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ON NEW MAJORIZATION THEOREMS

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ABSTRACT. The subject of majorization is treated extensively, see, for instance, [**1, 5**] and [**4**] and their references.

In 1947, L. Fuchs gave a weighted generalization of the well-known majorization theorem for convex functions and two sequences monotonic in the same sense, see [**4**, p. 419] or [**6**, p. 323]. For other related results, see [**4**, pp. 417–420] or [**6**, pp. 323–332].

In this paper we shall give related results in the case when only one sequence is monotonic. Moreover, while in Fuchs results we have real weights, in our results we need positive weights. These results form extensions of theorems of Kolumban and Mocanu [**2**], Toader [**8**] and Maligranda, Pečarić and Persson [**3**], as well as results from [**5**, pp. 337–338; 350] and [**7**, pp. 9 and 145–148].

1. Main results.

Theorem 1. *Let g be a strictly increasing function from (a,b) to (c,d), and let $f\circ g^{-1}$ be a concave function on $[c,d]$. Let the vectors $\mathbf{x}$ and $\mathbf{y}$ with elements from (a,b) satisfy*

$$\sum_{i=1}^{k} w_i g(x_i) \geq \sum_{i=1}^{k} w_i g(y_i), \quad k=1,\dots,n. \tag{1.1}$$

(a) *If*

(a_1) *f is decreasing*

(a_2) *the vector $\mathbf{y}$ is decreasing, then*

$$\sum_{i=1}^{k} w_i f(x_i) \leq \sum_{i=1}^{k} w_i f(y_i), \quad k=1,\dots,n. \tag{1.2}$$

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(b) *If*

$$\begin{aligned} &(b_1) \qquad \sum_{i=1}^{n} w_i g(x_i) = \sum_{i=1}^{n} w_i g(y_i), \quad \textit{and} \\ &(b_2) \qquad \textit{the vector } y \textit{ is decreasing, then} \end{aligned} \tag{1.3}$$

$$\sum_{i=1}^{n} w_i f(x_i) \leq \sum_{i=1}^{n} w_i f(y_i). \tag{1.4}$$

(c) *If*

$$\begin{aligned} &(c_1) \qquad f(x) \textit{is increasing} \\ &(c_2) \qquad \textit{the vector } \mathbf{x} \textit{ is increasing, then} \end{aligned}$$

$$\sum_{i=1}^{k} w_i f(x_i) \geq \sum_{i=1}^{k} w_i f(y_i), \quad k = 1, \dots, n. \tag{1.5}$$

(d) *If*

$$\begin{aligned} &(d_1) \qquad \sum_{i=1}^{n} w_i g(x_i) \leq \sum_{i=1}^{n} w_i g(y_i), \\ &(d_2) \qquad \textit{the vector } \mathbf{x} \textit{ is increasing, then} \end{aligned}$$

$$\sum_{i=1}^{n} w_i f(x_i) \geq \sum_{i=1}^{n} w_i f(y_i). \tag{1.6}$$

Proof. Without loss of generality, it is easy to see that it is enough to prove the special case $g(x) = x$ by substitution

$$g(x_i) = a_i, \qquad g(y_i) = b_i, \qquad f(x_i) = f \circ g^{-1}(a_i) = \bar{f}(a_i). \tag{1.7}$$

We will prove case (a). The other proofs are similar.

Because of the concavity of $f(x)$

$$\bar{f}(u) - \bar{f}(v) \geq \bar{f}' + (u)(u - v)$$

hence

$$\begin{aligned}\sum_{i=1}^{k} w_i(\bar{f}(b_i) - \bar{f}(a_i)) &\geq \sum_{i=1}^{k} w_i(b_i - a_i)\bar{f}'_+(b_i) \\ &= \bar{f}'_+(b_k)\bigg(\sum_{i=1}^{k} w_i(b_i - a_i) \\ &\quad - \sum_{i=1}^{k-1}\bigg(\sum_{m=1}^{i} w_m(b_m - a_m)(\bar{f}'_+(b_{i+1}) - \bar{f}'_+(b_i))\bigg)\bigg) \\ &\geq 0.\end{aligned}$$

The last inequality follows from (a_1), (a_2), (1.1), (1.7) and the concavity of $\bar{f}$, hence case (a) is proven. □

Corollary. *We get a special case of Theorem* 1 *for* $f(x) = x^p$, $g(x) = x^q$, $x > 0$ *and the positive vectors* $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$. *Part of it appears in* [**8**, Theorem 3].

Remark. In Theorem 1 cases (b) and (d), we may replace the given condition (1.1) by the condition $x_1/y_1 \geq x_2/y_{x2} \geq \cdots \geq x_n/y_{xn}$ because it leads to (1.1).

