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# SOME ISOMORPHIC PREDUALS OF $\ell_1$ WHICH ARE ISOMORPHIC TO $c_0$

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ABSTRACT. We introduce property $(FS)$, which asserts that a Banach space has many $c_0$-"sub-decompositions," and show that if $X$ is a Banach space with property $(FS)$ and $X^*$ is isomorphic to $\ell_1$, then $X$ itself is isomorphic to $c_0$.

**1. Introduction.** For quite some time it has been of interest to find out under what circumstances a separable $\mathcal{L}_\infty$-space is isomorphic to the simplest example of an $\mathcal{L}_\infty$-space, namely $c_0$.

Ghoussoub and Johnson [**3**] showed that a separable $\mathcal{L}_\infty$-space, which embeds into an order continuous Banach lattice, is isomorphic to $c_0$. (This result is based on earlier results by Johnson and Zippin [**6**] and Rosenthal [**9**].)

Using isometric methods, Godenfroy and Li [**5**] showed that a separable $\mathcal{L}_\infty$-space, which can be renormed into an $M$-ideal in its bidual, is also isomorphic to $c_0$.

Godefroy, Kalton and Saphar [**4**, Proposition 7.8] showed that $c_0$ is the only isometric predual of $\ell_1$, which is a $u$-ideal.

In [**5**] the authors pose the question whether an isometric predual of $\ell_1$, which has property $(u)$, is isomorphic to $c_0$. We give a partial result in this direction:

**Definition 1.** A separable Banach space $X$ has *property* $(FS)$, if every shrinking finite-dimensional decomposition $(F_n)$ of $X$ has the following property: Every increasing sequence $(m_n)$ of positive integers has a further subsequence $(k_n)$, so that $(F_{k_n})$ is a $c_0$-decomposition for its closed linear span $[F_{k_n}]$.

We call a sequence $(F_n)$ of subspaces of a Banach space $X$ a $c_0$-*decomposition for its closed linear span*, if it satisfies the following:

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There is a constant $C$ so that for all $N \in \mathbf{N}$, for all $(x_n)_{n=1}^N$ with $\|x_n\| \leq 1$ and $x_n \in F_n$ for all $n = 1, 2, \dots, N$,

$$\left\| \sum_{n=1}^{N} x_n \right\| \leq C.$$

Our main result is

**Theorem 2.** *A Banach space $X$ with property $(FS)$, whose dual $X^*$ is isomorphic to $\ell_1$, is isomorphic to $c_0$.*

Property $(FS)$ is an FDD-version of property $(S)$, which was introduced by Knaust and Odell [**7**]. A Banach space $X$ has *property* $(S)$, if every normalized weakly null sequence in $X$ contains a $c_0$-subsequence. It was shown in [**7**] that property $(S)$ is equivalent to the following: Every weak Cauchy sequence of elements $(x_n) \in Ba(X)$ has a subsequence $(x_{m_i})$, so that for some constant $C$,

$$(1) \qquad \sum_{i=1}^{\infty} |x^*(x_{m_i+1}) - x^*(x_{m_i})| \leq C \quad \text{for all } x^* \in Ba(X^*).$$

Thus property $(S)$ implies property $(u)$, where (1) holds for far out convex combinations of the sequence $(x_n)$ instead of for a subsequence.

Let us remark that Bourgain and Delbaen [**1**] constructed an example of a separable $\mathcal{L}_\infty$-space which satisfies the Schur property and thus properties $(u)$, $(S)$ and $(FS)$. Therefore, it is necessary to restrict the investigation from separable $\mathcal{L}_\infty$-spaces to isomorphic preduals of $\ell_1$.

Our notation is standard and can be found, e.g., in [**8**].

**2. Proof of the main result.** It suffices to show

**Proposition 3.** *Let $X$ be a separable Banach space with a shrinking finite dimensional decomposition. Then if $X$ has property $(FS)$, $X$ has a finite-dimensional decomposition, which is a $c_0$-decomposition* (*for the whole space*)*.*

We obtain Theorem 2 as follows.

