ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 2, Spring 1997

MATRIX INNER PRODUCT HAVING A MATRIX SYMMETRIC SECOND ORDER DIFFERENTIAL OPERATOR

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ABSTRACT. In this note we characterize those positive definite matrices of measures whose matricial inner product has a symmetric left-hand side matrix second order differential operator.

1. Introduction. Let W be an $N \times N$ positive definite matrix of measures (i.e., for any Borel set $A \subset \mathbf{R}$, $W(A)$ is a semi-definite positive numerical matrix). We put $\langle\ ,\ \rangle_W$ for the matrix inner product defined by W:

$$\langle P, Q\rangle_W = \int_{\mathbf{R}} P(t)\, dW(t) Q^*(t) \in M_{N\times N}, \qquad P, Q \in \mathbf{P}_{N\times N}, \tag{1.1}$$

where we denote by $M_{N\times N}$ the space of numerical $N \times N$ matrices, by $\mathbf{P}_{N\times N}$ the space of $N \times N$ matrix polynomials, and by Q^* the Hermitian adjoint of Q.

Orthogonal matrix polynomials on the real line with respect to a positive definite matrix of measures have been considered in detail in M.G. Krein [**12**] or, more recently, by Aptekarev and Nikishin [**1**], Geronimo [**7**], Sinap and Van Assche [**14**], Duran and Van Assche [**6**], Duran and Lopez-Rodriguez [**5**] and Duran [**3, 4**]. In [**3, 6**], a very close relationship between orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence relation has been established. However, as far as the author knows, no general results concerning orthogonal matrix polynomials and differential equations are known (for some examples of orthogonal matrix polynomials satisfying differential equations, see [**9, 10**].

In this note we characterize those positive definite matrices of measures whose matrix inner product, defined as before, has a symmetric

This work has been partially supported by DGICYT ref. PB 90-0893.
Received by the editors on July 25, 1994, and in revised form on May 24, 1995.

left-hand side matrix second order differential operator of the form

$$l_{2,L} = A_2 D'' + A_1 D' + A_0 D^0.$$

In Section 2 of this paper we establish a relationship between the existence of a symmetric left of right-hand side matrix second order differential operator for the matrix inner product defined by W, the existence of a second order differential equation which the sequence of orthonormal matrix polynomials with respect to W satisfies, and the existence of a recurrence formula for the moments of W.

In Section 3 we prove a characterization theorem, where by reducing the problem to the scalar case, and by applying the well-known Bochner classification theorem for sets of scalar orthogonal polynomials which satisfy a second order differential equation (see [**2**] or [**13**]), we show that the positive definite matrix of measures, for which a symmetric left-hand side second order differential operator exists, are of the form $W = D^*XD$ where D is a nonsingular matrix and $X = \nu I$, with ν a classical scalar weight, (i.e., ν is some of the Jacobi, Hermite or Laguerre weights up to a linear change of variable).

Finally, in Section 4 we show a counterexample which proves that the classification theorem does not work for right-hand side second order differential operators.

The difference in the behavior between left- and right-hand side is, of course, due to the noncommutativity of the matrix product.

2. Differential and moment equations. In this section we prove some general results showing the relationship between:

(i) (SO): the existence of a symmetric left of right-hand side matrix second order differential operator for the matrix inner product defined by the matrix of measures W,

(ii) (DE): the existence of a second order differential equation which the orthogonal matrix polynomials with respect to W satisfy, and

(iii) (ME): the existence of a recurrence formula for the moments of W.

In the scalar case the equivalence between these three properties is straightforward, but the noncommutativity of the matrix product makes the situation, in the matrix case, harder (for the relation between

(SO), (DE) and (ME) for scalar orthogonal polynomials and differential operators of higher order, see the important paper by H.L. Krall [**11**]).

First of all, we introduce some notations and definitions. The matrix second order differential operator associated to the matrix functions A_2, A_1, A_0 is defined by

$$\begin{aligned} l_{2,R} &= D'' A_2 + D' A_1 + D^0 A_0 \quad \text{right-hand side} \\ l_{2,L} &= A_2 D'' + A_1 D' + A_0 D^0 \quad \text{left-hand side.} \end{aligned}$$

We say that this operator is symmetric for the matrix inner product $\langle\ ,\ \rangle$ defined in $\mathbf{P}_{N\times N}$ if $\langle l_2 P, Q\rangle = \langle P, l_2 Q\rangle$ for any matrix polynomials P, Q.