Theorem 2. *Let* f *and* g *be integrable functions on* $[a, b]$, *and let* w *be a positive integrable function. Suppose that* ψ *is a strictly increasing function and* $\varphi \circ \psi^{-1}$ *is concave. Suppose that* f *is decreasing and that*

$$\int_x^b \psi(f(t))w(t)\,dt \geq \int_x^b \psi(g(t))w(t)\,dt, \quad \forall\, x \in [a, b].$$

(a) *If*

$$\int_a^b \psi(f(t))w(t)\,dt = \int_a^b \psi(g(t))w(t)\,dt,$$

then

$$\int_a^b \varphi(f(t))w(t)\,dt \geq \int_a^b \varphi(g(t))w(t)\,dt,$$

if g *is increasing, the inverse inequality holds.*

(b) *If* $\varphi \circ \psi^{-1}$ *is increasing, then*

$$\int_x^b \varphi(f(t))w(t)\,dt \geq \int_x^b \varphi(g(t))w(t)\,dt, \quad b \geq x \geq a.$$

Proof. As in Theorem 1, with no loss of generality, it is sufficient to prove Theorem 2 in case that $\Psi(t) = t$.

We will prove the theorem for $\varphi \in C'[c,d]$; the general case follows from the pointwise approximation of φ by a smooth function. We will prove case (b). The other proofs are similar.

Since φ is concave on $[c,d]$, it follows that

$$\varphi(u_1) - \varphi(u_2) \geq \varphi'(u_1)(u_1 - u_2).$$

If we set

$$F(x) = \int_x^b (f(t) - g(t))w(t)\,dt,$$

then $F(x) \geq 0$ for all $x \in [a,b]$.

Let $b \geq x \geq a$; then

$$\begin{aligned}\int_x^b (\varphi(f(t) - \varphi(g(t))w(t)\,dt &\geq \int_x^b \varphi'(f(t))(f(t) - g(t))w(t)\,dt \\ &= -\int_x^b \varphi'(f(t))\,dF(t) \\ &= [-\varphi'(f(t))F(t)]_x^b + \int_x^b F(t)d(\varphi'(f(t))) \\ &= \varphi'(f(x))\int_x^b (f(t) - g(t))w(t)\,dt \\ &\quad + \int_x^b F(t)\varphi''(t)f'(t)\,dt \\ &\geq 0.\end{aligned}$$

The last inequality follows from $\varphi \circ \psi^{-1}$ being concave increasing and f being decreasing. □

2. Some applications. In [**3**] the following lemma was proved:

Lemma 1 [**3**]. *Let $v(x)$ be a positive integrable function. If $h(x)$ is an increasing function on (a,b), then*

$$\int_a^x h(t)v(t)\,dt \int_a^b v(t)\,dt \leq \int_a^b h(t)v(t)\,dt \int_a^x v(t)\,dt$$

for all $x \in [a,b]$.

If $h(x)$ is a decreasing function on (a,b), then the reverse inequality holds.

Also in [**3**] the following lemma, which is another version of Theorem 2, was proved.

Lemma 2 [**3**]. *Let $w > 0$, f and g be integrable functions from $[a,b]$ to $[c,d]$. Suppose that $\varphi : [c,d] \to R$ is a convex function, and let*

$$\int_a^x f(t)w(t)\,dt \leq \int_a^x g(t)w(t)\,dt \quad \text{for all } x \in [a,b],$$

and

$$\int_a^b f(t)w(t)\,dt = \int_a^b g(t)w(t)\,dt.$$

(i) *If f is decreasing on $[a,b]$, then*

$$\int_a^b \varphi(f(t))w(t)\,dt \leq \int_a^b \varphi(g(t))w(t)\,dt.$$

(ii) *If g is increasing on $[a,b]$, then*

$$\int_a^b \varphi(g(t))w(t)\,dt \leq \int_a^b \varphi(f(t))w(t)\,dt.$$

From both Lemma 1 and Lemma 2 we get the following

Theorem 3. *Let f be an increasing function on $(0,1)$ and f/g a decreasing function on $(0,1)$. Let $w>0$, and let $g>0$ or $f>0$. Also let fw and gw be integrable functions on $[0,1]$.*

$$Z=\int_0^1 fw\,dt\Big/\int_0^1 gw\,dt\geq 0.$$

Let φ be a convex function; then, for every $k>0$,

$$\int_0^1 \varphi(kf(t))w(t)\,dt\leq\int_0^1 \varphi(kZg)w(t)\,dt.$$

Proof. Let $g(x)>0$. Using Lemma 1 [**3**] with

$$v(t)=g(t)w(t),\qquad h(t)=f/g,$$

we obtain

$$\int_0^x g(t)w(t)\,dt\int_0^1 f(t)w(t)\,dt\leq\int_0^x f(t)w(t)\,dt\int_0^1 g(t)w(t)\,dt$$

and, according to the definition of Z,

$$\int_0^x kZg(t)w(t)\,dt\leq\int_0^x kf(t)w(t)\,dt$$

and therefore, as $f(t)$ is increasing we get, as a result of Theorem 2, that

$$\int_0^1 \varphi(kZg(t))w(t)\,dt\geq\int_0^1 \varphi(kf(t))w(t)\,dt$$

the case where $f>0$ is proved by defining $h=g/f$, $\nu=f\cdot w$, where h is increasing. □

