ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 27, Number 2, Spring 1997

QUASI-MEASURES ON COMPLETELY REGULAR SPACES

JOHN P. BOARDMAN

ABSTRACT. Let X be a completely regular space. We give definitions of a Baire quasi-measure on X and a quasi-state on $C_b(X)$, the space of bounded, real-valued continuous functions on X. A representation theorem is developed for quasi-states on $C_b(X)$ in terms of Baire quasi-measures on X. We also define various notions of smoothness for quasi-measures and quasi-states, and then we furnish examples which demonstrate the different types of smoothness. Finally, by considering the space X to be embedded in its Stone-Čech compactification βX, the smoothness of a Baire quasi-measure μ on X is characterized by the behavior of $\bar{\mu}$, the corresponding Baire quasi-measure on βX.

1. Introduction. The theory of quasi-measures evolved from the study of certain nonlinear functionals (quasi-states) on commutative C^*-algebras. The goal here is to extend to completely regular spaces the theory of quasi-measures on compact Hausdorff spaces initiated by J. Aarnes [**1**]. For definitions and results regarding quasi-measures on compact Hausdorff spaces the reader is encouraged to consult [**1, 2**]. Our standard guide to work in topological measure theory is the survey paper by Wheeler [**11**].

One of the main goals of this paper is to develop a representation for quasi-states on $C_b(X)$ in terms of Baire quasi-measures on X. This is accomplished by generalizing results of Aarnes in [**1**] to completely regular spaces. We then state definitions of various types of smoothness for quasi-measures and quasi-states. Then, following work done by Varadarajan [**10**] for ordinary Baire measures, we demonstrate the connection between the smoothness of a Baire quasi-measure and the smoothness of its corresponding quasi-state.

Finally, using the techniques of Knowles [**6**], we characterize the smoothness of a Baire quasi-measure μ on X by the behavior of $\bar{\mu}$, the unique Baire quasi-measure on βX corresponding to μ. In Section 2

Received by the editors on October 27, 1994, and in revised form on April 20, 1995.

and Section 3 we present the work of J. Aarnes on quasi-states and quasi-measures [**1**, pp. 41–56] in the completely regular setting. We omit those proofs which follow exactly as in [**1**].

This work constitutes a portion of the author's doctoral dissertation at Northern Illinois University [**4**]. The author would like to thank Robert Wheeler and the referee for their helpful suggestions.

2. Definition and basic properties of a Baire quasi-measure. Throughout this paper X will denote a completely regular Hausdorff space. A subset Z of X is called a *zero set* if and only if it has the form $Z = f^{-1}(0)$, where $f \in C_b(X)$. A subset U of X is called a *cozero set* if and only if it has the form $U = f^{-1}(\mathbf{R}\backslash\{0\})$, where $f \in C_b(X)$. We reserve the letters Z and U to refer only to zero sets and cozero sets. The *algebra of Baire sets* is the least algebra of sets which contains the zero sets. It is denoted $\mathrm{Ba}^*(X)$. The *algebra of Borel sets* is the least algebra of sets which contains the closed sets. It is denoted $\mathrm{Bo}^*(X)$. For a subset $A \subseteq X$ we write $A \prec f$ if $f \in C_b(X)$, $f(A) = 1$ and $0 \leq f \leq 1$.

A *positive Baire measure* μ on a completely regular space X is a nonnegative, finite, finitely additive set function on $\mathrm{Ba}^*(X)$ which is zero-set regular (if $A \in \mathrm{Ba}^*(X)$, $\mu(A) = \sup\{\mu(Z) : Z \subseteq A\}$). A (signed) Baire measure is the difference of two positive Baire measures.

For a completely regular space X, let $\mathcal{Z}(X)$ and $\mathcal{U}(X)$ denote the collections of zero sets and cozero sets, respectively, or simply $\mathcal{Z}$ and $\mathcal{U}$ when no confusion will arise. Also, let $\mathcal{A} = \mathcal{Z} \cup \mathcal{U}$.

Definition 2.1. A real-valued, nonnegative function μ on $\mathcal{A}$ is called a *Baire quasi-measure on* X if the following conditions are satisfied:

(1) $\mu(Z) + \mu(X\backslash Z) = \mu(X)$, $Z \in \mathcal{Z}$;

(2) $Z_1 \subseteq Z_2 \Rightarrow \mu(Z_1) \leq \mu(Z_2)$, $Z_1, Z_2 \in \mathcal{Z}$;

(3) $Z_1 \cap Z_2 = \varnothing \Rightarrow \mu(Z_1 \cup Z_2) = \mu(Z_1) + \mu(Z_2)$, $Z_1, Z_2 \in \mathcal{Z}$;

(4) $\mu(U) = \sup\{\mu(Z) : Z \subseteq U; Z \in \mathcal{Z}\}$ for all $U \in \mathcal{U}$.

For convenience, we assume that $\mu(X) = 1$.

A similar definition, using closed and open sets, can be given for a

Borel quasi-measure on X.

Proposition 2.2 [**1**, 2.1]. *Let μ be a Baire quasi-measure on X.*

(a) $\mu(\varnothing) = 0$;

(b) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ *for all* $A, B \in \mathcal{A}$;

(c) *If* $A_1, A_2, \ldots, A_n$ *are mutually disjoint members of* $\mathcal{A}$ *whose union belongs to* $\mathcal{A}$, *then*

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i);$$

(d) $\mu(Z) = \inf\{\mu(U) : Z \subseteq U, U \in \mathcal{U}\}$ *for all* $Z \in \mathcal{Z}$.

A natural question which arises at this point is: What is the relationship between Baire quasi-measures on X and ordinary Baire measures on X? Just as in the case of (Borel) quasi-measures, the notion of a Baire quasi-measure is truly a generalization of that of a Baire measure. In other words, a Baire quasi-measure is not necessarily the restriction to $\mathcal{A}$ of a Baire measure. In Section 7 an example will be given of a Baire quasi-measure on $\mathbf{R}^2$ which is not subadditive (Example 7.3). If a Baire quasi-measure μ on X is subadditive, then it has a unique extension to a Baire measure γ on X as stated later in Theorem 4.4.

3. Definition and basic properties of a quasi-state. For $f \in C_b(X)$ we let $A(f)$ denote the smallest uniformly closed subalgebra of $C_b(X)$ containing f and 1 (the function identically equal to one on X).

Definition 3.1. A real-valued function ρ on $C_b(X)$ is called a *quasi-state* if the following conditions are satisfied:

(i) $\rho(g) \geq 0$ if $g \geq 0$, $g \in C_b(X)$;

(ii) ρ is linear on $A(f)$ for each $f \in C_b(X)$;

(iii) $\rho(1) = 1$.

Lemma 3.2 [**1**, 4.1]. *Let ρ be a quasi-state on $C_b(X)$, and let $0 \leq f$, $g \in C_b(X)$. Then*

(a) $\rho(f+g) = \rho(f) + \rho(g)$ *if* $f \cdot g = 0$;

(b) $\rho(f) \le \rho(g)$ *if* $f \le g$;

(c) $|\rho(f) - \rho(g)| \le \|f - g\|$.

Let μ be a Baire quasi-measure on X, and let $f \in C_b(X)$ be arbitrary. We establish the following notation: $\mathrm{Sp}\,(f) = \overline{f(X)}$ is the closure in $\mathbf{R}$ of the range of f, $Z_\alpha^f = \{x \in X : f(x) \ge \alpha\}$, $V_\alpha^f = \{x \in X : f(x) > \alpha\}$, $\hat{f}(\alpha) = \mu(Z_\alpha^f)$, $\check{f}(\alpha) = \mu(V_\alpha^f)$.

