ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 1, Winter 1997

ON THE MINIMAL FREE RESOLUTION OF GENERAL k-GONAL CURVES

E. BALLICO

ABSTRACT. Here we study the minimal free resolution of general embeddings of a curve in $\mathbf{P}^n$; the curve may be a general k-gonal curve and the embedding linearly normal (the aim is to prove condition N_p for suitable degree, genus and p) or not linearly normal (for low p the embedded curve has the same type of minimal free resolution).

0. Introduction. The main aim of this paper is to show that (as remarked in [**5**]) the proofs in [**5**] may be used to obtain several other nontrivial results. The topic is the minimal free resolution of a closed subscheme Z of $\mathbf{P}^n$. Here (as in [**5**]) we consider the case $\dim(Z) = 1$. We stress that [**5**] owes very much to [**11**]. To state our results we fix some notations and definitions. Let Z be a closed subscheme of $\mathbf{P}^n$, and let

$$(1) \qquad \cdots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow \cdots \longrightarrow E_1 \longrightarrow \mathbf{I}_Z \to 0$$

be the minimal free resolution of the ideal sheaf of Z. By definition this means that (1) is an exact sequence of sheaves with each E_i direct sum of line bundles, with $E_j = 0$ for $j > n$ and such that for every i if a line bundle, L, is a direct summand both in E_{i+1} and E_i the map $L \to L$ induced by the map $E_{i+1} \to E_i$ is zero.

Definition 0.1. Let $C \subset \mathbf{P}^n$ be a reduced curve; fix an integer $p \geq 1$. We will say that C satisfies the property N_p if C is arithmetically Cohen-Macaulay and for every integer i with $1 \leq i \leq p$ the ith-sheaf appearing in the minimal free resolution of the homogeneous ideal of C is the direct sum of line bundles of degree $-i-1$.

For instance, if we say that N_0 means C is *arithmetically Cohen-Macaulay*, then N_1 means that the curve C is N_0 and its homogeneous

Received by the editors on December 6, 1994, and in revised form on April 13, 1995.
The author was partially supported by MURST and GNSAGA of CNR (Italy). He is a member of Europroj.

ideal is generated by quadrics. Furthermore, if $p > 0$, then N_p implies N_{p-1}. For more on this concept, see [**8**] and [**11**].

Definition 0.2. Let $Z \subset \mathbf{P}^N$ be a closed subscheme; fix an integer $p \geq 1$. We will say that Z satisfies the property aN_p if, for every integer i with $1 \leq i \leq p$, the ith-sheaf appearing in the minimal free resolution of the homogeneous ideal of Z is the direct sum of line bundles of degree $-i-1$.

Hence, if Z is a reduced curve, property N_p is equivalent (for our conventions) to aN_p+ arithmetically Cohen-Macaulay.

We recall the following result [**5**, Theorem 0.2]; we will use often the notation $G(p, n)$ defined in its statement.

Theorem 0.3 [**5**, Theorem 0.2]. *Fix an integer $p \geq 1$. For every integer u, set*:

$$a_p(u) := (u^2)/(2p+2) - (u/2).$$

Fix an integer $n \geq 3$ with $n \geq p+1$, and set

$$G(p, n) := a_p((p+1)[n/(p+1)])$$

where $[y]$ is the greatest integer $\leq y$. Then for every integer $g \leq G(p, n)$ the general linearly normal nonspecial curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $\deg(C) = g + n$ *satisfies the property N_p.*

Here in Section 1 we will consider the same problem (see Theorems 1.1 and 1.2) not for curves with general moduli but for general k-gonal curves.

For the result about property aN_p, see Section 2 (Theorem 2.1). The main point to insert here in this section is to stress to the reader that sometimes it is possible to prove this kind of result using a hyperplane section containing embedded points (hence which is not the zero locus of a nonzero divisor of the homogeneous ring of the curve).

1. General k-gonal curves. In this section we will show (see respectively Theorems 1.1 and 1.2) how to extend for every integer

$k > n$, respectively $k \leq n$, Theorem 0.3 to curves not with general moduli, but just to general k-gonal curves.

