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GEOMETRY OF BANACH SPACES WITH (α, ε)-PROPERTY OR (β, ε)-PROPERTY

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ABSTRACT. Many authors investigated Banach spaces with property α or property β. They showed that a space with property α (property β) shares many geometrical properties of $l_1(l_\infty)$. We shall investigate the structure of the unit sphere of Banach spaces with property α or β in terms of points of local uniform rotundity, Fréchet differentiability and vertex points. As a consequence of this, we obtain that every Banach space can be renormed in such a way that there is no locally uniformly rotund point but the set of points of Fréchet differentiability for the norm is an open and norm dense subset of the space.

1. Introduction. Properties A and B were defined by J. Lindenstrauss [**7**] in the study of norm attaining operators. The Banach space X has property A if, for every Banach space Y, the norm attaining operators are dense in $L(X, Y)$ and the Banach space Y has property B if, for every Banach space X, the norm attaining operators are dense in $L(X, Y)$. He gave two geometric criteria for property A and B named property α and β [**7, 13**]. J. Partington [**12**] proved that every Banach space can be $(3+\varepsilon)$-equivalently renormed to have property β but, if the continuum hypothesis is assumed, a nonseparable Banach space is constructed in [**11**] which cannot be equivalently renormed to have property α. As a consequence, we observe that not every dual Banach space admits an equivalent dual norm with property β. Properties α and β generalize, in some sense, the geometric situation of l_1 and l_∞, as is pointed out in [**3, 5, 6** and **13**].

A *vertex* point of a closed bounded convex body C is a point which is strongly exposed by an open set of functionals. A *face* is the

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intersection of a hyperplane with the boundary of C, in the case that this intersection has nonempty interior in the relative topology to the hyperplane. The duality between both concepts is a good tool to study some geometrical properties shared by l_1 and l_∞ with spaces having property α and property β. One of these properties is that the unit ball contains no locally uniformly rotund point. From this fact, we deduce that the set of norms failing property α or property β is residual. Also, the set of points of Fréchet differentiability for the norm on a Banach space with property β is open and dense. This result generalizes the situation of l_∞.

A particular class of vertex points are the *strongly vertex points*. Some properties of these points are studied. For example, it is not difficult to see that strongly vertex points are isolated in the set of extreme points. Conversely, if the space has the Krein-Milman property, a denting point isolated in the set of extreme points is strongly vertex. Using this result, we prove that every extreme point of the unit ball of Lorentz sequence space $d(\omega, 1)$ is strongly vertex, and the dual norm is Fréchet differentiable in an open dense set.

Finally, we mention that a Banach space X having a biorthogonal system with cardinality equal to $\operatorname{dens} X$ can be $(1+\varepsilon)$-equivalently renormed to have property α [**6**]. It is proved in [**4**] that even $l^\infty(\Gamma)$ admits a fundamental biorthogonal system with the cardinality of Γ so we can say that, in a geometrical sense, $l^\infty(\Gamma)$ can be renormed close to $l_1(\Gamma)$. Throughout this paper, we use the notation (α, ε)-property or (β, ε)-property instead of property α or property β in order to make a reference to the parameter ε appearing in the definition of both properties.

2. Faces and vertex points of convex bodies. We only consider Banach spaces over the reals. Given a Banach space X, we denote by $B(X)$ the closed unit ball, by $S(X)$ the unit sphere and by X^* the dual space of X.

Definition 2.1. Given $f \in X^* \backslash \{0\}$ and $0 < \rho < 1$ the set

$$K(f, \rho) = \{x \in X : f(x) \geq \rho \|f\| \, \|x\|\}$$

is called a *ρ-cone*. It is not difficult to see that $K(f, \rho)$ is a closed convex set. Let C be a closed, bounded convex set and $x \in C$. The point x is said to be a *vertex point* of C if there exists a ρ-cone $K(f,\rho)$ so that

$$C \subseteq x - K(f,\rho).$$

We also say that x is a vertex point of C with respect to f.

Example 2.2. (i) Every point of the unit vector basis $\{e_n\}_{n\in\mathbf{N}}$ of l_1 is a vertex point of the unit ball $B(l_1)$.

(ii) Let X be an arbitrary Banach space, x a point of its unit sphere and $\lambda > 1$. If we consider the set

$$B_\lambda(X) = \text{conv}\,(\{\pm\lambda x\} \cup B(X)),$$

then the point λx is a vertex point of $B_\lambda(X)$.