We have $Z_\alpha^f \in \mathcal{Z}$, $V_\alpha^f \in \mathcal{U}$, so the functions $\hat{f}$ and $\check{f}$ are well defined for all real α. If $\mathrm{Sp}\,(f) \subseteq [\lambda_1, \lambda_2]$, then $\hat{f}$ and $\check{f}$ map $[\lambda_1, \lambda_2]$ into $[0,1]$, they are both decreasing, $\check{f} \le \hat{f}$, $\hat{f}(\lambda_1) = 1$, $\check{f}(\lambda_2) = 0$.

Proposition 3.3 [**1**, 3.1]. *Let* $f \in C_b(X)$ *be arbitrary.*

(a) $\hat{f}$ *is continuous at* α *if and only if* $\check{f}$ *is continuous at* α.

(b) $\hat{f}$ *and* $\check{f}$ *coincide at every mutual point of continuity, hence at all but countably many points of* $\mathbf{R}$.

Proof. Let $\alpha \in \mathbf{R}$ be fixed. If $\beta < \gamma < \alpha_1 < \alpha$, we have $Z_{\alpha_1}^f \subseteq V_\gamma^f \subseteq Z_\beta^f$. Therefore

$$\hat{f}(\alpha^-) = \check{f}(\alpha^-)$$

and

$$\hat{f}(\alpha^+) = \check{f}(\alpha^+)$$

hold. Since $\hat{f}$ and $\check{f}$ are both monotone, (a) follows. From the proof of (a), we see that $\hat{f}$ and $\check{f}$ must coincide at every mutual point of continuity since each is monotone. As for the second part of (b), $\hat{f}$ and $\check{f}$ must be continuous at all but countably many points of $\mathbf{R}$ since the jumps at points of discontinuity must be summable. $\square$

Remark. In the case when X is compact, Aarnes proved that $\hat{f}$ is continuous from the left and $\check{f}$ is continuous from the right. In the setting of a completely regular space this is not necessarily true, as we shall see in the following example.

Example 3.4. Let $X = (0,1]$ and define $f \in C_b(X)$ as follows: $f(x) = 0$ for all $x \geq 1/2$, $f(0^+) = 1$ and f is defined linearly on $(0, 1/2]$. Consider X to be embedded in its Stone-Čech compactification βX. Now the finite intersection property implies

$$\bigcap_n \mathrm{cl}_{\beta X}(0, 1/n] \neq \varnothing,$$

so let $\mu = \delta_{x_0}$ where $x_0 \in \cap_n \mathrm{cl}_{\beta X}(0, 1/n]$, $x_0 \notin X$. It is now easy to check that $\hat{f}(\alpha) = 1$ for all $\alpha < 1$ and $\hat{f}(1) = 0$. Therefore $\hat{f}$ is not left-continuous.

From this example we can also see that $\hat{f}$ is not necessarily the cumulative distribution function of a regular Borel measure. It is for this reason that we are unable to retain the spectral theory and functional calculus that Aarnes developed in the compact setting.

4. The representation theorem for quasi-states. In the following theorem we show that any Baire quasi-measure determines a quasi-state, and that all quasi-states arise in this way. We keep the notation and conventions of the two preceding sections. Compare the following with Theorem 4.1 of [**1**].

Theorem 4.1. *Let X be a completely regular space.*

(a) *To each quasi-measure μ in X satisfying $\mu(X) = 1$, there corresponds a unique quasi-state ρ such that, for any $f \in C_b(X)$, we have*

$$\rho(f) = \lambda_1 + \int_{\lambda_1}^{\lambda_2} \hat{f}(\alpha)\, d\alpha, \tag{1}$$

where $[\lambda_1, \lambda_2] \supseteq \mathrm{Sp}\,(f)$.

(b) *Conversely, for any quasi-state ρ on $C_b(X)$ there is a unique quasi-measure μ on X satisfying $\mu(X) = 1$ such that ρ is the quasi-state corresponding to μ. Specifically we have, for any zero set $Z \subseteq X$:*

$$\mu(Z) = \inf\{\rho(f) : Z \prec f; f \in C_b(X)\}. \tag{2}$$

Proof. Part (a) and most of part (b) can be proven in much the same manner as the compact case. Complete details are given in [**4**]. For part (b), let ρ be a quasi-state on $C_b(X)$. Using the techniques of Aarnes it can be shown that μ, as defined in (2), is a Baire quasi-measure on X. Now suppose ρ' is the quasi-state on $C_b(X)$ corresponding to μ as described in part (a). It is left to show that $\rho' = \rho$.

We first set notation to be used in the following two lemmas. Let $f \in C_b(X)$ be arbitrary. Choose $\lambda_1, \lambda_2 \in \mathbf{R}$ such that $(\lambda_1, \lambda_2) \supseteq \mathrm{Sp}\,(f)$; in this way $[\lambda_1, \lambda_2] \supseteq \mathrm{Sp}\,(f)$ and $\hat{f}$ is continuous at λ_1 and λ_2. Then

$$\rho'(f) = \lambda_1 + \int_{\lambda_1}^{\lambda_2} \hat{f}(\alpha)\, d\alpha.$$

On the other hand $\rho_f : \phi \to \rho(\phi(f))$ is a positive linear functional on $C([\lambda_1, \lambda_2])$ and determines a unique regular Borel measure ν_f on $[\lambda_1, \lambda_2]$ such that

$$\rho(\phi(f)) = \rho_f(\phi) = \int_{\mathrm{Sp}\,(f)} \phi(\alpha)\, d\nu_f(\alpha).$$

Let F denote the cumulative distribution function of ν_f so that $F(t) = \nu_f((-\infty, t])$ for $t \in \mathbf{R}$. It is easy to show that $\mathrm{supp}\,(\nu_f) \subseteq \mathrm{Sp}\,(f)$. By virtue of this fact, we have

$$F(\lambda_2) = \nu_f((-\infty, \lambda_2]) = \nu_f([\lambda_1, \lambda_2]) = \rho(\phi(f)) = \rho(1) = 1$$

where $\phi(t) = 1$ for all t.

Lemma 4.2. *Let $\lambda_1 \leq \alpha_0 \leq \lambda_2$ such that $\hat{f}(\alpha)$ is continuous at α_0. Then*

$$\nu_f([\alpha_0, \lambda_2]) = \hat{f}(\alpha_0).$$

Proof. For each $n = 1, 2, \dots$, let ϕ_n be a continuous function on $[\lambda_1, \lambda_2]$ satisfying $0 \leq \phi_n \leq 1$, $\phi_n(\alpha) = 0$ for $\alpha \leq \alpha_0 - 1/n$ and $\phi_n(\alpha) = 1$ for $\alpha \geq \alpha_0$. Then $\phi_n(f)(x) = \phi_n(f(x)) = 1$ for $f(x) \geq \alpha_0$ and $\phi_n(f)(x) = 0$ when $f(x) \leq \alpha_0 - 1/n$. Hence $Z^f_{\alpha_0} \prec \phi_n(f) \prec Z^f_{\alpha_0 - 1/n}$.