We assume that the positive matrix of measures W or, in short, the matrix weight W, is nondegenerate, i.e., for any matrix polynomial $P \not\equiv \theta$,

$$\int_{\mathbf{R}} P dW P^* \neq \theta. \tag{2.1}$$

(Here and in the rest of this paper, we write θ for the null matrix, the size of which can be determined from the context. For instance, here θ is the $N \times N$ null matrix.)

The moments of the matrix weight W are defined by $\mu_n = \int_{\mathbf{R}} t^n\, dW(t)$, $n \geq 0$, and they are Hermitian matrices. As in the scalar case, it has been proved (see [**11**]) that a sequence of Hermitian matrices $(\mu_n)_n$ are the moments of a positive definite matrix of measures if and only if $(\mu_n)_n$ is positive definite, i.e., for any sequence of vectors $v_0, v_1, \ldots \in \mathbf{C}^N$,

$$\sum_{j,k=0}^{n} v_j \mu_{j+k} v_k^* \geq 0, \quad n = 0, 1, 2, \ldots .$$

The nondegenerate condition (see (2.1)) implies that the equality holds only for $v_n = \theta$, $n = 0, 1, \ldots$.

A generalization of the Gram-Schmidt orthonormalization process for the set

$$\{I, tI, t^2 I, \ldots\}$$

with respect to the matrix inner product $\langle\ ,\ \rangle_W$ will give a set of orthonormal matrix polynomials $(P_n)_n$ which satisfies

$$\int P_n(t)\, dW(t) P_k^*(t) = \delta_{n,k} I, \quad n, k = 0, 1, \dots .$$

Moreover, $P_n(t)$ is a matrix polynomial of degree n, with nonsingular leading coefficient.

We now prove the relationship between (SO) and (DE).

Lemma 2.1. *If l_2 is symmetric for $\langle\ ,\ \rangle_W$, then the sequence of orthonormal matrix polynomials with respect to W satisfies the matrix second order differential equation*

$$l_2(P_n) = \Gamma_n P_n, \quad n = 0, 1, \dots$$

for certain Hermitian matrices Γ_n.

For $l_{2,R}$ the converse is also true, but for $l_{2,L}$ the converse is false.

Proof. We write $l_2(P_n) = \sum_{k=0}^n \Gamma_{n,k} P_k$. Since l_2 is symmetric, we have

$$\Gamma_{n,k} = \langle l_2(P_n), P_k\rangle_W = \langle P_n, l_2(P_k)\rangle_W = \theta \quad \text{if } k < n,$$

and, for $k = n$, $\Gamma_{n,n} = \Gamma_{n,n}^*$.

Now let us assume that $(P_n)_n$ satisfy $l_{2,R}(P_n) = \Gamma_n P_n$ where Γ_n are Hermitian matrices. Since Γ_n are Hermitian we have $\langle l_{2,R}(P_n), P_m\rangle_W = \langle P_n, l_{2,R}(P_m)\rangle_W$. We now extend by linearity.

It is worthy to remark that to extend by linearity we need $l_{2,R}(CP) = C l_{2,R}(P)$ for any numerical matrix C and any matrix polynomial P. Since the matrix product is noncommutative, in general, $l_{2,L}(CP) \neq C l_{2,L}(P)$. In Section 4, we will show a counterexample proving that for $l_{2,L}$ the converse is false. □

From this proposition we deduce that if l_2 is symmetric for $\langle\ ,\ \rangle_W$, then the matrix functions A_2, A_1, A_0 are matrix polynomials of degree at most 2, 1, 0, respectively. Indeed, for $n = 0, 1, 2$, the differential equation for the orthonormal polynomials gives (left-hand side)

$$\begin{aligned} A_0(t)P_0(t) &= \Gamma_0 P_0(t) \\ A_1(t)P_1'(t) + A_0 P_1(t) &= \Gamma_1 P_1(t) \\ A_2(t)P_2''(t) + A_1 P_2'(t) + A_0 P_2(t) &= \Gamma_2 P_2(t). \end{aligned}$$

Since $P_0(t), P_1'(t), P_2''(t)$ are nonsingular numerical matrices (they are the leading coefficients of $P_0(t), P_1(t), 2P_2(t)$), we have

$$\begin{aligned} &A_0(t) = \Gamma_0 \\ &A_1(t) = (\Gamma_1 P_1(t) - A_0 P_1(t))(P_1')^{-1}(t) \\ &A_2(t) = (\Gamma_2 P_2(t) - A1 P_2'(t) - A_0 P_2(t))(P_2'')^{-1}(t). \end{aligned}$$

The result for the right-hand side can be proved in the same way.

