-1}(z),$$

and by substitution for c_{n-1} from (3.3), this gives

$$\frac{1 - \bar{z}_{n-1}z}{z - z_n}B_n(z) = \frac{\bar{z}_n - \bar{z}_{n-1}}{\kappa_n(1 - |z_n|^2)}\phi_n(z) - \eta_n \frac{(1 - \bar{z}_{n-1}z_n)\overline{\phi_n^*(z_{n-1})}}{\kappa_n(1 - |z_n|^2)}B_{n-1}(z) - \frac{\bar{z}_n - \bar{z}_{n-1}}{\kappa_n(1 - |z_n|^2)}L_{n-2}(z). \qquad \square$$

4. Proofs.

Proof of Theorem 2.1. We first prove

$$\int_0^{2\pi} \phi_k(z)\overline{\phi_j(z)}\,d\mu_n = 0, \qquad j \neq k, \quad z = e^{i\theta}. \tag{4.1}$$

We use the backward induction for $k = n, n-1, \dots, 1, 0$.

Case $k = n$. For $0 \leq j \leq n-1$,

$$\begin{aligned}\int_0^{2\pi} \phi_n(z)\overline{\phi_j(z)}\,d\mu_n &= \frac{1}{2\pi}\int_0^{2\pi} \left[\overline{\frac{\phi_j(z)}{\phi_n(z)}}\right] P(z, z_n)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\phi_j^*(z)B_n(z)}{\phi_n^*(z)B_j(z)} P(z, z_n)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\phi_j^*(z)}{\phi_n^*(z)} b_{j+1}(z)b_{n-1}(z)P(z, z_n)\,d\theta \\ &= \frac{\phi_j^*(z)}{\phi_n^*(z)} b_{j+1}(z)b_n(z)|_{z=z_n} = 0,\end{aligned}$$

because of (i) in Lemma 3.1, $\phi_n^*(z) \neq 0$ for $|z| \leq 1$ and so $(\phi_j^*(z)/\phi_n^*(z))$ $b_{j+1}(z)b_n(z)$ is analytic when $|z| \leq 1$.

Now assume that the lemma holds for some $k \leq n$, i.e.,

$$\int_0^{2\pi} \phi_k(z)\overline{\phi_j(z)}\,d\mu_n = 0, \qquad 0 \leq j < k, \quad z = e^{i\theta}.$$

We are to prove (4.1) for $k-1$. For $0 \leq j < k-1$. We have, from (iv) in Lemma 1,

$$\begin{aligned}\frac{1-|z_k|^2}{1-|z_{k-1}|^2}\frac{\kappa_{k-1}}{\kappa_k}\int_0^{2\pi} \phi_{k-1}(z)\overline{\phi_j(z)}\,d\mu_n \\ = \bar{e}_n \int_0^{2\pi} \frac{1-\bar{z}_k z}{z - z_{k-1}} \phi_k(z)\overline{\phi_j(z)}\,d\mu_n \\ + \frac{a_k e_k}{\eta_k}\int_0^{2\pi} \frac{1-\bar{z}_k z}{z - z_{k-1}} \phi_k^*(z)\overline{\phi_j(z)}\,d\mu_n.\end{aligned}$$

For the first term above we get

$$\bar{e}_n \int_0^{2\pi} \frac{1-\bar{z}_k z}{z-z_{k-1}} \phi_k(z)\overline{\phi_j(z)}\, d\mu_n = \bar{e}_n \int_0^{2\pi} \phi_k(z)\overline{\phi_j(z)\frac{z-z_k}{1-\bar{z}_{k-1}}}\, d\mu_n = 0,$$

since the induction for k and $(z-z_k)/(1-\bar{z}_{k-1}z)\phi_j(z) \in \mathcal{R}_{k-1}$. For the second term we have