It now follows from the definition of μ from ρ in equation (2) and Lemma 3.2(b) that

$$\mu(Z^f_{\alpha_0}) \leq \rho(\phi_n(f)) \leq \mu(Z^f_{\alpha_0 - 1/n}).$$

By assumption $\hat{f}$ is continuous at α_0 which implies $\lim_{n\to\infty} \rho(\phi_n(f)) = \hat{f}(\alpha_0)$. On the other hand, we clearly have $\nu_f([\alpha_0, \lambda_2]) = \lim_{n\to\infty} \rho_f(\phi_n)$. $\square$

Lemma 4.3. *The following three conditions are equivalent*:

(1) $\hat{f}$ *is continuous at* α_0;

(2) F *is continuous at* α_0;

(3) $\nu_f(\{\alpha_0\}) = 0$.

In addition, each of these implies:

(4) $\mu\{x \in X : f(x) = \alpha_0\} = 0$.

Proof. It is straightforward to show the equivalence of (1)–(3) using the additivity properties of the regular Borel measure ν_f and the fact that F is necessarily right-continuous.

To complete the proof it is enough to show that (1) $\Rightarrow$ (4). Suppose $\hat{f}$ is continuous at α_0; then $\hat{f}(\alpha_0) = \check{f}(\alpha_0)$ by Proposition 3.3. Therefore, by Proposition 2.2(c),

$$\begin{aligned} \mu\{x \in X : f(x) = \alpha_0\} &= \mu(Z^f_{\alpha_0} \backslash V^f_{\alpha_0}) \\ &= \mu\{x \in X : f(x) \geq \alpha_0\} \\ &\quad - \mu\{x \in X : f(x) > \alpha_0\} \\ &= \hat{f}(\alpha_0) - \check{f}(\alpha_0) \\ &= 0. \qquad \square \end{aligned}$$

Remark. In the lemma above, (4) does not necessarily imply (1)–(3). For instance, (4) does not insure (1) as demonstrated in Example 3.4.

As defined previously, $\rho(\phi(f)) = \rho_f(\phi) = \int_{\mathrm{Sp}\,(f)} \phi(\alpha)\, d\nu_f(\alpha)$. By letting $\phi(t) = t$, we get $\rho(\phi(f)) = \rho(f) = \int_{\mathrm{Sp}\,(f)} \alpha\, d\nu_f(\alpha)$. This integral is an ordinary Riemann-Stieltjes integral over the interval $\mathrm{Sp}\,(f)$. Applying the remarks after the statement of Proposition 3.2 in [**1**], we obtain

$$\rho(f) = \lambda_2 + \int_{\lambda_1}^{\lambda_2} -F(\alpha)\,d\alpha. \tag{3}$$

Finally we will show $\rho' = \rho$. By Lemmas 4.2 and 4.3, along with the fact that $F(\lambda_2) = 1$, we have

$$\hat{f}(\alpha_0) = \nu_f([\alpha_0, \lambda_2]) = F(\lambda_2) - F(\alpha_0^-) = 1 - F(\alpha_0^-)$$

for all $\lambda_1 \le \alpha_0 \le \lambda_2$ with $\hat{f}$ continuous at α_0. Also, by Lemma 4.3, we know that $\hat{f}$ is continuous at α_0 if and only if F is continuous at α_0. Therefore, for all $\lambda_1 \le \alpha_0 \le \lambda_2$ with $\hat{f}$ continuous at α_0,

$$\hat{f}(\alpha_0) = 1 - F(\alpha_0).$$

Finally, using equation (3) and the fact that $\hat{f}$ is continuous at all but a countable number of $\alpha \in [\lambda_1, \lambda_2]$, it can be shown that $\rho'(f) = \rho(f)$. Since $f \in C_b(X)$ was arbitrary, $\rho = \rho'$. The uniqueness of μ in part (b) follows as in [**1**].

Remark. The representation theorem presented here is indeed a generalization of the Alexandroff representation theorem [**11**, Theorem 5.1], i.e., a quasi-state on $C_b(X)$ is linear if and only if the corresponding Baire quasi-measure extends to a measure on the entire Baire algebra $\mathrm{Ba}^*(X)$. In that way, Example 7.3 also provides us with a quasi-state on $C_b(X)$ which is not linear.

In Section 6 of [**1**] Aarnes provides an example of a quasi-measure on the unit square in $\mathbf{R}^2$ which is not subadditive, thus demonstrating that some quasi-measures are not the restrictions of ordinary measures. In the next theorem we note that subadditivity is not only necessary, but also sufficient to establishing the existence of an extension of a Baire quasi-measure to the entire Baire algebra.

This result is due to Wheeler [**12**, 3.1, 3.3] for Borel quasi-measures and is stated here for the convenience of the reader.

Theorem 4.4 [**12**, 3.1, 3.3]. *Let X be a completely regular space. Let μ be a Baire quasi-measure on X and let ρ be the quasi-state on $C_b(X)$ corresponding to μ. Then the following are equivalent*:

(1) *ρ is linear (hence a positive linear functional on $C_b(X)$)*;

(2) *If $f, g \geq 0$ in $C_b(X)$, then $\rho(f+g) \leq \rho(f) + \rho(g)$*;

(3) *if $Z_1, Z_2 \in \mathcal{Z}$, then $\mu(Z_1 \cup Z_2) \leq \mu(Z_1) + \mu(Z_2)$*;

(4) *if $U_1, U_2 \in \mathcal{U}$, then $\mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2)$*;

(5) *μ admits an extension to a finitely-additive zero-set regular measure γ on the Baire algebra of X (the extension is unique, by regularity)*.

Remark. In [**12**] Wheeler presents his result for normal spaces and Borel quasi-measures; the proof is virtually identical. For this and other results about quasi-measures on normal spaces, consult [**12**].

It is often useful to extend a quasi-measure defined on a subspace to a quasi-measure on the entire space. This arises in a couple of our examples in Section 7 and the next result addresses this issue.

Theorem 4.5. *Let X be a normal space. If A is a closed subset of X and μ is a Borel quasi-measure on A, then $\lambda(B) = \mu(B \cap A)$ is a Borel quasi-measure on X.*

Proof. Properties (1), (2) and (3) of a quasi-measure are immediate. If U is open in X and $\varepsilon > 0$, choose a closed (in A) subset F of $U \cap A$ such that $\mu(U \cap A) < \mu(F) + \varepsilon$. But F is closed in X (since A is), and $\lambda(U) < \lambda(F) + \varepsilon$. Hence (4) holds. $\square$

Notice that the theorem is stated for spaces X which are normal. This method of proof cannot be easily extended to the completely regular setting since, for X completely regular, the closure in X of a set $Z \in \mathcal{Z}(A)$ is not necessarily a zero set of X.

5. Smoothness properties of quasi-measures. Let X be a completely regular space. Throughout this section we use the term quasi-measure to mean either a Baire quasi-measure or a Borel quasi-measure, defined on the open and closed subsets of X as in Definition 2.1.

Definition 5.1. Let μ be a Baire quasi-measure on X and $A \subset X$. Then
$$\mu_*(A) = \sup\{\mu(Z) : Z \subseteq A, Z \in \mathcal{Z}\}$$
is called the *inner quasi-measure of* μ.

Definition 5.2. Let μ be a Baire quasi-measure on X and $B \subset X$. Then
$$\mu^*(B) = \inf\{\mu(U) : U \supseteq B, U \in \mathcal{U}\}$$
is called the *outer quasi-measure of* μ.

Similar definitions can be constructed for Borel quasi-measures by replacing the zero sets Z and the cozero sets U with closed sets and open sets, respectively.