We write $A_2(t) = t^2 A_{2,2} + t A_{2,1} + A_{2,0}$ and $A_1(x) = t A_{1,1} + A_{1,0}$ where $A_{i,j}$ are numerical matrices.

We now prove the relationship between (SO) and (ME).

Lemma 2.2. *We assume $A_0 \mu_n = \mu_n A_0^*$, $n \geq 0$. If l_2 is symmetric for $\langle\ ,\ \rangle_W$, then the moments $(\mu_n)_n$ of W satisfy the following equations*

$$(2.2)\quad (n-1)(A_{2,2}\mu_n + A_{2,1}\mu_{n-1} + A_{2,0}\mu_{n-2} + A_{1,1}\mu_n + A_{1,0}\mu_{n-1}) = \theta, \quad n \geq 1,$$

$$(2.3)\quad A_{2,2}\mu_n + A_{2,1}\mu_{n-1} + A_{2,0}\mu_{n-2} = \mu_n A_{2,2}^* + \mu_{n-1} A_{2,1}^* + \mu_{n-2} A_{2,0}^*, \quad n \geq 2,$$

$$(2.4)\quad A_{1,1}\mu_n + A_{1,0}\mu_{n-1} = \mu_n A_{1,1}^* + \mu_{n-1} A_{1,0}^*, \quad n \geq 1.$$

For $l_{2,R}$ the converse is also true, but for $l_{2,L}$ the converse is false.

Proof. The symmetry of l_2 gives

$$(2.5)\quad \langle l_2(t^n I), t^m I\rangle_W = \langle t^n I, l_2(t^m I)\rangle_W, \quad n, m \geq 0.$$

We write

$$\begin{aligned}B_{n+m} &= A_{2,2}\mu_{n+m} + A_{2,1}\mu_{n+m-1} + A_{2,0}\mu_{n+m-2},\\ C_{n+m} &= A_{1,1}\mu_{n+m} + A_{1,0}\mu_{n+m-1},\\ D_{n+m} &= A_0\mu_{n+m}.\end{aligned}$$

Then, from (2.5) we have the following moment equations:

$$\begin{aligned}n(n-1)B_{n+m} &+ nC_{n+m} + D_{n+m}\\ &= m(m-1)B^*_{n+m} + mC^*_{n+m} + D^*_{n+m}, \quad n, m \geq 0.\end{aligned} \tag{2.6}$$

For $m = 0$, we get

$$n(n-1)B_n + nC_n + D_n = D^*_n, \tag{2.7}$$

and for $n-1$, $m = 1$, we have $(n-1)(n-2)B_n + (n-1)C_n + D_n = C^*_n + D^*_n$. This and (2.7) give $2(n-1)B_n + C_n = -C^*_n$, i.e.,

$$B_n = -\frac{1}{2(n-1)}(C_n + C^*_n), \quad n \geq 2.$$

Hence we have $B_n = B^*_n$, $n \geq 2$, and so (2.3) follows.

By hypothesis, we have $D_n = D^*_n$. Hence, from (2.7), (2.2) and (2.4) follow.

We finally prove the converse for $l_{2,R}$. From (2.2) for $n+m$, we get

$$\begin{aligned}(n-1)B_{n+m} + C_{n+m} + mB_{n+m} &= \theta,\\ (m-1)B_{n+m} + C_{n+m} + nB_{n+m} &= \theta.\end{aligned}$$

And so we have

$$n(n-1)B_{n+m} + nC_{n+m} = m(m-1)B_{n+m} + mC_{n+m}.$$

From (2.3), (2.4) and the assumption, we deduce that (2.6) holds. Hence, we have

$$\langle l_{2,R}(t^nI), t^mI\rangle_W = \langle t^n, l_{2,R}(t^mI)\rangle_W, \quad n, m \geq 0.$$

We now extend by linearity.