*Proof of Theorem* 2. Let $X$ be an isomorphic predual of $\ell_1$ with property $(FS)$. Since $\ell_1$'s standard basis is boundedly complete, it follows that $X$ itself has a shrinking basis $(x_n)$ (see [**8**, pp. 8–10]). Trivially, $(x_n)$ is a shrinking FDD; thus, $X$ possesses a $c_0$-decomposition $(G_n)$ by Proposition 3. Consequently, $X$ is isomorphic to the $c_0$-sum of the spaces $(G_n)$. An $\mathcal{L}_\infty$-space of this form is isomorphic to $c_0$ (see [**9**, pp. 102–103]). □

*Proof of Proposition* 3. Let $(F_n)$ be a shrinking FDD for $X$. We will show that an appropriately chosen blocking of the given FDD is a $c_0$-decomposition for $X$.

Given an increasing sequence $M = (m_n)$ of positive integers, we let $(G_n^M)_{n=1}^\infty$ be the shrinking FDD obtained by setting

$$G_n^M = \operatorname{span}\{F_{m_{n-1}+1}, F_{m_{n-1}+2}, \dots, F_{m_n}\}.$$

(Here we let $m_0 = 0$.)

We define

$$\mathcal{A} = \{M = (m_n)_{n=1}^\infty \mid (G_{2n-1}^M)_{n=1}^\infty \text{ is a } c_0\text{-decomposition}\}.$$

The set

$$\mathcal{A}(C,N) := \left\{ M = (m_n)_{n=1}^\infty \mid \left\| \sum_{n=1}^{N} x_{2n-1} \right\| \le C \quad \text{for all } x_{2n-1} \in Ba(G_{2n-1}^M) \right\}$$

is a closed (and open) set for all $C$, $N \in \mathbf{N}$, when the infinite subsets of $\mathbf{N}$ are endowed with the relative topology, induced by the product topology on $2^{\mathbf{N}}$. Since $\mathcal{A} = \cup_{C\in\mathbf{N}} \cap_{N\in\mathbf{N}} \mathcal{A}(C,N)$, $\mathcal{A}$ is a Ramsey set [**2, 10**].

Consequently, there is a subsequence $L = (l_n) \subset \mathbf{N}$ so that either $M \in \mathcal{A}$ for all subsequences $M \subset L$ or $M \notin \mathcal{A}$ for all subsequences $M \subset L$.

By property $(FS)$, the second Ramsey alternative must fail; indeed, applying property $(FS)$ to the FDD $(G_n^L)$, we find a subsequence $(n_k)$ so that $(G^L_{n_k})$ is a $c_0$-decomposition. We may assume that $n_{k+1} > n_k + 1$. We let

$$M = \{l_{n_1}, l_{n_2-1}, l_{n_2}, l_{n_3-1}, \dots\},$$

and observe that $M \subset L$ and $(G^M_n) \in \mathcal{A}$.

So the first Ramsey alternative holds. Let $L' = \{l_2, l_3, \dots\}$. Both $L$ and $L'$ are in $\mathcal{A}$. This means that there are constants $C_1$ and $C_2$ so that, for all $N \in \mathbf{N}$,

$$\left\| \sum_{n=1}^{N} x_{2n-1} \right\| \leq C_1 \quad \text{for all } x_{2n-1} \in Ba(G^L_{2n-1}),$$

and

$$\left\| \sum_{n=1}^{N} x_{2n-1} \right\| \leq C_2 \quad \text{for all } x_{2n-1} \in Ba(G^{L'}_{2n-1}).$$

Since $G^{L'}_{2n-1} = G^L_{2n}$ for all $n \geq 2$, $(G^L_n)$ is a $c_0$-decomposition for the whole space $X$. □

**3. Remarks.**

1. Naturally the questions arise, whether $\ell_1$-preduals with property $(S)$ have property $(FS)$, and whether property $(u)$ implies property $(S)$ among $\ell_1$-preduals.

2. It is easy to check that $c_0$ has property $(FS)$.

3. Not every separable Banach space with property $(S)$ has property $(FS)$; indeed, we have the following example.

**Example 4.** There is a Banach space with a shrinking unconditional basis, which satisfies property $(S)$, but fails property $(FS)$.