The example (ii) motivates the following definition. Let C be a closed, bounded convex set and $x \in C$. The point x is said to be a *strongly vertex point* of C if there exists a closed, bounded convex subset $D \subset C$ with $x \notin D$ satisfying

$$C = \text{conv}\,(\{x\} \cup D).$$

Let $f \in X^*\backslash\{0\}$ attaining its maximum in C. The set

$$C_f = \{x \in C : f(x) = \sup_C f\}$$

is called a *face* (with respect to f) whenever it has nonempty interior in the relative topology of $\{x \in X : f(x) = \sup_C f\}$. We denote this interior by $\text{int}\,(C_f)$ and the boundary of C by ∂C. The point $x \in C$ is said to be a strongly exposed point provided there is $f \in X^*\backslash\{0\}$ with $f(x) = \sup_C f$ satisfying that $\lim_n \|x_n - x\| = 0$ whenever $\{x_n\} \subset C$ and $\lim_n f(x_n) = f(x)$. We also say that f strongly exposes x in C.

It is clear that every strongly vertex point is vertex and that every vertex is a strongly exposed point. The next propositions actually prove more, namely, the duality between vertex and faces. A vertex of a closed, bounded convex body C induces a face in its polar set C^0, and a face in C is induced by a vertex in C^0.

Lemma 2.3. *Given $f \in X^*$ and $0 < \rho < 1$, define the set*

$$\tilde{K}(f,\rho) = \{g \in X^* : \text{ there exists } 0 < \gamma < 1 \text{ with } K(f,\rho) \subseteq K(g,\gamma)\}.$$

Then:

(i) *$\tilde{K}(f,\rho)$ is an open set.*

(ii) $\lim_{\rho\to 0} \operatorname{diam}(\tilde{K}(f,\rho) \cap S(X^*)) = 0$.

(iii) *$g \in \tilde{K}(f,\rho)$ if and only if $-g$ strongly exposes $0 \in K(f,\rho)$.*

Proof. (i) Let $g \in \tilde{K}(f,\rho)$ and $0 < \gamma < 1$ be such that $K(f,\rho) \subseteq K(g,\gamma)$. We are going to prove that, for every $0 < \eta < \gamma$, we have $K(g,\gamma) \subseteq K(h,\eta)$ whenever $\|g - h\| < (1/2)\|g\|(\gamma - \eta)$. Indeed, if $y \in K(g,\gamma)$, then

$$\begin{aligned} h/\|h\|(y/\|y\|) &= g/\|g\|(y/\|y\|) \\ &\quad + (h/\|h\| - g/\|g\|)(y/\|y\|) \\ &\geq \gamma - \|h\|g\| - \|h\|g\|\,(\|h\|\,\|g\|)^{-1} \\ &\geq \gamma - 2\|h - g\|\,\|g\|^{-1} \\ &\geq \eta. \end{aligned}$$

(ii) Assume that $\|f\| = 1$. To prove (ii) it is enough to show that $\lim_n \|g_n - f\| = 0$ whenever

$$g_n \in \tilde{K}(f, n^{-1}) \cap S(X^*).$$

For every $n \in \mathbf{N}$, there exists $\gamma_n \in (0,1)$ such that $K(f,n^{-1}) \subseteq K(g_n,\gamma_n)$. First we see

$$\operatorname{Ker} g_n \subset \{x \in X, |f(x)| < n^{-1}\|x\|\}, \qquad n \in \mathbf{N}. \tag{2.1}$$

Let $y \in \operatorname{Ker} g_n$ so that $|f(y)| \geq n^{-1}\|y\|$. This implies that $\alpha y \in K(f,n^{-1})$, where $\alpha = \operatorname{sign} f(y)$. Hence, $\alpha y \in K(g_n,\gamma_n)$, which is a contradiction.

Since $\tilde{K}(f,n^{-1})$ is an open set, according to the Bishop-Phelps theorem (cf. [**1**, p. 13]) we may assume that g_n attains its norm at some point $y_n \in S(X)$, i.e., $g_n(y_n) = 1$. It is easy to see that

$$B(X) \subseteq \{z + \lambda y_n : z \in \operatorname{Ker} g_n, \|z\| \leq 2, |\lambda| \leq 1\}, \qquad n \in \mathbf{N}. \tag{2.2}$$