In Section 4, we will show a counterexample proving that for $l_{2,L}$ the converse is false. □

The equations for the moments, which have been found in the previous lemma, can be written in terms of functional equations for the matrix of measures W. Indeed, we can consider this matrix of measures as an operator W acting on $\mathbf{P}_{N\times N}$ by

$$\langle W, P\rangle = \int dW(t)P^* \in M_{N\times N}, \quad \text{for } P \in \mathbf{P}_{N\times N}.$$

Given a matrix polynomial P, we define the operators PW, WP and W' by

$$\langle PW, Q\rangle = \int P\, dWQ^*, \quad \text{for } Q \in \mathbf{P}_{N\times N},$$
$$\langle WP, Q\rangle = \int dW(t)PQ^*, \quad \text{for } Q \in \mathbf{P}_{N\times N},$$

and

$$\langle W', P\rangle = -\langle W, P'\rangle, \quad \text{for } P \in \mathbf{P}_{N\times N}. \tag{2.8}$$

With these definitions, Lemma 2.2 can be rewritten as follows

Lemma 2.3. *We assume $A_0W = WA_0^*$. If l_2 is symmetric for $\langle\ ,\ \rangle_W$, then W (as an operator acting on $\mathbf{P}_{N\times N}$) satisfies*

$$\begin{aligned}(A_2W)' &= A_1W,\\ A_2W &= WA_2^*,\\ A_1W &= WA_1^*.\end{aligned} \tag{2.9}$$

For $l_{2,R}$ the converse is also true, but for $l_{2,L}$ the converse is false.

3. Left-hand side differential operators. In this section we establish the main result in this note, classifying those positive definite matrices of measures whose matrix inner product has a symmetric left-hand side second order differential operator.

We assume that W is a matrix weight for which a symmetric left-hand side second order differential operator exists. First of all, we show that we can assume the first moment μ_0 of W equal to the identity matrix and the second moment μ_1 of W being a diagonal matrix. Indeed, since μ_0 is positive definite and μ_1 is Hermitian, a nonsingular matrix C exists such that $C^*\mu_0 C$ is the identity matrix and $C^*\mu_1 C$ is a diagonal matrix (see [**8**, p. 466]). We define a new matrix weight X by $X = C^*WC$. The first moment of X is the identity matrix, and its second moment is a diagonal matrix. It will be enough to prove that the operator $l_{2,L}$ is also symmetric for X. Clearly, $\langle P,Q\rangle_X = \langle PC^*, QC^*\rangle_W$ for any matrix polynomials P,Q. Then we have

$$\begin{aligned}\langle l_{2,L}(P), Q\rangle_X &= \langle l_{2,L}(P)C^*, QC^*\rangle_W\\ &= \langle l_{2,L}(PC^*), QC^*\rangle_W\\ &= \langle PC^*, l_{2,L}(QC^*)\rangle_W\\ &= \langle PC^*, l_{2,L}(Q)C^*\rangle_W\\ &= \langle P, l_{2,L}(Q)\rangle_X.\end{aligned}$$

The following lemma will be the key to establish the classification theorem

Lemma 3.1. *If $l_{2,L}$ is symmetric for $\langle\ ,\ \rangle_W$, then the coefficients of A_2, A_1 and A_0 must be the identity matrix up to scalar multiplicative constants.*

Proof. For a unitary matrix Γ we define the operator

$$l_{2,L,\Gamma}(P) = \Gamma^* l_{2,L}(\Gamma P)$$

then we have

$$\begin{aligned}\langle l_{2,L,\Gamma}(P), Q\rangle_W &= \langle \Gamma^* l_{2,l}(\Gamma P), Q\rangle_W = \Gamma^*\langle l_{2,L}(\Gamma P), Q\rangle_W\\ &= \Gamma^*\langle \Gamma P, l_{2,L}(Q)\rangle_W = \langle P, l_{2,L}(Q)\rangle_W.\end{aligned}$$