Definition 5.3. A quasi-measure μ on X is said to be:

(i) *σ-smooth* if whenever $\{Z_n\}$ is a sequence of zero sets with $Z_n \downarrow \varnothing$ then $\mu(Z_n) \to 0$;

(ii) *τ-smooth* if whenever $\{Z_\alpha\}$ is a net of zero sets with $Z_\alpha \downarrow \varnothing$ then $\mu(Z_\alpha) \to 0$;

(iii) *tight* if for any $\delta > 0$ there exists a compact subset K of X such that $\mu_*(X \backslash K) \leq \delta$.

Remark. In contrast to the setting of ordinary Baire measures, it is not clear whether a set function demonstrating the σ-smoothness property and satisfying (1)–(3) of Definition 2.1 necessarily satisfies (4) of Definition 2.1, the regularity condition. More can be said when a stronger smoothness property is assumed. See the remark after Definition 5.10.

Definition 5.4. A quasi-state ρ on $C_b(X)$ is said to be:

(i) *σ-smooth* if whenever $\{f_n\}$ is a sequence in $C_b(X)$ with $f_n \downarrow 0$ then $\rho(f_n) \to 0$;

(ii) τ*-smooth* if whenever $\{f_\alpha\}$ is a net in $C_b(X)$ with $f_\alpha \downarrow 0$ then $\rho(f_\alpha) \to 0$;

(iii) *tight* if whenever $\{f_\alpha\}$ is a net in $C_b(X)$ such that $\|f_\alpha\| \le 1$ for all α and $f_\alpha \to 0$ uniformly on compact subsets of X then $\rho(f_\alpha) \to 0$.

Before getting to the first theorem of this section we introduce the concept of a regular sequence. We say that a sequence $\{Z_n\}$ is *regular* if the following two conditions are satisfied: (1) $Z_n \uparrow X$ and (2) for any n there exists a cozero set U_n such that $Z_n \subset U_n \subset Z_{n+1}$.

The following theorem is essentially due to Varadarajan.

Theorem 5.5 [**10**, pp. 171–172]. *If μ is a quasi-measure such that for any regular sequence $\{Z_n\}$, $\mu(X\backslash Z_n) \to 0$ as $n \to \infty$, then μ is σ-smooth.*

Theorem 5.6. *A quasi-state ρ is σ-smooth if and only if its corresponding quasi-measure μ is σ-smooth.*

Proof. $\Rightarrow$. Let ρ be a σ-smooth quasi-state and $\{Z_n\}$ a regular sequence. By [**10**, Theorem 14] there is a sequence $\{f_n\}$ in $C_b(X)$ such that $f_n \downarrow 0$, $0 \le f_n \le 1$ and $X\backslash Z_{n+1} \subseteq f_n^{-1}(1)$. So $\rho(f_n) \ge \mu(X\backslash Z_{n+1})$ by Theorem 4.1(a) and since ρ is σ-smooth, $\rho(f_n) \to 0$ as $n \to \infty$. Thus it follows that $\mu(X\backslash Z_n) \to 0$ as $n \to \infty$.

$\Leftarrow$. Let $\{f_n\} \subset C_b(X)$ so that $f_n \downarrow 0$. Let $\lambda_2 \in \mathbf{R}$ such that $\mathrm{Sp}\,(f_1) \subseteq [0, \lambda_2]$ which implies $\mathrm{Sp}\,(f_n) \subseteq [0, \lambda_2]$ for all n. By Theorem 4.1,

$$\rho(f_n) = \int_0^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha.$$

Let $\varepsilon > 0$ be given. Since $0 \le \hat{f}_n \le 1$ for all n, we have

$$\begin{aligned}\rho(f_n) &= \int_0^{\varepsilon} \hat{f}_n(\alpha)\, d\alpha + \int_\varepsilon^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha \\ &\le \varepsilon + \int_\varepsilon^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha.\end{aligned}$$

Fix $\alpha \in [\varepsilon, \lambda_2]$ and let $Z_\alpha^n = \{x \in X : f_n(x) \ge \alpha\}$. Now $Z_\alpha^n \downarrow \varnothing$ as $n \to \infty$, so $\mu(Z_\alpha^n) \to 0$ as $n \to \infty$ since μ is σ-smooth. Therefore

$\hat{f}_n(\alpha) \to 0$ as $n \to \infty$ for all $\alpha \in [\varepsilon, \lambda_2]$. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\varepsilon}^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha = \int_{\varepsilon}^{\lambda_2} \lim_{n\to\infty} \hat{f}_n(\alpha)\, d\alpha = 0.$$

Thus $\lim_{n\to\infty} \rho(f_n) \le \lim_{n\to\infty}(\varepsilon + \int_{\varepsilon}^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha) = \varepsilon$. Therefore, $\lim_{n\to\infty} \rho(f_n) = 0$ since ε was arbitrary. □

In the second half of the proof above we split the definite integral $\int_0^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha$ into the sum of two definite integrals, namely $\int_0^{\varepsilon} \hat{f}_n(\alpha)\, d\alpha$ and $\int_{\varepsilon}^{\lambda_2} \hat{f}_n(\alpha)\, d\alpha$. This is the only significant difference between our proof and that of Varadarajan for ordinary Baire measures [**10**, p. 172]. This technique allows us to use Varadarajan's arguments for the following two theorems so we omit the proofs.

Theorem 5.7. *A quasi-state ρ is τ-smooth if and only if its corresponding quasi-measure μ is τ-smooth.*

Theorem 5.8. *A quasi-state ρ is tight if and only if its corresponding quasi-measure μ is tight.*

Let $\mathcal{QM}_\sigma(X)$, $\mathcal{QM}_\tau(X)$ and $\mathcal{QM}_t(X)$ denote the spaces of σ-smooth quasi-measures, τ-smooth quasi-measures and tight quasi-measures, respectively. We have

$$\mathcal{QM}_t(X) \subset \mathcal{QM}_\tau(X) \subset \mathcal{QM}_\sigma(X) \subset \mathcal{QM}(X).$$

The first inclusion is a consequence of Dini's theorem, which states that a decreasing net $\{f_\alpha\}$ in $C_b(X)$ tends to zero if and only if it tends to zero uniformly on compact subsets of X. The other inclusions are obvious.

At this point we mention some difficulties which may arise in the case of quasi-measures that do not appear in the theory of ordinary measures. Because of the lack of linearity of quasi-states it is still unknown whether $f_n \downarrow f$ necessarily implies $\rho(f_n) \to \rho(f)$ for every σ-smooth quasi-state ρ. A similar problem arises for τ-smoothness. We therefore make the following definitions with this uncertainty in mind.

Definition 5.9. A quasi-measure μ is *strongly σ-smooth* if and only if $Z_n \downarrow Z$ implies $\mu(Z_n) \to \mu(Z)$.

Definition 5.10. A quasi-measure μ is *strongly τ-smooth* if and only if $Z_\alpha \downarrow Z$ implies $\mu(Z_\alpha) \to \mu(Z)$.

Remark. It is easy to show that a set function which is strongly σ-smooth and which satisfies (1)–(3) of Definition 2.1 is necessarily a Baire quasi-measure, i.e., satisfies condition (4) of Definition 2.1.

Let $\mathcal{QM}_{S\sigma}$ and $\mathcal{QM}_{S\tau}$ denote the spaces of strongly σ-smooth quasi-measures and strongly τ-smooth quasi-measures, respectively.