Now choose $x_n \in S(X)$ such that

$$f(x_n) \geq 1 - n^{-1}. \tag{2.3}$$

For $n \geq 2$ we have $f(x_n) \geq 1 - n^{-1} \geq n^{-1}$. Then $x_n \in K(f, n^{-1}) \subseteq K(g_n, \gamma_n)$ and, hence, $g_n(x_n) \geq \gamma_n$. Using (2.2) we can find $z_n \in \operatorname{Ker} g_n$ with $x_n = z_n + \lambda_n y_n$, $\|z\| \leq 2$, and $|\lambda_n| \leq 1$. Therefore,

$$g_n(x_n) = \lambda_n \geq \gamma_n > 0,$$

that is, $\lambda_n > 0$, $n \in \mathbf{N}$. Also, by (2.1) it is clear that $|f(z_n)| \leq 2n^{-1}$, $n \in \mathbf{N}$. Thus, it follows from (2.3) that

$$1 - n^{-1} \leq f(z_n + \lambda_n y_n) \leq 2n^{-1} + \lambda_n f(y_n) \tag{2.4}$$

and, as $0 < \lambda_n \leq 1$, then

$$f(y_n) \geq \lambda_n f(y_n) \geq 1 - 3n^{-1}. \tag{2.5}$$

Let $x \in B(X)$. By (2) we can find $z \in \operatorname{Ker} g_n$, $\|z\| \leq 2$, so that $x = z + \lambda y_n$, $|\lambda| \leq 1$. Using (1), (4) and (5), we obtain the following estimate:

$$\begin{aligned} |(f - g_n)x| &= |(f - g_n)(z + \lambda y_n)| \\ &= |f(z) + \lambda(f(y_n) - 1)| \\ &\leq |f(z)| + |\lambda|(1 - f(y_n)) \\ &\leq 2n^{-1} + 3n^{-1} = 5n^{-1}. \end{aligned}$$

So $\|f - g_n\| \leq 5n^{-1}$ and (ii) is proved.

(iii) If g strongly exposes 0 in $K(f, \rho)$, there exists $\gamma > 0$ such that $g(x/\|x\|) < -\gamma$ for every $x \in K(f, \rho)\backslash\{0\}$ and, equivalently, $-g(x) > \gamma\|x\|$. This implies that

$$K(f, \rho) \subseteq K(-g, \gamma\|g\|^{-1})$$

so $-g \in \tilde{K}(f, \rho)$. Assume now that $-g \in \tilde{K}(f, \rho)$, and let $\gamma \in (0, 1)$ so that $K(f, \rho) \subseteq K(-g, \gamma)$. If g does not strongly expose 0 in $K(f, \rho)$, we can find a sequence $\{x_n\}_1^\infty \subset K(f, \rho)$ with $\|x_n\| > \delta > 0$ and $\lim_n g(x_n) = 0$. This implies that $\lim g(x_n/\|x_n\|) = 0$, which contradicts the fact that $x_n/\|x_n\| \in K(-g, \gamma)$. □

Lemma 2.4. *Let $f, g \in X^*$ and $0 < \rho < 1$.*

(i) *If there is a $\gamma \in \mathbf{R}$ so that $g(x) < \gamma$ for every $x \in K(f,\rho)$, then $g(x) \leq 0$, $x \in K(f,\rho)$.*

(ii) *If $g(x) \leq 0$ for every $x \in K(f,\rho)$ and $0 < \rho < \sigma < 1$, then g strongly exposes 0 in $K(f,\sigma)$.*

Proof. We shall only prove (ii). Suppose that g does not strongly expose 0 in $K(f,\sigma)$. Then there exists a sequence $\{x_n\}_1^\infty \subset K(f,\sigma)$ such that $\lim g(x_n) = 0$ and $\|x_n\| \geq \delta > 0$. Set $y_n = x_n/\|x_n\|$. We have $y_n \in K(f,\rho)$, $\lim g(y_n) = 0$ and $\|y_n\| = 1$. Without loss of generality, we may assume that $\|f\| = \|g\| = 1$. Let $x \in S(X)$, $g(x) > 0$, and consider $a = (\sigma - \rho)/2$. Since

$$\begin{aligned} f(y_n + ax) &= f(y_n) + af(x) \geq \sigma\|y_n\| - a\|x\| \\ &= \sigma - a = \rho + a \geq \rho\|y_n + ax\| \end{aligned}$$

we get $y_n + ax \in K(f,\rho)$. Then $g(y_n + ax) \leq 0$, a contradiction with the fact that $\lim g(y_n + ax) = ag(x) > 0$. □

Proposition 2.5. *Let C be a closed, bounded convex body with $0 \in \operatorname{int}(C)$. Consider a functional $f \in C^0$, the polar of C, and suppose that the set $C_f = \{x \in C : f(x) = 1\}$ is a face of C. Then f is a vertex point of C^0.*