But also

$$\begin{aligned}\langle l_{2,L,\Gamma}(P), Q\rangle_W &= \langle \Gamma^* l_{2,l}(\Gamma P), Q\rangle_W = \Gamma^*\langle l_{2,L}(\Gamma P), \Gamma Q\rangle_W\Gamma\\ &= \Gamma^*\langle \Gamma P, l_{2,L}(\Gamma Q)\rangle_W\Gamma = \langle P, \Gamma^* l_{2,L}(\Gamma Q)\rangle_W\\ &= \langle P, l_{2,L,\Gamma}(Q)\rangle_W.\end{aligned}$$

So we have $\langle P, l_{2,L}(Q) - l_{2,L,\Gamma}(Q)\rangle_W = \theta$ for any matrix polynomials P,Q. Since for every matrix polynomial Q, $l_{2,L}(Q) - l_{2,L,\Gamma}(Q)$ is also a matrix polynomial, we conclude (see (2.1)) that for every matrix polynomial Q, $l_{2,L}(Q) - l_{2,L,\Gamma}(Q) = \theta$, and hence $A_0 = \Gamma^* A_0 \Gamma$, $A_1 = \Gamma^* A_1 \Gamma$, $A_2 = \Gamma^* A_2 \Gamma$, which gives $\Gamma A_0 = A_0 \Gamma$, $\Gamma A_1 = A_1 \Gamma$, $\Gamma A_2 = A_2 \Gamma$. This means that the coefficients of A_2, A_1 and A_0 commute with any unitary matrix, so they must be the identity matrix up to scalar multiplicative constants. □

We are now ready to prove the classification theorem

Theorem 3.2. *The following conditions are equivalent*:

(i) *W is a matrix weight whose matrix inner product defined as in* (1.1) *has a symmetric left-hand side second order differential operator.*

(ii) *A nonsingular matrix D exists for which $W = D^* X D$ where $X = \nu I$, with ν a classical scalar weight* (*i.e., ν is some of the Jacobi, Hermite or Laguerre weights up to a linear change of variable*).

Proof. (i) ⇒ (ii). Proceeding as before, we get a nonsingular matrix D such that the first moment of the weight $X = D^* W D$ is the identity matrix, and its second moment is a diagonal matrix. We write $(\rho_n)_n$ for the moments of X. Since the operator $l_{2,L}$ is also symmetric for the inner product defined by X, from Lemma 3.1 it follows that the coefficients of A_2, A_1 and A_0 must be the identity matrix up to scalar multiplicative constants. Because $A_0 = \Gamma_0$ (Γ_0 is the Hermitian matrix which appears in Proposition 2.1), and Γ_0 is Hermitian, we have that $A_0 = a_0 I$, with $a_0 \in \mathbf{R}$. So the assumption in Lemma 2.2 holds. Let us write $A_{i,j} = a_{i,j} I$, with $a_{i,j} \in \mathbf{C}$.

Then we have

$$(n-1)A_{2,2} + A_{1,1} = ((n-1)a_{2,2} + a_{1,1})I. \tag{3.1}$$

We assume the following claim, which we prove later

Claim. *For every $n \geq 1$, $(n-1)a_{2,2} + a_{1,1} \neq 0$.*

For $n = 1$ this gives $a_{1,1} \neq 0$. Then, from equation (2.2), for $n = 1$, we get that $\rho_1 = -a_{1,1}^{-1} a_{1,0} I$, and so ρ_1 is the identity matrix up to a multiplicative constant.

Again, from equation (2.2), we have

$$\rho_n = -((n-1)a_{2,2} + a_{1,1})^{-1}((n-1)a_{2,1}\rho_{n-1} + (n-1)a_{2,0}\rho_{n-2} + a_{1,0}\rho_{n-1}). \tag{3.2}$$

Taking into account that the moments ρ_0, ρ_1 are the identity matrix up to scalar multiplicative constants, it follows that a sequence of numbers $(b_n)_n$ exists for which $\rho_n = b_n, I$, $n \geq 0$.