$$\begin{aligned} &\frac{a_k e_k}{\eta_n}\int_0^{2\pi} \frac{1-\bar{z}_k z}{z-z_{k-1}} \phi_k^*(z)\overline{\phi_j(z)}\, d\mu_n \\ &= \frac{a_k e_k}{\eta_n}\int_0^{2\pi} \frac{1-\bar{z}_k z}{z-z_{k-1}} B_k(z)\overline{\phi_k(z)\phi_j(z)}\, d\mu_n \\ &= \frac{a_k e_k}{\eta_n}\int_0^{2\pi} \frac{1-\bar{z}_k z}{z-z_{k-1}} b_{j+1}(z)\cdots b_k(z)\overline{\phi_k(z)}\phi_j^*(z)\, d\mu_n \\ &= -\frac{a_k e_k}{\eta_n}\int_0^{2\pi} b_{j+1}(z)\cdots b_{k-2}(z)\eta_{k-1}\eta_k \frac{z_k - z}{1-\bar{z}_{k-1}z}\overline{\phi_k(z)}\phi_j^*(z)\, d\mu_n \\ &= 0, \end{aligned}$$

since $b_{j+1}(z)\cdots b_{k-2}(z)((z_k-z)/(1-\bar{z}_{k-1}z))\phi_j^*(z) \in \mathcal{R}_{k-1}$. According to induction, we prove (4.1).

Next we show

$$\int_0^{2\pi} |\phi_j(z)|^2\, d\mu_n = 1, \quad j = 0, \ldots, n. \tag{4.2}$$

We will use the backward induction for $k = n, n-1, \ldots, 1, 0$ again.

Case $k = n$. It is easy to see, since

$$\int_0^{2\pi} |\phi_n(z)|^2\, d\mu_n = \frac{1}{2\pi}\int_0^{2\pi} P(z, z_n)\, d\theta = 1.$$

Now, assume that (4.2) holds for some $k \le n$, i.e.,

$$\int_0^{2\pi} |\phi_k(z)|^2\, d\mu_n = 1. \tag{4.3}$$

We are to prove (4.2) for $k-1$. One can say that, for $j=0,1,\dots,n$,

$$\|\phi_j\|^2_{\mu_n} := \int_0^{2\pi} |\phi_j(z)|^2\, d\mu_n = \kappa_j \int_0^{2\pi} \phi_j(z)\overline{B_j(z)}\, d\mu_n. \tag{4.4}$$

Multiplying (iv) in Lemma 3.1 for $n=k$ by $\overline{B_k(z)}$ and taking the integral, we have

$$\begin{aligned}\bar{e}_k \int_0^{2\pi} \phi_k(z)\overline{B_k(z)}\, d\mu_n &\\ &= \frac{1-|z_k|^2}{1-|z_{k-1}|^2}\frac{\kappa_{k-1}}{\kappa_k}\int_0^{2\pi} \frac{z-z_{k-1}}{1-\bar{z}_k z}\phi_{k-1}(z)\overline{B_k(z)}\, d\mu_n\\ &\quad - \frac{a_k e_k}{\eta_n}\int_0^{2\pi} \phi_k^*(z)\overline{B_k(z)}\, d\mu_n.\end{aligned}$$

Notice from (4.3) and (4.4), we have

$$\int_0^{2\pi} \phi_k(z)\overline{B_k(z)}\, d\mu_n = \frac{\|\phi_k\|^2_{\mu_n}}{\kappa_k} = \frac{1}{\kappa_k},$$

and

$$\int_0^{2\pi} \phi_k^*(z)\overline{B_k(z)}\, d\mu_n = \int_0^{2\pi} \overline{\phi_k(z)}\, d\mu_n = 0$$

because of (4.1). Then we get

$$\frac{\bar{e}_k}{\kappa_k} = \frac{1-|z_k|^2}{1-|z_{k-1}|^2}\frac{\kappa_{k-1}}{\kappa_k}\int_0^{2\pi} \frac{z-z_{k-1}}{1-\bar{z}_k z}\phi_{k-1}(z)\overline{B_k(z)}\, d\mu_n,$$

i.e.,

$$\bar{e}_k = \frac{1-|z_k|^2}{1-|z_{k-1}|^2}\kappa_{k-1}\int_0^{2\pi} \phi_{k-1}(z)\overline{\frac{1-\bar{z}_{k-1}z}{z-z_k}B_k(z)}\, d\mu_n. \tag{4.5}$$