Theorem 1.1. *Fix integers n, k and p with $k > n \geq p$ and $n \geq 3$. Then for every integer $g \leq G(n,p)$ the general linearly normal k-gonal nonspecial curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $\deg(C) = g+n$ satisfies the property N_p.*

Fix integers p, n and g in the interval covered by Theorem 0.3 and an integer $k > n$. Note that in each inductive step of the quoted proof of 0.3 given in [**5**] for the integer (p,n,g) the rational normal curve, D, meets the other component in at most k points. We claim that the same proof gives that a general nonspecial embedding in $\mathbf{P}^n$ of a general k-gonal curve has the property N_p. To prove the claim, it is sufficient to show that at each inductive step the reducible curve is the limit (in the ambient projective space) of a family of smooth k-gonal curves. In principle, this was checked, for instance, in [**1**, Section 2]; here we will give the only changes needed. Everything goes as in [**1**, pp. 338–339], until we need to use the theory of admissible coverings. Now we have the following reducible curve. A smooth curve (call it W) of genus a with a fixed g^1_k, hence a map $u: W \rightarrow \mathbf{P}^1$, a point $P \in \mathbf{P}^1$ such that the fiber $A := f^{-1}(P) := \{P_1, \dots, P_k\}$ is reduced, plus a smooth rational curve D intersecting W only at s points with $3 \leq s \leq k$, all in A (say $P_1, \dots, P_s$) and with $B := W \cup D$ with only ordinary nodes as singularities. Hence B is a stable curve of genus $a+s-1$, and we want to prove that it is in the closure of the set of smooth k-gonal curves. Take another copy of $\mathbf{P}^1$ and call T the union of it with the first $\mathbf{P}^1$ glued at the point P, i.e., let T be a reducible conic and call P its singular point and T' one component and see the morphism u as a morphism $u: W \rightarrow T'$; call T'' the other component of T. We choose a noncomplete g^1_s on D with $\{P_1, \dots, P_s\}$ as one of its group of points. We take $k-s$ disjoint copies of $\mathbf{P}^1$, say A_j, $s+1 \leq j \leq k$, and we call Y the reducible curve with only nodes as singularities and union of B and all A_j's, with $B \cap A_j = P_j$ for every $s+1 \leq j \leq k$. We extend the morphism $u: W \rightarrow T'$ to the following morphism $v: Y \rightarrow T$; v sends D to T'' using the given g^1_s; v sends isomorphically every A_j onto T''. The triple (Y, T, v) (plus a choice of a few points on T) is an admissible s-covering in the sense of [**9**, Section 4]. Note that B is the stable reduction of Y. Hence, by the theory of admissible coverings, i.e., the irreducibility of the corresponding moduli space, Y is in the closure of the set of smooth genus $a+s-1$ k-gonal curves, as wanted.

In the same way (but adding only rational curves meeting the previous component in at most k points) we obtain the following result.

Theorem 1.2. *Fix integers n, k and p with $n \geq k \geq p$ and $n \geq 3$. Set $G(p,n;k) := \max(0, G(p,n-k)) + (k-1)(n-k)$. Then for every integer $g \leq G(p,n;k)$ the general linearly normal k-gonal nonspecial curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $\deg(C) = g+n$ satisfies the property N_p.*

2. Nonlinearly normal embeddings. In this section we will consider the case of nonlinearly normal embeddings and study the property aN_p. We will explain why the proofs of Theorem 0.3 (given in [**5**]) and of [**3**, Theorem 2.3] (which depends heavily upon [**2**]) give the following theorem.

Theorem 2.1. *Fix integers n, p, g, g' and k with $n \geq 3$, $4n \leq g \leq g' \leq G(p,n)$, $3(g'-g) < g'$, $2(g'-g)+p+2 \leq n$, $k > 0$, $2k+1 \leq g$, $g'+n \geq 2+n(n+1)$, $g'+n \geq (g'(k+1)/k)+k+1$. Then a general nonspecial degree $g'+n$ embedding into $\mathbf{P}^n$ of a general curve of genus g has the property aN_p.*