Definition 5.11. A quasi-state ρ is *strongly σ-smooth* if and only if $f_n \downarrow f$ implies $\rho(f_n) \to \rho(f)$.

Definition 5.12. A quasi-state ρ is *strongly τ-smooth* if and only if $f_\alpha \downarrow f$ implies $\rho(f_\alpha) \to \rho(f)$.

Here all functions f_n, f_α and f are assumed to belong to $C_b(X)$.

Theorem 5.13. *A quasi-state ρ is strongly σ-smooth if and only if its corresponding quasi-measure μ is strongly σ-smooth.*

Proof. $\Rightarrow$. Let $Z_n \downarrow Z$. Let $\varepsilon > 0$ and choose $f \in C_b(X)$, $0 \leq f \leq 1$, $f|Z \equiv 1$. For each n, choose a sequence of functions $\{g_i^{(n)}\}_{i=1}^{\infty}$ in $C_b(X)$ with the following properties: (1) $0 \leq g_i^{(n)} \leq 1$ for every i, (2) $g_i^{(n)}|Z_n \equiv 1$, and (3) $g_i^{(n)} \downarrow \chi_{Z_n}$ as $i \to \infty$. Let $h_n = \min\{g_n^{(1)}, g_n^{(2)}, \ldots, g_n^{(n)}\}$, $n = 1, 2, \ldots$. Then $h_n \downarrow \chi_Z$ and so $[\max(h_n, f)] \downarrow f$. Since ρ is strongly σ-smooth, $\rho(h_n \vee f) \to \rho(f)$.

Now $Z_n \prec h_n$, and so $\mu(Z_n) \leq \rho(h_n) \leq \rho(h_n \vee f)$, which implies that $\lim_{n\to\infty} \mu(Z_n) \leq \rho(f)$. This holds for any $f \succ Z$, and so $\lim_{n\to\infty} \mu(Z_n) \leq \mu(Z)$. Thus $\lim_{n\to\infty} \mu(Z_n) = \mu(Z)$.

$\Leftarrow$. Let $f_n \downarrow f$. If $t \in \mathbf{R}$, then

$$\{x \in X : f(x) \geq t\} = \bigcap_{n=1}^{\infty} \{x \in X : f_n(x) \geq t\}$$

so that (since μ is strongly σ-smooth) $\hat{f}_n(t) \downarrow \hat{f}(t)$ as $n \to \infty$. Choose $\lambda_1 < \lambda_2$ such that $\mathrm{Sp}\,(f) \cup \cup_{n=1}^{\infty} \mathrm{Sp}\,(f_n) \subseteq [\lambda_1, \lambda_2]$. Then

$$\begin{aligned} \rho(f) &= \lambda_1 + \int_{\lambda_1}^{\lambda_2} \hat{f}(t)\,dt \\ &= \lim_{n\to\infty} \lambda_1 + \int_{\lambda_1}^{\lambda_2} \hat{f}_n(t)\,dt \\ &= \lim_{n\to\infty} \rho(f_n) \end{aligned}$$

by the bounded convergence theorem. $\square$

We do not know if the analogous result for strong τ-smoothness is valid.

In the case of ordinary measures, the space of σ-smooth measures is the same as the space of strongly σ-smooth measures. Also the space of τ-smooth measures is the same as the space of strongly τ-smooth measures. This can be seen most easily by considering the linear functionals which correspond to the measures.

Theorem 5.14. *If X is pseudocompact (i.e., every continuous real-valued function on X is bounded), then every quasi-measure on X is strongly σ-smooth.*

Proof. Let $Z_n \downarrow Z$, and suppose $\lim_n \mu(Z_n) = \mu(Z)+\delta$, $\delta > 0$. Choose a cozero set $U \supseteq Z$ with $\mu(U) < \mu(Z)+\delta/2$. Then $Z_n \cap (X \setminus U) \downarrow \varnothing$, and so, by Theorem 16 of [**10**], there exists an n_0 such that $Z_n \cap (X\setminus U) = \varnothing$ for $n \geq n_0$. Hence for $n \geq n_0$, $Z_n \subseteq U$ so that $\mu(Z_n) \leq \mu(U) < \mu(Z) + \delta/2$. This is a contradiction. $\square$

Theorem 5.15. *If X is discrete, then a quasi-measure on X is σ-smooth if and only if it is strongly σ-smooth.*

Proof. Let μ be a σ-smooth quasi-measure on X, and let $Z_n \downarrow Z$. Now $X\backslash Z$ is a zero set, since X is discrete, and $Z_n \cap (X\backslash Z) \downarrow \varnothing$, so $\mu(Z_n \cap (X\backslash Z)) \to 0$. Since $Z_n = Z \cup [Z_n \cap (X\backslash Z)]$, where the union is disjoint and all sets here are zero sets, we have $\mu(Z_n) \to \mu(Z)$. □

Remark. If X is discrete, then a quasi-measure on X is a finitely-additive zero-set regular Baire measure, since every set is a zero set. This yields an alternate proof of Theorem 5.15.

Theorem 5.16. *If $\mu \in \mathcal{QM}_{S\sigma}(X)$, then μ is τ-smooth if and only if for all cozero covers $\{U_\alpha\}$ of X there exists a countable subcollection $\{U_{\alpha_n}\}$ such that $\mu(X\backslash \cup_{n=1}^\infty U_{\alpha_n}) = 0$.*

Proof. Let $\mu \in \mathcal{QM}_{S\sigma}(X)$.

$\Rightarrow$. Let $\{U_\alpha\}$ cover X, $U_\alpha \in \mathcal{U}$ for all α. Let $\mathcal{F}$ be the set containing all finite unions of U_α's and order $\mathcal{F}$ by set inclusion. Now $V_\beta \uparrow X$ where $V_\beta \in \mathcal{F}$ for all β. By τ-smoothness $\mu(V_\beta) \to \mu(X) = 1$. So, for each n, there exists a β_n such that

$$\mu(V_{\beta_n}) > 1 - 1/n.$$

Notice that $\cup_{n=1}^\infty V_{\beta_n}$ is actually a countable union of sets U_α. So for all n we have

$$\mu\left(\bigcup_{n=1}^\infty V_{\beta_n}\right) \geq \mu(V_{\beta_n}) > 1 - 1/n.$$

This implies that $\mu(\cup_{n=1}^\infty V_{\beta_n}) = 1$ so, in fact, $\mu(X\backslash \cup_{n=1}^\infty V_{\beta_n}) = 0$.