Examples. Let $h(x)$ be increasing on $(0,1]$, $w>0$, and

$$\int_0^1 th(t)w(t)\,dt\geq 0.$$

Define

$$Z = \int_0^1 th(t)w(t)\,dt \Big/ \int_0^1 tw(t)\,dt;$$

then, if φ is convex,

$$\int_0^1 \varphi(Zt)w(t)\,dt \le \int_0^1 \varphi(th(t))w(t)\,dt,$$

the inequality obtained by defining $f = t$, $g = th(t)$. This last inequality is satisfied especially by all convex functions g on $[0,1]$ satisfying $0 \ge g(0)$

$$\int_0^1 g(t)w(t)\,dt \ge 0,$$

and because $g(t)/t$ is increasing on $(0,1]$.

Moreover, if $h(t) \ge 0$, then $1 \ge \alpha \ge \beta \ge 0$,

$$\int_0^1 \varphi(Zt\alpha + h(t)t(1-\alpha))w(t)\,dt \le \int_0^1 \varphi(Zt\beta + h(t)t(1-\beta))w(t)\,dt.$$

As for positive concave functions $f(x)$, $f(x)/x$ is decreasing. Therefore, we get for increasing positive concave functions f and for convex functions φ, the following inequality:

$$\int_0^1 \varphi(Zt)w(t)\,dt \ge \int_0^1 \varphi(f(t))w(t)\,dt,$$

where

$$Z = \int_0^1 f(t)w(t)\,dt \Big/ \int_0^1 tw(t)\,dt.$$

In this case we get Theorem 1 [**3**], which is an extension of Favard's theorem quoted in [**3**] as follows:

Theorem. *Let f be a positive, continuous concave function on $[a,b]$, and let φ be a convex function. Define*

$$\bar{f} = \frac{1}{b-a}\int_a^b f(t)\,dt,$$

then

$$\frac{1}{b-a}\int_a^b \varphi(f(t))\,dt \le \int_0^1 \varphi(2\bar{f}s)\,ds.$$

Of course, there are non concave functions f that satisfy the same inequality. For instance,

$$f(x) = (1+x^p)^{1/p}, \qquad p>0, \quad x\ge 0$$

because for such f, $f(x)/x$ is decreasing.

The extension of Theorem 3 and Theorem 2 [**3**] is as follows:

Theorem 3*. *Let f be increasing on (a,b) and f/g be a decreasing function on (a,b). Let $w>0$, and let $g>0$. Suppose that fw and gw are integrable functions on $[a,b]$, ψ is a strictly increasing function and $\varphi\circ\psi^{-1}$ is concave.*

Let Z be such that

$$\int_0^1 \psi(Zg(t))w(t)\,dt = \int_0^1 \psi(f(t))w(t)\,dt \ge 0.$$

Then

$$\int_0^1 \varphi(f(t))w(t)\,dt \ge \int_0^1 \varphi(Zg(t))w(t)\,dt \ge 0.$$

Proof. Because f/g is decreasing and ψ is strictly increasing, and because

$$\int_0^1 \psi(f(t))w(t)\,dt = \int_0^1 \psi(Zg(t))w(t)\,dt,$$

there is an $x_0 \in [0,1]$ such that $f(x)/g(x) \ge 1$, $0\le x\le x_0$ and $f(x)/g(x)\le 1$, $x_0\le x\le 1$, hence

$$\int_x^1 \psi(Zg(t))w(t)\,dt \le \int_x^1 \psi(f(t))w(t)\,dt$$

and as $\varphi\circ\psi^{-1}$ is concave and f is increasing, we get from Theorem 2

$$\int_0^1 \varphi(Zg(t))w(t)\,dt \le \int_0^1 \varphi(f(t))w(t)\,dt. \qquad \square$$

This is an extension of Theorem 2 [**3**] for $g(t) = t$ where $f(t)$ is a positive concave function. Theorem 2 [**3**] is an extension of Berwald's theorem, which is quoted in [**3**] as follows:

Theorem. *Let f be a positive continuous concave function on $[a,b]$. Let ψ be a strictly increasing function on $[a,\propto)$. Assume that $\varphi\circ\psi^{-1}$ is a convex function on $[a,\propto)$. If*

$$\int_0^1 \psi(Zt)\,dt = \frac{1}{b-a}\int_a^b \psi(f(t))\,dt,$$

then

$$\int_0^1 \varphi(Zt)\,dt \geq \frac{1}{b-a}\int_a^b \varphi(f(t))\,dt.$$

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