Proof. First of all, we see that $x \in \operatorname{int}(C_f)$ implies that x strongly exposes f in C^0. Indeed, assume that there exists a sequence $\{f_n\}_1^\infty \subset C^0$, $\lim f_n(x) = 1$ and $\|f_n - f\| > \varepsilon > 0$. We claim that, for every n verifying $|(f_n - f)x| < (\varepsilon/2)\|f\|^{-1}$ there is $y_n \in \operatorname{Ker} f \cap S(X)$, $f_n(y_n) \geq \varepsilon/(2 + 2\|f\|)$. Since

$$B(x) \subseteq \{z + \lambda x, z \in \operatorname{Ker} f, \|z\| \leq 1 + \|f\|, |\lambda| \leq \|f\|\}$$

and $\|f - f_n\| > \varepsilon$, there exist $z \in \operatorname{Ker} f$, $\|z\| \leq 1 + \|f\|$ and $\lambda \in \mathbf{R}$, $|\lambda| \leq \|f\|$ such that

$$|f_n(z) + \lambda(f_n - f)x| > \varepsilon;$$

hence

$$|f_n(z)| > \varepsilon - \|f\|\,|(f_n - f)x| > \varepsilon/2,$$

so setting $y_n = \alpha z/\|z\|$, $\alpha = \operatorname{sign}(f_n(z))$, we have $y_n \in \operatorname{Ker} f_n \cap S(X)$ and $f_n(y_n) \geq \varepsilon/(2+2\|f\|)$, which proves the claim. Now, take $0 < \delta$ verifying $\{y \in X : f(y) = 1, \|x - y\| < \delta\} \subset C_f$ and $n_0 \in \mathbf{N}$ such that

$$f_{n_0}(x) > 1 - \frac{\delta\varepsilon}{4 + 4\|f\|}.$$

Then, $z = x + \delta y_{n_0} \in C$ and

$$f_{n_0}(z) = f_{n_0}(x + \delta y_{n_0}) \geq 1 - \frac{\delta\varepsilon}{4+4\|f\|} + \frac{\delta\varepsilon}{2+2\|f\|},$$

a contradiction. Thus, x strongly exposes f in C^0.

By Lemma 2.3 (ii), we can choose $0 < \rho < 1$ so small that $y \in \operatorname{int}(C_f)$ whenever $y \in \tilde{K}(x,\rho) \cap \partial C$. Suppose that $C^0 \backslash f - K(x,\sigma) \neq \varnothing$ for some $0 < \sigma < \rho$, and let $g \in C^0 \backslash f - K(x,\sigma)$. The set $f - K(x,\sigma)$ is w^*-closed so, by the Hahn-Banach theorem, there exist $z \in X$ and $\tau > 0$ such that

$$g(z) > \tau > h(z)$$

for every $h \in f - K(x,\sigma)$. Lemma 2.4 (i) shows that z attains its maximum over the set $f - K(x,\sigma)$ at f. Hence, by Lemma 2.4 (ii), z strongly exposes f in $f - K(x,\rho)$, and Lemma 2.3 (iii) implies that $z \in \tilde{K}(x,\rho)$. To finish the proof, it is enough to consider

$$\eta = \sup\{\varphi(z) : \varphi \in C^0\} < \infty$$

and $\omega = z/\eta$. From the bipolar theorem, it follows that $\omega \in \partial C$ so $\omega \in \operatorname{int}(C_f)$, $f(\omega) = 1$ and it is not possible that $g(\omega) > f(\omega)$ because $g \in C^0$. □

Corollary 2.6. *Let C be a closed, bounded convex set with $0 \in \operatorname{int}(C)$. Let $x \in C$ and suppose that the set $\{f \in C^0 : f(x) = 1\}$ is a face of C^0. Then x is a vertex point of $C^{00} \subset X^{**}$ and hence of C.*

Proposition 2.7. *Let x be a vertex point of a closed, bounded convex set C with $0 \in \operatorname{int}(C)$. Then*

(i) *The set $C^0_x = \{f \in C^0 : f(x) = 1\}$ is a face of C^0.*

(ii) *The point x is a vertex of C with respect to every $g \in \operatorname{int}(C^0_x)$.*

Proof. Lemma 2.3 (i) shows that $\tilde{K}(f,\rho)$ is an open set. To prove (i) it is enough to show that

$$\tilde{K}(f,\rho) \cap \partial C^0 \subset C^0_x.$$

Let $g \in \tilde{K}(f,\rho) \cap \partial C^0$. There exists $\gamma \in (0,1)$ such that $K(f,\rho) \subseteq K(g,\gamma)$. Therefore, $C \subset x - K(g,\gamma)$ so it is clear that g exposes x in $x - K(g,\gamma)$ and, hence, in C. Thus

$$g(x) = \sup\{g(y) : y \in C\} = 1$$

and $g \in C^0_x$. Part (ii) follows directly from Proposition 2.5. □

In other words, we have proved that, given a closed, bounded convex set C with $0 \in \operatorname{int}(C)$ and a vertex point x of C, x is vertex with respect to an open set of functionals in X^* which also strongly expose x at C. The closure of this set consists of functionals attaining their maximum over C in x.