$\Leftarrow$. Let $\{Z_\alpha\} \subseteq \mathcal{Z}$ such that $Z_\alpha \downarrow \varnothing$. Then if we let $U_\alpha = X\backslash Z_\alpha$ we have $U_\alpha \uparrow X$. By assumption there is a countable subcollection $\{U_{\alpha_n}\}_{n=1}^\infty$ such that $\mu(X\backslash \cup_{n=1}^\infty U_{\alpha_n}) = 0$. Now let $V_i = \cup_{n=1}^i U_{\alpha_n}$, $i = 1, 2, \ldots$. Then $V_i \uparrow \cup_{n=1}^\infty U_{\alpha_n}$ so, by strong σ-smoothness, $\mu(V_i) \to \mu(\cup_{n=1}^\infty U_{\alpha_n}) = 1$. Since for each V_i we can find a $U_\alpha^i \supseteq V_i$, ($\{U_\alpha\}$ is an increasing family of sets) we have

$$\lim_\alpha \mu(U_\alpha) \geq \lim_i \mu(V_i) = 1.$$

Therefore, $\lim_\alpha \mu(U_\alpha) = 1$ so $\lim_\alpha \mu(Z_\alpha) = 0$. □

Corollary 5.17. *If X is Lindelöf, then $\mathcal{QM}_\sigma(X) = \mathcal{QM}_\tau(X)$.*

Proof. If X is Lindelöf we can choose the countable subcollection to be a cover of X so we only need σ-smoothness in the proof of sufficiency for the theorem. $\square$

In the following theorem we consider a space X which is locally compact and paracompact. Due to the nature of these spaces, the question of smoothness of quasi-measures on X reduces to a cardinality argument. Let Y be a discrete space with $\operatorname{card}(Y) = \alpha$. We say α is not real-valued measurable if for every countably additive measure μ on the power set of Y with $\mu(\{y\}) = 0$ for all $y \in Y$, $\mu \equiv 0$.

Theorem 5.18 (cf. [**10**, pp. 177–178]). *Let X be a locally compact, paracompact space. Then the following are equivalent*:

(1) $\mathcal{QM}_\sigma(X) = \mathcal{QM}_\tau(X)$;

(2) *if $Y \subseteq X$ is closed and discrete, then* $\operatorname{card}(Y)$ *is not real-valued measurable.*

Proof. Let $X = \cup_{\alpha\in A} X_\alpha$, where each X_α is σ-compact, locally compact and the X_α's are disjoint open subsets of X [**5**, p. 241]. We will need the following lemma:

Lemma 5.19. *With A as above,* $\operatorname{card}(A)$ *is not real-valued measurable if and only if for all closed discrete sets $Y \subseteq X$,* $\operatorname{card}(Y)$ *is not real-valued measurable.*

We leave the proof of this lemma to the reader and proceed with the proof of the theorem.

(1) $\Rightarrow$ (2). Assume $\mathcal{QM}_\sigma(X) = \mathcal{QM}_\tau(X)$. By the lemma it will be enough to show that $\operatorname{card}(A)$ is not real-valued measurable. Let μ be a σ-smooth measure defined on $\mathcal{P}(A)$ with $\mu(\{\alpha\}) = 0$ for each $\alpha \in A$. Choose $x_\alpha \in X_\alpha$ for all α, and let $Y = \{x_\alpha\}_{\alpha\in A}$. For $Z \subseteq Y$ define $\lambda(Z) = \mu(B)$ where $Z = \{x_\alpha\}_{\alpha\in B}$. As stated previously, $\lambda \in \mathcal{M}_\sigma(Y)$. Now $i : Y \hookrightarrow X$ induces a Baire measure $i(\lambda)$ on X [**8**, 2.4]. Then $i(\lambda)$ is τ-smooth and $i(\lambda)(X_\alpha) = 0$ for all α, so $i(\lambda) \equiv 0$. Therefore, $\lambda \equiv 0$,

and so $\mu \equiv 0$.

(2) $\Rightarrow$ (1). Let $\mu \in \mathcal{QM}_\sigma(X)$. We need to show $\mu \in \mathcal{QM}_\tau(X)$.

Claim. $\mu(X) = \sum_{\alpha\in A} \mu(X_\alpha)$.

Define $\lambda : \mathcal{P}(A) \to \mathbf{R}$ by

$$\lambda(B) = \mu\bigg(\bigcup_{\alpha\in B} X_\alpha\bigg).$$

It is easy to show that λ is nonnegative and finite. We will show that λ is countably additive. Suppose $B \subseteq A$, B the disjoint union of sets B_n. Let $D_1 = A\backslash B$, $D_2 = (A\backslash B) \cup B_1$, $D_3 = (A\backslash B) \cup B_1 \cup B_2$, etc. Then $D_n \uparrow A$ so that $\{\cup_{\alpha\in D_n} X_\alpha\} \uparrow X$. By the σ-smoothness of μ,

$$\lim_{n\to\infty} \mu\bigg(\bigcup_{\alpha\in D_n} X_\alpha\bigg) = \mu(X).$$

This implies that

$$\mu\bigg(\bigcup_{\alpha\in A\backslash B} X_\alpha\bigg) + \sum_{i=1}^{\infty} \mu\bigg(\bigcup_{\alpha\in B_i} X_\alpha\bigg) = \mu(X).$$

Therefore

$$\begin{aligned}
\lambda(A\backslash B) + \sum_{i=1}^{\infty} \lambda(B_i) &= \mu\bigg(\bigcup_{\alpha\in A\backslash B} X_\alpha\bigg) + \sum_{i=1}^{\infty} \mu\bigg(\bigcup_{\alpha\in B_i} X_\alpha\bigg) \\
&= \mu(X) \\
&= \mu\bigg(\bigcup_{\alpha\in A\backslash B} X_\alpha\bigg) + \mu\bigg(\bigcup_{\alpha\in B} X_\alpha\bigg) \\
&= \lambda(A\backslash B) + \lambda(B).
\end{aligned}$$

Thus $\sum_{i=1}^{\infty} \lambda(B_i) = \lambda(B)$, so in fact λ is countably additive. Since A is discrete and $\operatorname{card}(A)$ is not real-valued measurable we have $\lambda \in \mathcal{M}_\tau(A)$ (see [**7**, Theorem 2.1]) and $\lambda(A) = \sum_{\alpha\in A} \lambda(\{\alpha\})$, i.e., $\mu(X) = \sum_{\alpha\in A} \mu(X_\alpha)$. The claim is done.

Since $\sum_{\alpha\in A}\mu(X_\alpha)\in\mathbf{R}$ there exists a sequence $\{\alpha_n\}_{n=1}^\infty\subseteq A$ such that

$$\mu(X)=\sum_{n=1}^{\infty}\mu(X_{\alpha_n}). \tag{4}$$

Let $Y=\cup_{n=1}^\infty X_{\alpha_n}$, and let $\lambda=\mu|Y$. Check to see that λ is a σ-smooth quasi-measure on Y. By Corollary 5.17, λ is τ-smooth. By virtue of equation (4), $\mu(Z)=\lambda(Z\cap Y)$ for all $Z\in\mathcal{Z}$. Therefore μ is τ-smooth. $\square$

6. Characterization of smoothness. In this section we refer the reader to a paper by Knowles [**6**] where the ideas for this section originated. Again we will omit proofs which are virtually identical to those of Knowles for ordinary Baire measures.

Let X be a completely regular space and let βX denote the Stone-Čech compactification of X. Consider a quasi-state ρ on $C_b(X)$. Define the function $\bar\rho$ on $C(\beta X)$ by:

$$\bar\rho(\bar f)=\rho(f)$$

where $\bar f$ is the unique extension of f to βX (or, equivalently, f is the restriction of $\bar f$ to X). Since $C_b(X)$ and $C(\beta X)$ are isomorphic as C^*-algebras, $\bar\rho$ is a quasi-state on $C(\beta X)$. By Theorem 4.1 above and Theorem 4.1 of [**1**], $\bar\rho$ induces a Baire quasi-measure, $\bar\mu$, and a Borel quasi-measure, $\bar\nu$, on βX. As we will see later in this section, $\bar\mu$ is the restriction of $\bar\nu$ to the zero sets and cozero sets of βX. Our present objective is to investigate the smoothness of μ by examining the measurability of X with respect to $\bar\mu$ and $\bar\nu$ (regarding X as a subset of βX).