From Lemma 3.2, we obtain

$$\begin{aligned}\int_0^{2\pi} \phi_{k-1}(z)\overline{\frac{1-\bar{z}_{k-1}z}{z-z_k}B_k(z)}\, d\mu_n &\\ &= \frac{z_k - z_{k-1}}{\kappa_k(1-|z_k|^2)}\int_0^{2\pi} \phi_{k-1}(z)\overline{\phi_k(z)}\, d\mu_n\\ &\quad + \int_0^{2\pi} \phi_{k-1}(z)\overline{Q_{n-2}(z)}\, d\mu_n\\ &\quad - \eta_k\frac{(1-z_{k-1}\bar{z}_k)\phi_k^*(z_{k-1})}{\kappa_k(1-|z_k|^2)}\int_0^{2\pi} \phi_{k-1}(z)\overline{B_{k-1}(z)}\, d\mu_n.\end{aligned}$$

From (4.1) and (4.4), then

$$\int_0^{2\pi} \phi_{k-1}(z)\overline{\frac{1-\bar{z}_{k-1}z}{z-z_k}B_k(z)}\,d\mu_n$$
$$= -\eta_k \frac{(1-z_{k-1}\bar{z}_k)\phi_k^*(z_{k-1})}{\kappa_k(1-|z_k|^2)}\frac{\|\phi_{k-1}\|_{\mu_n}^2}{\kappa_{k-1}}.$$

Combining with (4.5), we get

$$\bar{e}_k = -\frac{1-|z_k|^2}{1-|z_{k-1}|^2}\eta_k \frac{(1-z_{k-1}\bar{z}_k)\phi_k^*(z_{k-1})}{\kappa_k(1-|z_k|^2)}\|\phi_{k-1}\|_{\mu_n}^2.$$

Together with (iii) for $n=k$ in Lemma 3.1, we obtain $\|\phi_{k-1}\|_{\mu_n} = 1$. We prove (4.2). This completes the proof of the theorem. □

Proof of Theorem 2.2. The existence. Since

$$\mu_n(t) := \int_0^t d\mu_n$$

is increasing and uniformly bounded, by Helly's selection theorem and convergence theorem, there is a subsequence $\{n_k\}$ and an increasing function μ on $[0, 2\pi)$ such that

$$\lim_{k\to\infty} \mu_{n_k}(\theta) = \mu(\theta)$$

and

$$\lim_{k\to\infty} \int_0^{2\pi} f(z)\,d\mu_{n_k}(\theta) = \int_0^{2\pi} f(z)\,d\mu(\theta)$$

for $f \in C(T)$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_l(z)\overline{\phi_j(z)}\,d\mu(\theta) = \lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi} \phi_l(z)\overline{\phi_j(z)}\,d\mu_{n_k} = \delta_{l,j}.$$

The uniqueness comes from the unique representation of bounded linear functions on $C(T)$. □

Proof of Theorem 2.3. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of complex in $|z| < 1$ and dense in $|z| \le 1$. We now define $\{a_n\}_{n=1}^{\infty}$ and $\{\Phi_n(z)\}_{n=0}^{\infty}$ as follows

$$\Phi_0(z) = 1, \qquad z_0 = 0,$$

for $n \geq 1$, define

$$a_n = \frac{z_{n-1} - v_n}{1 - \bar{z}_n v_n} \frac{\Phi_{n-1}(v_n)}{\Phi^*_{n-1}(v_n)}$$

and

$$\psi_n(z) = \frac{z - z_{n-1}}{1 - \bar{z}_n z} \Phi_{n-1}(z) + a_n \frac{1 - \bar{z}_{n-1} z}{1 - \bar{z}_n z} \Phi^*_{n-1}(z),$$

$$\Phi_n(z) = \frac{\psi_n(z)}{\psi^*_n(z_n)}.$$

Then it is easy to see that $|a_n| < 1$ and $\Phi_n(v_n) = 0$. From Lemma 3.1, Theorems 2.1 and 2.2, we can find a measure $d\mu \geq 0$ such that $\{\phi_n(z)\}_{n=0}^{\infty}$ is a sequence of orthonormal rational functions with respect to $d\mu$ and $\phi_n(v_n) = 0$. □

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