Definition 2.8. A point x of a closed, bounded convex body C is said to be *locally uniformly rotund* (lur) if $\lim_n \|x_n - x\| = 0$ whenever there exists f_n, $f \in X^*$ such that

(i)
$$\sup\{f_n(c) : c \in C\} \le \sup\{f(c) : c \in C\} = f(x)$$

(ii)
$$\lim_n f_n((x_n + x)/2) = f(x), \qquad x_n \in C.$$

If C is the unit ball, the summation of (i) and (ii) is equivalent to

(iii)
$$\lim_n \|(x_n + x)/2\| = 1, \qquad \|x\| = 1, \qquad \|x_n\| \le 1.$$

Also we say that x is a *denting* point of C if, for every $v > 0$ there exists $f \in X^*$ and $0 < \delta < f(x)$ such that $\operatorname{diam}\{y \in C : f(y) \ge \delta\} < v$. We denote by $\operatorname{lur} C$ the set of locally uniformly rotund points of C, by ext C the set of extreme points, and so on. A norm is said to be locally uniformally rotund (LUR) if every point of its unit ball is lur.

Observation 2.9. It is clear that not every vertex point is strongly vertex. The main reason to distinguish both concepts is because strongly vertex points are not lur points.

3. Banach spaces with vertex or faces in its unit sphere.

Definition 3.1. The Banach space X has (α, ε)-*property* if there exists a system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ and $0 \leq \varepsilon < 1$ such that

$$x_i^*(x_i) = \|x_i\| = \|x_i^*\| = 1, \qquad |x_i^*(x_j)| \leq \varepsilon, \qquad i \neq j. \tag{3.1}$$

$$B(X) = \overline{\text{conv}}(\{\pm x_i\}_{i\in I}). \tag{3.2}$$

The Banach space X has (β, ε)-*property* if there exists a system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ and $0 \leq \varepsilon < 1$ satisfying (3.1) and

$$\|x\| = \sup_{i\in I} |x_i^*(x)|, \qquad x \in X. \tag{3.3}$$

In the following proposition, we examine the geometry of Banach spaces with (β, ε)-property by using an idea of [**13**].

Proposition 3.2. *Let X be a Banach space with (β, ε)-property. Then the set of faces of the unit ball with diameter greater than or equal to $1 - \varepsilon$ is dense in the unit sphere.*

Proof. Let $\{x_i, x_i^*\}_{i\in I}$ be a system verifying (3.1) and (3.3). For every $i \in I$, define

$$F_i = \{x \in B(X) : x_i^*(x) = 1\}.$$

If we denote by $\text{int}(F_i)$ the interior of F_i in the relative topology to the set $\{x \in X : x_i^*(x) = 1\}$, then $x_i \in \text{int}\, F_i$. Indeed, if $\|x\| = 1$ and $\|x - x_i\| < (1 - \varepsilon)/2$ we have

$$\begin{aligned}
x_i^*(x) &= x_i^*(x_i) + x_i^*(x - x_i) \\
&> 1 - (1 - \varepsilon)/2 = (1 + \varepsilon)/2 \\
|x_j^*(x)| &= |x_j^*(x_i + x - x_i)| \leq \varepsilon + (1 - \varepsilon)/2 \\
&= (1 + \varepsilon)/2 < 1, \qquad j \neq i,
\end{aligned}$$

and this implies, by (3.3), that $x_i^*(x) = 1$, so $x \in F_i$. Therefore, F_i is a face with $\operatorname{diam}(F_i) \geq 1 - \varepsilon$ (F_i contains the relative ball $(x_i + 2^{-1}(1-\varepsilon)B(X)) \cap \{x \in X : x_i^*(x) = 1\}$).

We only need to prove that the unit sphere $S(X)$ is the closure of $\cup_{i \in I} F_i$. Given $x \in S(X)$ and $0 < \delta < 1$, there exists $i \in I$ such that $|x_i^*(x)| > 1 + \delta\varepsilon - \delta$. Setting $z = x + (\operatorname{sign} x_i^*(x))\delta x_i$, then

$$|x_i^*(z)| > 1 + \delta\varepsilon - \delta + \delta = 1 + \delta\varepsilon$$
$$|x_j^*(z)| \leq 1 + \delta\varepsilon, \qquad i \neq j,$$

so $z/\|z\| \in \pm F_i$ and $\|x - z/\|z\|\| < 2\delta$. □

A first consequence of the previous proposition is that, if we assume the continuum hypothesis, the question of whether every dual Banach space admits an equivalent dual norm with property β has a negative answer. Indeed, a norm on X^* having a face in its unit sphere with respect to a functional of $X^{**} \backslash X$ is not dual. Then, if a dual norm has property β with respect to the system $\{x_i^*, x_i^{**}\}_{i \in I} \subset X^* \times X^{**}$, necessarily $x_i^{**} \in X$, $i \in I$, and thus X has property α. As it was observed in [**5**], the Kunen space cannot be equvalently renormed with property α.