Theorem 6.1. *The following are equivalent for a quasi-measure* μ:

(i) μ *is* σ*-smooth*;

(ii) $\bar\mu^*(X)=\bar\mu(\beta X)$;

(iii) $\bar\mu(Z)=0$ *for all* $Z\subseteq\beta X\backslash X$, Z *a zero set of* βX.

Theorem 6.2. *The following are equivalent for a quasi-measure* μ:

(i) *μ is τ-smooth*;

(ii) $\bar{\nu}^*(X) = \bar{\nu}(\beta X)$;

(iii) *every compact set K in βX disjoint from X is $\bar{\nu}$-null.*

In order to provide our characterization for tight quasi-measures we need the following proposition, which is due to Wheeler.

Proposition 6.3 [**12**, 2.3]. *Let X be a normal space. If $\mu : \mathcal{Z} \to \mathbf{R}^+$ is a Baire quasi-measure on X, then $\nu(F) = \inf\{\mu(Z) : F \subseteq Z\}$ is a Borel quasi-measure on X such that $\nu|\mathcal{Z} = \mu$. Moreover, ν is the unique extension of μ to a Borel quasi-measure on X.*

Theorem 6.4. *A quasi-measure μ is tight if and only if $\bar{\nu}_*(X) = \bar{\nu}(\beta X)$.*

Proof. $\Leftarrow$. Suppose $\bar{\nu}_*(X) = \bar{\nu}(\beta X)$. Let $\{f_\alpha\}$ be a net in $C_b(X)$, $\|f_\alpha\| \leq 1$ which converges to zero uniformly on compact sets. Let $\varepsilon > 0$ be arbitrary. By hypothesis there exists a compact set $K \subseteq X$ such that $\bar{\nu}(K) > \bar{\nu}(\beta X) - \varepsilon/2 = 1 - \varepsilon/2$. Notice that $\bar{\nu}(K) + \bar{\nu}(\beta X \backslash K) = 1$ so

$$\bar{\nu}(\beta X \backslash K) = 1 - \bar{\nu}(K) < 1 - (1 - \varepsilon/2) = \varepsilon/2.$$

Consider $\bar{f}_\alpha \in C(\beta X)$, the unique extension of f_α to βX, and let

$$U_\alpha = \{x \in \beta X : \bar{f}_\alpha(x) < \varepsilon/2\}.$$

Since $f_\alpha \to 0$ uniformly on K, there exists an α_0 such that $\bar{f}_\alpha(K) < \varepsilon/2$ for all $\alpha > \alpha_0$. Therefore $K \subseteq U_\alpha \Rightarrow \beta X \backslash K \supseteq \beta X \backslash U_\alpha$. Since $\bar{\nu}(\beta X \backslash K) < \varepsilon/2$ we have $\bar{\nu}(\beta X \backslash U_\alpha) < \varepsilon/2$ for all $\alpha > \alpha_0$. So, for any $\alpha > \alpha_0$,

$$\begin{aligned}
\rho(f_\alpha) &= \bar{\rho}(\bar{f}_\alpha) \\
&= \int_0^1 (\bar{f}_\alpha)^\wedge(t)\,dt \\
&= \int_0^{\varepsilon/2} (\bar{f}_\alpha)^\wedge(t)\,dt + \int_{\varepsilon/2}^1 (\bar{f}_\alpha)^\wedge(t)\,dt \\
&\le \varepsilon/2 + (1-\varepsilon/2)\cdot(\bar{f}_\alpha)^\wedge(\varepsilon/2) \\
&= \varepsilon/2 + (1-\varepsilon/2)\cdot\bar{\nu}(\beta X\backslash U_\alpha) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, $\rho(f_\alpha) \to 0$ and so, by Theorem 5.8, μ is tight.

$\Rightarrow$. Suppose μ is tight. It suffices to show that, given $\varepsilon > 0$, there exists an open set G with $G \supseteq \beta X\backslash X$ and $\bar{\nu}(G) < \varepsilon$ (since then $\bar{\nu}^*(\beta X\backslash X) < \varepsilon$, so $\bar{\nu}_*(X) > 1-\varepsilon$). Let $\varepsilon > 0$ be given. Since μ is tight there exists a compact set $K \subseteq X$ such that $\mu_*(X\backslash K) < \varepsilon$.

Now we have $\bar{\mu}^*(K) = \inf(\{\bar{\rho}(\bar{f}) : K \prec \bar{f}\} = \inf\{\rho(f) : K \prec f\} = \mu^*(K)$, using Lemma 4.2 of [**1**] and Proposition 6.3 and Theorem 4.1(b) from above. It follows that

$$\bar{\mu}_*(\beta X\backslash K) = \mu_*(X\backslash K) < \varepsilon.$$

Since K is also compact in βX, $\beta X\backslash X \subseteq \beta X\backslash K = G$ where G is open in βX. Finally, since $\bar{\nu}$ is an extension of $\bar{\mu}$ on the compact Hausdorff space βX (compact Hausdorff implies normal), $\bar{\nu}(G) = \bar{\mu}_*(G) < \varepsilon$ by Proposition 6.3. □

7. Examples of smoothness properties for quasi-measures. In this section we will furnish examples to demonstrate the smoothness properties introduced in the preceding sections.

We first sketch an example due to Aarnes of a quasi-measure which is not the restriction of a regular Borel measure. We will refer back to this example throughout this section.

Example 7.1 [**1**, Section 6]. Let $X = S$ be the closed unit square in $\mathbf{R}^2$, equipped with the relative topology. Fix a point $p \in X$, say

$p = (1/2, 1/2)$. A subset D of X is *co-connected* if $X \backslash D$ is connected. A subset D of X is *solid* if it is connected and co-connected. For a solid set $D \subseteq X$, let $\mu_0(D) = 1$ if either (1) $\partial X \subseteq D$ or (2) $D \cap \partial X \neq \varnothing$, $D^c \cap \partial X \neq \varnothing$, and $p \in D$. Otherwise set $\mu_0(D) = 0$.

According to Theorem 5.1 of [**3**], μ_0 has a unique extension to a quasi-measure μ on S which only takes the values 0 and 1. Aarnes showed that μ is not subadditive so it cannot be the restriction to $\mathcal{A}$ of an ordinary Baire measure. It can also be shown that μ is minimal in the sense that no positive, nonzero, finitely additive measure is dominated by it.

The next example demonstrates that the definition of σ-smoothness for quasi-measures must only be stated in terms of decreasing sequences of zero sets and not arbitrary sets from $\mathcal{A}$.

Example 7.2. Let μ be the Aarnes quasi-measure on the unit square S in $\mathbf{R}^2$ (Example 7.1) with fixed point $(1/2, 1/2)$. Construct a sequence of zero sets $\{Z_n\}$, each with two connected components, which increase to S as follows:

Let $R_n = \{(x, y) : 1/2 - 1/n < x < 1/2 + 1/n; 1/2 - 1/n < y < 1\}$, $Z_{n,1} = (R_n)^c \backslash \{(x, 1) : 1/2 < x < 1/2 + 1/n\}$ and $Z_{n,2} = \{(1/2, y) : 1/2 \leq y \leq 1 - 1/n\}$. Then set $Z_n = Z_{n,1} \cup Z_{n,2}$.

Now $Z_n \uparrow S$ but $\mu(Z_n) = 0$ for all n.