Unfortunately Theorem 2.1 has two defects: its proof and its statement. Its proof is just to see that the quoted proofs of Theorem 0.3 and [**3**, Theorem 2.3] produce (under the very strong numerical conditions of Theorem 2.1) a reducible nonreduced but smoothable curve, Z, satisfying both the statement of Theorem 0.3 (for aN_p) and the one of [**3**, Theorem 2.3] in the following sense. We have $\deg(Z) = g'+n$, $p_a(Z) = g$, $p_a(Z_{\text{red}}) = g'$ and Z is the schematic union of Z_{red} and $g'-g$ suitable zero-dimensional schemes (see [**3**, Section 2]). Z_{red} satisfies N_p (see later) and the homogeneous ideal of Z is generated by a subset of the quadrics which generates the homogeneous ideal of Z_{red}; hence, the inclusion holds for higher syzygies and Z satisfies aN_p. Since Z is smoothable inside $\mathbf{P}^n$ and it has nonspecial hyperplane line bundle, we obtain in this way Theorem 2.1. Z_{red} is built as partial smoothing of two reduced reducible curves, say A and B, with A given by the proof of [**3**, Theorem 2.3] and B given by the quoted proof of Theorem 0.3. Z_{red} is the union of a smooth nondegenerate curve W of degree $g' + n - 3(g' - g)$, $g' - g$ lines G_i, $1 \leq i \leq g' - g$, each G_i secant to W, and $g' - g$ pairs of lines (L_i, L_i'), $1 \leq i \leq g' - g$ such that every line L_i, L_j' intersects $G_i \backslash (G_i \cap W)$, every line L_i intersects $W \backslash (W \cap G_i)$ at one point and quasi-transversally, every pair (L_i, L_i') has a common point $P(i)$ and the components of Z_{red} have no other intersections; the names of the lines of Z_{red} are the same as the names of the corresponding lines in the proof of [**3**, Theorem 2.3] except that now we use (easier but more expensive numerically) $g' - g$ lines G_i instead of one line G; all the other components of the reduced curve constructed in the proof of [**3**, Theorem 2.3] are smoothed obtaining the curve W. Each of the nilpotents of Z is supported by a different point $P(i)$ (as in [**3**]). To obtain B with Z_{red} as partial smoothing, note that at each step of the quoted proof of Theorem 0.3 we may take a bunch of lines (see, e.g., [**13**] or [**10**]) with a common vertex as curve D; at a later step we add something meeting one of these lines (and call it L_1) so that, after the partial smoothing L_1 intersects W. Then we may repeat the construction $g' - g - 1$ times; each time we get G_i, L_i and L_i'; only at the very last step we link the curves L_i to save steps. As in the proof of Theorem 0.3 we start the construction inductively starting from $\mathbf{P}^{p+1}$; hence, we need the condition $n \geq p + 1 + 2(g' - g) + 1$. The main drawback of the statement of Theorem 2.1 is just the very strong assumption $n \geq p + 2 + 2(g' - g)$, which, as seen, comes because we add in one step only part of one pair (L_i, L_i') and not a part of several such pairs. In particular cases we are able to do that obtaining stronger statements, and we see Theorem 2.1 just as a "philosophical" statement showing that the proofs (and in particular to be able to carry it over without smoothing except at the very end of the proof) may give very good information on these kinds of problems. In particular, the interested reader may try new cases or just consider (as in Section 1) general curves with a certain gonality.

REFERENCES

1. E. Ballico, *On general k-gonal curves*, Arch. Math. **44** (1985), 336–339.

2. ———, *Generators for the homogeneous ideal of s general points in* $\mathbf{P}^3$, J. Algebra **106** (1987), 46–52.

3. ———, *On the homogeneous ideal of projectively normal curves*, Ann. Mat. Pura Appl. **154** (1989), 83–90.

4. ———, *On the minimal free resolution of some projective curves*, Ann. Mat. Pura Appl., to appear.

5. ———, *On the minimal free resolution of general embeddings of curves*, Pacific J. Math. **172** (1996), 315–319.

6. E. Ballico and Ph. Ellia, *On the existence of curves with maximal rank in* $\mathbf{P}^n$, J. Reine Angew. Math. **397** (1989), 1–22.

7. M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), 125–171.

8. M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67** (1988), 301–314.

9. J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23–86.

10. R. Hartshorne and A. Hirschowitz, *Smoothing algebraic space curves*, Lecture Notes in Math. **1124** (1985), 98–131.

11. R. Lazarsfeld, *A sampling of vector bundles techniques in the study of linear series*, in *Lectures on Riemann surfaces*, World Scientific, 1989.

12. U. Nagel and Y. Pitteloud, *On graded Betti numbers and geometrical properties of projective varieties*, Queen's Papers in Pure and Applied Math. **95** (1993), E1–E22.

13. E. Sernesi, *On the existence of certain families of curves*, Invent. Math. **75** (1984), 25–57.

Department of Mathematics, University of Trento, 38050 Povo (TN), Italy
E-mail address: ballico@science.unitn.it