Proposition 3.3. *Let X be a Banach space with (β, ε)-property. Then* $\operatorname{lur} B(X) = \varnothing$ *and the set of points of Fréchet differentiability for the norm is open and norm dense.*

Proof. Let $\{x_i, x_i^*\}_{i \in I} \subset X \times X^*$ be a system verifying (3.1), (3.3) and F_i, as in the preceding proposition. Clearly, if $x \in \operatorname{int}(\pm F_i)$, then $x \notin \operatorname{lur} B(X)$. In the general case, given $x \in S(X)$ and $0 < \delta < (1-\varepsilon)/2$, there exists $i \in I$ and $y \in \pm F_i$ such that $\|x - y\| < \delta$. By the preceding proposition we can find $z \in F_i$, $\|y - z\| > (1-\varepsilon)/2$. Then

$$\|x - z\| \geq \|y - z\| - \|x - y\| = (1-\varepsilon)/4 > 0$$

and, also,

$$\|x + z\| \geq \|y + z\| - \|x - y\| \geq 2 - (1-\varepsilon)/4$$

which proves the first part of the proposition. For the second part, the norm is Fréchet differentiable at every point of int (F_i), $i \in I$, and the set $\cup_{i\in I}(\pm\text{int}\, F_i)$ has been proved to be dense in the unit sphere. On the other hand, if $x \in S(X)\setminus \cup_{i\in I} (\pm\text{int}\, F_i)$, then either there is a sequence $\{x_n^*\} \subset \{x_i^*\}_{i\in I}$ of pairwise different functionals satisfying $\lim_n x_n^*(x) = 1$ or there exist $i, j \in I$, $i \neq j$, with $x_i^*(x) = x_j^*(x) = 1$. Using the Šmulyan test, c.f., e.g., [**1**, p. 3], we obtain that, in both cases, x is not a point of Fréchet fifferentiability for the norm. □

Corollary 3.4. *Every Banach space admits a* $(3+\varepsilon)$*-equivalent norm Fréchet differentiable in an open dense set with no locally uniformly rotund point.*

Proof. The proof follows directly from Corollary 3.2, and the crucial result of Partington [**12**] stated in the introduction. □

We mention now that every Banach space admitting an LUR norm has the following property: every equivalent norm can be uniformly approximated by a norm Fréchet differentiable in an open dense set [**9**]. See also [**10**] for this subject. However, if X is a non-Asplund space, then X admits no LUR norm Fréchet differentiable in an open set. Moreover, the existence of a norm Fréchet differentiable in an open set which contains a lur point implies the existence of a Fréchet differentiable norm in the whole space except at the origin. The proof of this fact is based on the following lemma.

Lemma 3.5. *Let X be a Banach space and $\|\cdot\|$ an equivalent norm Fréchet differentiable in an open set $U \subset S(X)$. Suppose that there exists $f \in S(X^*)$ and $0 < \delta < 1$ such that $\{x \in S(X) : f(x) = \delta\} \subseteq U$. Then X admits a Fréchet differentiable norm.*

Proof. Consider the set $V = \{x \in B(X) : f(x) = \delta\}$ and a point $x_0 \in V$, with $\|x_0\| < 1$. Let $H = \text{Ker}\, f$ and define

$$W = \{y - x_0 : y \in V\} \subset H.$$

It is clear that W is a closed convex and bounded neighborhood of 0 in the relative topology of H. Let p be the Minkowski functional on

H relative to the set W. The implicit function theorem used in the equation

$$\|x_0 + p^{-1}(h)h\| = 1, \qquad h \in H$$

asserts that p is Fréchet differentiable, so $q(h) = p(h) + p(-h)$ is an equivalent Fréchet differentiable norm on H. Since X is isomorphic to $H \oplus \mathbf{R}$, we can define a new equivalent norm

$$|(h,r)| = (q^2(h) + r^2)^{1/2}, \qquad h \in H,\ r \in \mathbf{R}$$

which is Fréchet differentiable. $\square$

Notice that in separable Asplund spaces there exist norms which are locally uniformly rotund and Fréchet differentiable. Also, there exist some non Asplund spaces admitting a rotund norm Fréchet differentiable in an open dense set. In the next proposition we turn back to examine the geometrical structure of Banach spaces with property α.