It is straightforward to see that the scalar sequence $(b_n)_n$ is positive definite. Let ν be a positive measure for which $b_n = \int t^n d\nu(t)$, $n \geq 0$. The equation (3.2) gives

$$b_n = -((n-1)a_{2,2} + a_{1,1})^{-1}((n-1)a_{2,1}b_{n-1} + (n-1)a_{2,0}b_{n-2} + a_{1,0}b_{n-1}). \tag{3.2}$$

And this is equivalent to the symmetry of the second order differential operator defined by

$$(a_{2,2}t^2 + a_{2,1}t + a_{2,0})D'' + (a_{1,1}t + a_{1,0})D',$$

with respect to the scalar inner product $\langle p, q\rangle = \int p(t)\overline{q(t)}\,d\nu(t)$. To finish the proof, we apply the classical Bochner classification theorem for orthogonal polynomial sets which satisfy a second order differential equation (cf. [**2**] or [**3**]).

We finally prove the claim. Let us consider the Hermitian matrices Γ_n, $n \geq 0$, which appear in Proposition 1.1. Since $l_{2,L}(P_n) = \Gamma_n P_n$, and the leading coefficient of P_n is a nonsingular matrix, we have that

$$\Gamma_n = n(n-1)A_{2,2} + nA_{1,1} + A_0. \tag{3.3}$$

Suppose that, for certain $n \geq 1$, $((n-1)a_{2,2} + a_{1,1}) = 0$, i.e., $(n-1)A_{2,2} + A_{1,1} = \theta$. From (3.3) we have $\Gamma_n = A_0$. Then, for any $k \geq 0$,

$$\begin{aligned} A_0 &\int P_n(t)\,dW(t)P_k^*(t) \\ &= \Gamma_n \int P_n(t)\,dW(t)P_k^*(t) \\ &= \int \Gamma_n P_n(t)\,dW(t)P_k^*(t) \\ &= \int l_{2,L}(P_n(t))\,dW(t)P_k^*(t) \\ &= \int (A_2(t)P_n''(t) + A_1(t)P_n'(t) + A_0P_n(t))\,dW(t)P_k^*(t). \end{aligned}$$

Hence, for any $k \geq 0$, we get

$$\theta = \int (A_2(t)P_n''(t) + A_1(t)P_n'(t))\, dW(t)P_k^*(t).$$

Since the coefficients of A_2 and A_1 are the identity matrix up to multiplicative constants, they commute with any matrix, so we have

$$\begin{aligned}\theta &= \int (P_n''(t)A_2(t) + P_n'(t)A_1(t))\, dW(t)P_k^*(t)\\ &= \int P_n''(t)A_2(t)\, dW(t)P_k^*(t) + \int P_n'(t)A_1(t)\, dW(t)P_k^*(t).\end{aligned}$$

Taking into account the equations (2.9), we have

$$\begin{aligned}\theta = &\int P_n''(t)A_2(t)\, dW(t)P_k^*(t)\\ &+ \int P_n'(t)(A_2(t)\, dW(t))'P_k^*(t), \quad k \geq 0.\end{aligned} \tag{3.4}$$

We now prove that, for any matrix of measures Z and any matrix polynomials P and Q, then

$$\int (P(dZ)' + P'\, dZ)Q^* = \int (P\, dZ)'Q^*. \tag{3.5}$$

Indeed, it will be enough to prove this equality for $Q(t) = t^n I$, $n \geq 0$. From the definition (2.8) we have that

$$\begin{aligned}\int (P\, dZ)'t^n I &= -\int nt^{n-1}P\, dZ\\ &= -\int ((t^nP)' - t^nP')\, dZ\\ &= -\int (t^nP)'\, dZ + \int t^nP'\, dZ\\ &= \int t^nP(dZ)' + \int t^nP'\, dZ\\ &= \int P(dZ)'t^nI + \int P'\, dZt^nI.\end{aligned}$$

Hence, from (3.4) and (3.5), we have

$$\theta = \int (P_n(t)'(A_2W(t))'P_k^*(t), \quad k \geq 0.$$

Since the sequence $(P_k)_k$ is a basis in $\mathbf{P}_{N\times N}$, we conclude that as an operator defined in $\mathbf{P}_{N\times N}$, $(P_n(t)'(A_2W(t))' = 0$. Taking into account the definition 2.8, we have that $P_n(t)'A_2W(t) = 0$. Since $P_n'A_2$ is a polynomial, the condition (2.1) gives that $P_n'A_2 = \theta$. Since the coefficients of A_2 are the identity matrix up to scalar multiplicative constants, and $A_2 \neq \theta$, we conclude that

$$P_n' = \theta. \tag{3.6}$$

Since $n \geq 1$, (3.6) is a contradiction, and the claim is proved.