Claim. *If $\{Z_n\}$ is a sequence of solid zero sets such that $Z_n \uparrow S$, then $\mu(Z_n) \to 1$.*

Proof. Suppose Z_n is a sequence of solid zero sets such that $Z_n \uparrow S$ but $\mu(Z_n) = 0$ for all n. The point $(1/2, 1/2)$ must be in Z_k for some k since $Z_n \uparrow S$. However, $Z_{k'}$ cannot contain any points on the boundary of S for any $k' > k$ since $\mu(Z_{k'}) = 0$. This contradicts $Z_n \uparrow S$. The claim is done. □

This claim illustrates the fact that the sequence of zero sets constructed above is minimal in the sense that we cannot construct a counterexample with solid zero sets.

This next example provides a Baire quasi-measure on $\mathbf{R}^2$ which is not the restriction of a positive Baire measure and also is not σ-smooth.

Example 7.3. Let S_n denote the solid square in $\mathbf{R}^2$ which is centered at the origin and contains the points $(n, 0)$ and $(0, n)$ on its boundary. Define μ_n on S_n as Aarnes (see Example 7.1) does using the point $(0, 0)$, for $n = 1, 2, \dots$. By Theorem 4.5, μ_n can be considered as a quasi-measure on $\mathbf{R}^2$. Let ρ_n be the quasi-state corresponding to μ_n, and let $\bar{\rho}_n$ be the quasi-state on $C(\beta\mathbf{R}^2)$ defined by

$$\bar{\rho}_n(\bar{f}) = \rho_n(f) \quad \text{for all } \bar{f} \in C(\beta\mathbf{R}^2)$$

where f is the restriction of $\bar{f}$ to $\mathbf{R}^2$. Since the set of all quasi-states on $C(\beta\mathbf{R}^2)$ is compact in the topology of pointwise convergence on $C(\beta\mathbf{R}^2)$ (see [**2**]), the sequence $\bar{\rho}_n$ has a cluster point $\bar{\rho}_0$. Let ρ_0 be the quasi-state on $C_b(\mathbf{R}^2)$ defined by

$$\rho_0(f) = \bar{\rho}_0(\bar{f}) \quad \text{for all } f \in C_b(\mathbf{R}^2)$$

where $\bar{f}$ is the unique extension of f to $\beta\mathbf{R}^2$. It can be shown that ρ_0 is a quasi-state on $C_b(\mathbf{R}^2)$ which is not a state and that ρ_0 is not σ-smooth.

In the following example we create a σ-smooth Baire quasi-measure which is not the restriction of a positive Baire measure and also is not τ-smooth.

Example 7.4. Let L denote the extended long line (cf. [**9**]) and let $X = L \times L \backslash \{(\omega_1, \omega_1)\}$ with the relative product topology. Here ω_1 denotes the first uncountable ordinal. It can be shown that $\beta X = L \times L$.

Let S_α denote the square with vertices $(0, 0)$, $(0, \alpha)$, $(\alpha, 0)$, (α, α) and define μ_α on S_α in the same manner as Aarnes using the fixed point $(1/2, 1/2)$. This is possible as in Example 7.3. Let ρ_α be the quasi-state corresponding to μ_α, and let $\bar{\rho}_\alpha$ be the quasi-state on $C(\beta X)$ defined by

$$\bar{\rho}_\alpha(\bar{f}) = \rho_\alpha(f) \quad \text{for all } \bar{f} \in C(\beta X)$$

where f is the restriction of $\bar{f}$ to X.

Since the set of all quasi-states on $C(\beta X)$ is compact in the topology of pointwise convergence on $C(\beta X)$, there exists a cluster point of the net $\{\bar{\rho}_\alpha\}$, say $\bar{\rho}_0$. Now let ρ_0 be the quasi-state defined by

$$\rho_0(f) = \bar{\rho}_0(\bar{f}) \quad \text{for all } f \in C_b(X)$$

where $\bar{f}$ is the unique extension of f to βX. It can be shown that ρ_0 is a quasi-state on $C_b(X)$ which is not a state and that ρ_0 is (strongly) σ-smooth but not τ-smooth.

We now provide an example of a τ-smooth Baire quasi-measure γ which is not tight (γ is also not subadditive so it cannot be the restriction of an ordinary Baire measure). This example involves a product quasi-measure, so we refer the reader to Chapter 5 of [**4**] for details.

Example 7.5. Let X be the Sorgenfrey line, and let $Y = [0,1]\times[0,1]$. Let λ denote Lebesgue measure on X, and let μ denote the Aarnes quasi-measure on Y (cf. Example 7.1). For each set $E \subseteq X \times Y$ and $x \in X$, let $E_x = \{y \in Y : (x,y) \in E\}$. Define γ on $X \times Y$ as follows:

$$\gamma(E) = \int_X \mu(E_x)\, d\lambda$$

for each set $E \subseteq X \times Y$ which is either open or closed. Then γ is a τ-smooth Baire quasi-measure on $X \times Y$ which is not tight. See Example 5.3.4 of [**4**] for details.

Our last example in this section demonstrates that the Aarnes quasi-measure μ on the unit square S (Example 7.1) is minimal in the sense that no positive, nonzero, finitely additive measure is dominated by it.

Example 7.6. Let μ denote the Aarnes quasi-measure on the unit square S with the fixed point $(1/2, 1/2)$ (Example 7.1).

Claim. *If λ is a finitely additive measure such that $0 \leq \lambda \leq \mu$, then $\lambda \equiv 0$.*

Proof. Let λ be a finitely additive measure with $0 < \lambda \leq \mu$. Since S is compact λ is necessarily τ-smooth (even tight). Therefore, supp$(\lambda) \neq \varnothing$ [**7**, 2.1]. Let $x \in \operatorname{supp}(\lambda)$; then choose an open set $G \ni x$ such that $\mu(G) = 0$, which can be done by the nature of μ. This implies that $\lambda(G) = 0$, contradicting $x \in \operatorname{supp}(\lambda)$. Thus $\lambda \equiv 0$. □

REFERENCES

1. J. Aarnes, *Quasi-states and quasi-measures*, Advances in Math. **86** (1991), 41–67.

2. ———, *Pure quasi-states and extremal quasi-measures*, Math. Ann. **295** (1993), 575–588.

3. ———, *Construction of non-subadditive measures and discretization of Borel measures*, Mathematics Preprint No. 7, University of Trondheim, Norway, 1993.

4. J. Boardman, *Quasi-measures on completely regular spaces*, Dissertation, Northern Illinois University, 1994.

5. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

6. J. Knowles, *Measures on topological spaces*, Proc. London Math. Soc. **17** (1967), 139–156.

7. W. Moran, *The additivity of measures on completely regular spaces*, J. London Math. Soc. **43** (1968), 633–639.

8. ———, *Measures and mappings on topological spaces*, Proc. London Math. Soc. **19** (1969), 493–508.

9. L.A. Steen and J.A. Seebach, Jr., *Counterexamples in topology*, Springer-Verlag, New York, 1978.

10. V.S. Varadarajan, *Measures on topological spaces*, Mat. Sb. **55**, 33–100 (Russian); Amer. Math. Soc. Transl. Ser. 2**48** (1965), 161–228.

11. R. Wheeler, *A survey of Baire measures and strict topologies*, Expositiones Math. **2** (1983), 97–190.

12. ———, *Quasi-measures and dimension theory*, Topology Appl., to appear.

Morehead State University, 150 University Blvd., U.P.O. Box 592, Morehead, KY 40351