Proposition 3.6. *Let X be a Banach space with (α, ε)-property. Then its unit ball $B(X)$ is the closed, convex hull of its strongly vertex points and*

(a) $$\operatorname{dent} B(X) = \operatorname{strver} B(X)$$

(b) $$\operatorname{lur} B(X) = \varnothing$$

Proof. If the Banach space X has (α, ε)-property, then there exists a system $\{x_i, x_i^*\}_{i \in I} \subset X \times X^*$ satisfying (3.1) and (3.2). Let $x \in S(X) \backslash \{\pm x_i\}_{i \in I}$. Suppose that, for every $n \in \mathbf{N}$, we can find $f_n \in S(X^*)$ and $\delta_n \in (0,1)$ such that $\operatorname{diam}\{y \in B(X) : f(y) \geq \delta_n\} < 1/n$ and $f_n(x) \geq \delta_n$. From property (3.2) we can choose, for every $n \in \mathbf{N}$, $i_n \in I$ so that $x_{i_n} \in \{y \in B(X) : f(y) \geq \delta_n\}$. We may assume that $x_{i_n} \neq x_{i_m}$, $n \neq m$ (taking a suitable subsequence) so $\{x_{i_n}\}$ is a Cauchy sequence, which is a contradiction and (a) is proved.

To prove (b), every lur point is a denting point; thus, we only need to check the set $\{\pm x_i\}_{i \in I}$. For every $i_0 \in I$, we can write

$$B(X) = \overline{\operatorname{conv}}(\{\pm x_i\}) = \operatorname{conv}(\{\pm x_{i_0}\} \cup \overline{\operatorname{conv}}(\{\pm x_i\}_{i \in I \backslash \{i_0\}})) \tag{3.4}$$

By property (3.1), $\operatorname{dist}(x_{i_0}, \overline{\operatorname{conv}}(\{\pm x_i\}_{i\in I\setminus\{i_0\}})) \geq 1-\varepsilon$ so x_{i_0} is a strongly vertex point of $B(X)$ and, therefore, x_{i_0} is the extreme point of many segments contained in $S(X)$. Thus $x_{i_0} \notin \operatorname{lur} B(X)$ and (b) is proved. □

Having in mind the fact that the Banach space X has (α,ε)-property with respect to the system $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ if and only if its dual space X^* has (β,ε)-property with respect to the system $\{x_i^*, x_i\}_{i\in I} \subset X^* \times X \subset X^* \times X^{**}$, it is clear that a Banach space with (α,ε)-property verifies

$$\operatorname{lur} B(X) = \operatorname{lur} B(X^*) = \operatorname{lur} B(X^{**}) = \varnothing.$$

On the other hand, if X is a Banach space having (β,ε)-property with respect to the system $\{x_i, x_i^*\}_{i\in I}$, then X^* does not necessarily have (α,ε)-property with respect to $\{x_i^*, x_i\}_{i\in I}$. However, we can insure, for every $i \in I$, that x_i^* is a vertex point of $B(X^*)$ with respect to x_i. This fact is an immediate consequence of Proposition 3.2.

Given a Banach space X, let $(N(X),\rho)$ be the metric space of all equivalent norms with the uniform metric. The space $(N(X),\rho)$ is a complete metric space so it is also a Baire space.

Corollary 3.7. *The set of norms failing to have (α,ε)-property or (β,ε)-property for every $\varepsilon \in [0,1)$ is residual in $(N(X),\rho)$.*

Proof. From the preceding results, it is enough to prove that the set $\{|\cdot| \in (N(X),\rho) : \operatorname{lur} B_{|\cdot|} \neq \varnothing\}$ is residual in $(N(X),\rho)$. In [**1**] it is proved that the set of lur norms in $(N(X),\rho)$ is either empty or residual. The same proof given in [**1**] applies to show that the set of norms locally uniformly rotund in a fixed point $x \in X$ is also either empty or residual, but actually this set is never empty as the following simple geometrical construction proves. Let $x_0 \in X$, $f \in X^*$, $f(x_0) = 1 = \|x\| = \|f\|$. Define, by induction,

$$\begin{aligned} B_0 &= B(X)\setminus\{x \in B(X) : |f(x)| > 0\} \\ \hat{B}_0 &= \operatorname{conv}(\{\pm x_0\} \cup B_0) \\ B_1 &= \hat{B}_0\setminus\{x \in \hat{B}_0 : |f(x)| > 1/2\} \end{aligned}$$