The proof of (ii) $\Leftarrow$ (i) is straightforward from the fact (proved at the beginning of this section) that if $l_{2,L}$ is symmetric for the inner product defined by W, so is for the inner product defined by C^*WC, for any nonsingular matrix C. $\square$

Appendix

In this section we show that the classification result, which has been proved in the previous section (Theorem 3.2), does not work for right-hand side second order differential operators. Indeed, we take the weight

$$W(t) = \begin{pmatrix} e^{-t^2} & 0 \\ 0 & e^{-(t-1)^2} \end{pmatrix}, \quad a, b > 0,\ a \neq b,$$

and the matrix polynomials $A_2 = I$,

$$A_1 = \begin{pmatrix} -2t & 0 \\ 0 & -2(t-1) \end{pmatrix},$$

and $A_0 = 0$. Since the conditions of Lemma 2.3 are fulfilled, the right-hand side differential operator defined by A_2, A_1 and A_0 is symmetric for the inner product defined by W. But the weight W is not of the form given in Theorem 3.2.

Moreover, this is a counterexample to show that the converse of Lemmas 2.1, 2.2 and 2.3 is not true for a left-hand side second order differential operator. Indeed, this weight satisfies the condition of Lemma 2.3, and so its moments satisfy the condition of Lemma 2.2; it is also straightforward to see that the sequence of orthonormal matrix polynomials for this weight satisfies a second order differential equation, but the left-hand side differential operator defined by A_2, A_1, A_0 is not symmetric for the inner product defined by W, because W is not of the form given in Theorem 3.2 (ii).

As the last result in this note, we show that the example given in this Section is essentially the only possible when A_2 is a nonsingular constant and A_0 is the identity matrix up to a multiplicative constant:

Proposition 4.1. *Let W be a matrix weight whose matrix inner product defined as in* (1.1) *has a symmetric right-hand side second order differential operator for which A_2 is a nonsingular numerical matrix and A_0 is the identity matrix up to a multiplicative constant. Then a nonsingular matrix D exists for which $W = D^*XD$, where X is a diagonal matrix whose entries in the diagonal are classical Hermite weights up to a linear change of variables* (*possibly a different change in every entry*).

To prove this proposition, we need the following lemma

Lemma 4.2. *In the hypothesis of Lemma* 2.2*, if $A_{1,1}$ is a singular matrix and v is an eigenvector of $A_{1,1}$ associated to* 0*, then v is also an eigenvector of the matrices $A_{2,2}$, $A_{2,1}$, $A_{2,0}$ and $A_{1,0}$ associated to* 0.

Proof of Lemma 4.2. By hypothesis we have $vA_{1,1} = \theta$ and so, from the moment equation (2.2) for $n = 1$, we have $vA_{1,0}\mu_0 = \theta$, since the matrix μ_0 is positive definite, and hence nonsingular, we have $vA_{1,0} = \theta$.

The moment equation (2.2) for $n = 2, 3, 4$ gives that

$$v(A_{2,2}\mu_n + A_{2,1}\mu_{n-1} + A_{2,0}\mu_{n-2}) = \theta, \quad n = 2, 3, 4.$$

We can write these equations as follows:

$$(vA_{2,0} \quad vA_{2,1} \quad vA_{2,2}) \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} \begin{pmatrix} A_{2,0}^* v^* \\ A_{2,1}^* v^* \\ A_{2,2}^* v^* \end{pmatrix} = \theta.$$

Since the matrix

$$\begin{pmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix}$$

is positive definite, we get $vA_{2,0} = \theta$, $vA_{2,1} = \theta$ and $vA_{2,2} = \theta$, and the lemma is proved. □

Proof of Proposition 4.1. We know that $A_0 = \Gamma_0$ (see Proposition 2.1). Since Γ_0 is Hermitian, we have that A_0 is the identity matrix up to a real multiplicative constant. By using Proposition 2.1, we can assume $A_0 = 0$. So the assumption of Lemma 2.2 is fulfilled.