*Proof.* We use a tree-space example constructed by Talagrand [**11**]. Let $\mathcal{D} = \cup_{n=0}^{\infty}\{0,1\}^n$ denote a binary tree. An element $\varphi \in \mathcal{D}$ is called a node. If $\varphi = (\varepsilon_i)_{i=1}^n$ with $\varepsilon_i \in \{0,1\}$, we say $\varphi$ has length $|\varphi| = n$. As usual, we define a partial order on $\mathcal{D}$ as follows: $\varphi \preceq \psi$ if, writing

$\varphi = (\varepsilon_i)_{i=1}^n$ and $\psi = (\delta_i)_{i=1}^m$, $n \leq m$ and $\varepsilon_i = \delta_i$ for all $i = 1, \dots, n$. If $\varphi$ and $\psi$ are incomparable, we write $\varphi \| \psi$.

For $x \in c_{00}$, let

$$\|x\| = \sup_{n \in \mathbf{N}} \left\{ \left( \sum_{|\varphi|=n} \sup_{\psi \succeq \varphi} |x(\psi)|^2 \right)^{1/2} \right\}.$$

The Banach space $T$ is the completion of $c_{00}$ under this norm.

Let $(e_\varphi)$ denote the standard basis in its lexicographical order, defined by $e_\varphi(\psi) = 1$, if $\varphi = \psi$ and $e_\varphi(\psi) = 0$ otherwise. Obviously, $(e_\varphi)$ is a 1-unconditional basis. Since $\ell_1$ does not embed into $T$ [**11**], it follows that the basis is shrinking. Talagrand showed that $T$ has property $(S)$.

We claim that $T$ fails property $(FS)$. Consider the finite-dimensional decomposition $(F_n)$, defined by

$$F_n = [e_\varphi]_{|\varphi|=n} \quad \text{for all } n \in \mathbf{N}.$$

Clearly $(F_n)$ is a shrinking decomposition for $T$. We show that $(F_n)$ has no $c_0$-sub-decomposition.

Let $(m_n)$ be a subsequence of $\mathbf{N}$. We choose inductively a sequence of nodes $\varphi_n \in F_{m_n}$. Let $\varphi_1$ be such that $|\varphi_1| = m_1$.

Once the $\varphi_n$'s have been chosen for $n = 2^{k-1}, \dots, 2^k - 1$, we proceed as follows: Let $\alpha_n$ and $\beta_n$ denote the two distinct successor nodes of $\varphi_n$, i.e., $\alpha_n \neq \beta_n$, $\alpha_n \succeq \varphi_n$, $\beta_n \succeq \varphi_n$ and $|\alpha_n| = |\beta_n| = m_n + 1$. Then choose $\varphi_\ell$, $\ell = 2^k, \dots, 2^{k+1} - 1$ so that $|\varphi_\ell| = m_\ell$ for all $\ell = 2^k, \dots, 2^{k+1} - 1$, and so that $\varphi_{2n} \succeq \alpha_n$, $\varphi_{2n+1} \succeq \beta_n$ for $n = 2^{k-1}, \dots, 2^k - 1$. The construction ensures that $\varphi_{2n} \| \varphi_{2n+1}$ for $n = 2^{k-1}, \dots, 2^k - 1$, and that $\varphi_n = \min(\varphi_{2n}, \varphi_{2n+1})$ for $n = 2^{k-1}, \dots, 2^k - 1$, i.e., whenever a node $\psi \in \mathcal{D}$ satisfies $\varphi_{2n} \succeq \psi$ and $\varphi_{2n+1} \succeq \psi$, then $\varphi_n \succeq \psi$.

Clearly, $e_{\varphi_n} \in F_{m_n}$ and $\|e_{\varphi_n}\| = 1$ for all $n \in \mathbf{N}$; on the other hand, it is easy to see that

$$\left\| \sum_{n=2^k}^{2^{k+1}-1} e_{\varphi_n} \right\| \geq 2^{k/2} \quad \text{for all } k \in \mathbf{N}.$$

Thus $(F_{m_n})$ fails to be a $c_0$-decomposition. $\square$

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