$$\hat{B}_1 = \operatorname{conv}\left(\left\{\pm\left(1-\frac{1}{4}\right)x_0\right\}\cup B_1\right)$$
$$B_n = \hat{B}_{n-1}\Big\backslash\left\{x\in\hat{B}_{n-1} : |f(x)| > \frac{1}{2}+\cdots+\frac{1}{2^{2n-1}}\right\}$$
$$\hat{B}_n = \operatorname{conv}\left(\left\{\pm\left(1-\cdots-\frac{1}{4^n}\right)x_0\right\}\cup B_n\right)$$
$$B = \bigcap_n \hat{B}_n.$$

The Minkowski gauge relative to the set B is a norm $|\cdot|\in(N(X),\rho)$ such that $(2/3)x_0\in\operatorname{lur}B_{|.|}$. Observe also that $(2/3)x_0$ is an isolated point in the set $\operatorname{lur}B_{|.|}$. ◻

Let C be a closed, convex subset of a Banach space. Then C is said to have the *Krein-Milman property*, KMP, if each closed bounded convex subset K of C satisfies $K=\overline{\operatorname{conv}}(\operatorname{ext}K)$ where $\operatorname{ext}K$ is the set of extreme points of K. Equivalently, C has the KMP if each closed, bounded convex subset of C has an extreme point. The following proposition can be used to recognize strongly vertex points in some special sets.

Proposition 3.8. *Let C be a closed, bounded convex set with the* KMP. *Then $x\in C$ is a strongly vertex point of C if and only if x is denting and there exists a neighborhood V of x such that $V\cap\operatorname{ext}C=\{x\}$.*

Proof. If $x\in\operatorname{strver}C$, then there exists a closed, convex subset $D\in C$, $x\notin C$, such that $C=\operatorname{conv}(\{x\}\cup D)$. By using a separation argument, we can find $f\in X^*$ and $\sigma>0$ verifying

$$f(x)>\sigma>\sup\{f(y):y\in D\}$$

so we can take $V=\{c\in C: f(c)>\sigma\}$.

On the other hand, suppose that $x\in C$ is a denting point with a neighborhood $x\in V$ so that $\operatorname{ext}C\cap V=\{x\}$. There exists $f\in X^*$ and $0<\sigma<f(x)$ with $S=\{c\in C: f(c)>\sigma\}\subset V$. We claim that $C=\operatorname{conv}(\{x\}\cup(C\backslash S))\}$. Indeed, assume that $y\in C\backslash\operatorname{conv}(\{x\}\cup(C\backslash S))$.

The Bishop-Phelps theorem insures the existence of a functional $g \in X^*$ attaining its supremum over C, and a positive number ζ satisfying

$$g(y) > \zeta > \{g(z) : z \in \operatorname{conv}(\{x\} \cup (C \backslash S))\}.$$

Let $\vartheta = \sup\{g(c) : c \in C\}$. By using the KMP, the set $\{c \in C : g(c) = \vartheta\}$ contains an extreme point, a contradiction. □

Given a nonincreasing sequence of positive numbers $\omega = \{\omega_n\}_{n=1}^{\infty}$, the Lorentz sequence space $d(\omega, 1)$ consists of all sequences of scalars $x = (a_1, a_2, \dots)$ for which

$$\|x\| = \sup_{\pi} \left(\sum_{n=1}^{\infty} |a_{\pi(n)}| \omega_n \right) < \infty$$

where the supremum is taken over the set of all permutations π of the positive integers.

Corollary 3.9. *Consider the space* $d(\omega, 1)$*. Then*:

(i) *Every extreme point of the unit ball* $B_{d(\omega,1)}$ *is a strongly vertex point and*

$$B_{d(\omega,1)} = \overline{\operatorname{conv}}(\operatorname{strver} B_{d(\omega,1)}).$$

(ii) *The dual norm is Fréchet differentiable in an open dense set.*

Proof. Since $d(\omega, 1)$ is a separable dual space, cf., e.g., [**8**, p. 9], then, according to Bessaga-Pelczinski's theorem, it has the KMP, cf., e.g., [**2**, p. 160]. Also, every point of $\operatorname{ext} B_{d(\omega,1)}$ is denting point, and it is isolated in the set $\operatorname{ext} B_{d(\omega,1)}$.

To prove the second part, we use the KMP to obtain that, for every $f \in S(X^*)$ attaining its norm, there exists $i \in I$ such that $f(x_i) = 1$, so f belongs to the face $F_i = \{f \in d(\omega, 1)^* : \|f\| = f(x_i) = 1\}$. □

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