First of all we show that we can assume the first moment of W to be the identity matrix. Indeed, we take the polynomials $Q_n(t) = P_n(t)P_0^{-1}(t)$, $n \geq 0$, which are orthonormal with respect to the matrix weight $X = P_0WP_0^*$. It is straightforward to see that the sequence $(Q_n)_n$ satisfies the matrix second order differential equation

$$Q_n''(P_0A_2P_0^{-1}) + Q_n'(P_0A_1P_0^{-1}) = \Gamma_n Q_n.$$

Hence Lemma 2.2 gives that the right-hand side second order differential operator

$$l_{2,R} = D''(P_0A_2P_0^{-1}) + D'(P_0A_2P_0^{-1})$$

is symmetric for the matrix inner product defined by X. But it is clear that the first moment of X is the identity matrix. We write $B_2 = (P_0A_2P_0^{-1})$, $B_1 = (P_0A_1P_0^{-1})$, $B_{i,j} = (P_0A_{i,j}P_0^{-1})$ and $(\rho_n)_n$ for the moment sequence of X. We remark that since A_2 is a nonsingular constant matrix, so is B_2.

From Lemma 4.2, the nonsingularity of the matrix B_2 implies the nonsingularity of the matrix $B_{1,1}$. We write the moment equation (2.2) as follows:

$$(n-1)B_{1,1}^{-1}B_{2,0}\rho_{n-2} + \rho_n + B_{1,1}^{-1}B_{1,0}\rho_{n-1} = \theta. \tag{4.1}$$

We will prove that the moments $(\rho_n)_n$ and the matrix $B_{1,1}^{-1}B_{2,0}$ and $B_{1,1}^{-1}B_{1,0}$ commute.

For $n = 1$, (4.1) automatically gives that $B_{1,1}^{-1}B_{1,0} = -\rho_1$, and so $B_{1,1}^{-1}B_{1,0}$ is Hermitian. Now (4.1) for $n = 2$ gives

$$B_{1,1}^{-1}B_{2,0} = -\rho_2 + \rho_1^2,$$

and so $B_{1,1}^{-1}B_{2,0}$ is also Hermitian. We now prove that $B_{1,1}^{-1}B_{2,0}$ and $B_{1,1}^{-1}B_{1,0}$ commute. It will be enough to prove that ρ_1 and ρ_2 also commute. But (4.1) for $n = 3$, and the formulas for $B_{1,1}^{-1}B_{2,0}$ and $B_{1,1}^{-1}B_{1,0}$ give

$$-2\rho_2\rho_1 + 2\rho_1^3 + \rho_3 - \rho_1\rho_2 = \theta,$$

and from here it is easy to conclude that ρ_1 and ρ_2 commute.

The equation (4.1) shows that all the Hermitian matrices $B_{1,1}^{-1}B_{2,0}$, $B_{1,1}^{-1}B_{1,0}$ and ρ_n, $n \geq 0$ commute. So they can be diagonalized by the same unitary matrix C. We write $B_{1,1}^{-1}B_{2,0} = C^*D_{2,0}C$, $B_{1,1}^{-1}B_{1,0} = C^*D_{1,1}C$ and $\rho_n = C^*T_nC$, where $D_{2,0}, D_{1,0}$ and T_n are diagonal matrices. Then, from (4.1), we get

$$(n-1)D_{2,0}T_{n-2} + T_n + D_{1,0}T_{n-1} = \theta. \tag{4.2}$$

We take the sequences in the diagonal of T_n, i.e., $(T_{n,k})_n$, $k = 1, \dots, N$. It is clear that all these sequences are positive definite. We write ν_k for the positive measure satisfying $\int t^n \, d\nu_k = T_{n,k,k}$, $n \geq 0$.

From (4.2) we deduce that the moments $(T_{n,k})_n$ satisfy the moment equation

$$(n-1)D_{2,0,k,k}T_{n-2,k,k} + T_{n,k,k} + D_{1,0,k,k}T_{n-1,k,k} = 0,$$

which implies that the second order differential operator

$$D_{2,0,k,k}D'' + (x + D_{1,0,k,k})D',$$

is symmetric for the scalar inner product defined by ν_k, $k = 1, \dots, N$. Again, from the Bochner classification theorem, we deduce that the measure ν_k are Hermite weights up to linear changes of variable, possibly a different change for every k. And so the proof is finished. $\square$

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