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A CLASS OF CAUCHY PROBLEMS THAT INVOLVE FACTORABLE DIFFERENTIAL OPERATORS

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ABSTRACT. Let $x = (x_1, \dots, x_p)$, $D = (D_1, \dots, D_p)$, and let $P(D)$ be a linear partial differential operator with constant coefficients that can be factored over the complex field into a product of linear combinations of the D_j. Using the simple quasi inner product (qip), we obtain representations of solutions of a class of Cauchy problems that includes $\partial^n w(x,t)/\partial t^n = P(D)w(x,t)$, $\partial^j w(x,t)/\partial t^j|_{t=0} = \phi_j(x)$ for $j = 0, 1, \dots, n-1$ as multiple integrals of complex translations of the data functions. The factor switching property of the qip plays a central role in constructing these representations and imposing smoothness restrictions on the data. Examples are given to illustrate the flexibility that the qip permits in altering solution forms to fit in with the data or in determining optimal growth conditions for entire data.

1. Introduction. Let p be a positive integer with $p \geq 2$, let $x = (x_1, x_2, \dots, x_p)$, and let $D = (D_1, D_2, \dots, D_p)$ in which $D_j f(x) = \partial f(x)/\partial x_j$. Next, let $P(D)$ be a partial differential operator with constant coefficients which can be factored over the complex field into a product of linear combinations of the D_j. We will be concerned with a class of higher order Cauchy problems that includes the following

$$(1.1)\qquad \begin{aligned} &\frac{\partial^n}{\partial t^n} w(x,t) = P(D)w(x,t),\\ &\frac{\partial^j}{\partial t^j} w(x,0) = \phi_j(x),\\ &\qquad j = 0, 1, \dots, n-1 \end{aligned}$$

as well as ones in which the underlying equation is a higher order generalization of the Euler Poisson Darboux equation. The primary objectives of this paper are: (a) to develop representations of solutions of these problems as multiple integrals of complex translations of the

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data functions and (b) to determine optimal analyticity conditions on the data functions for which these representations are valid. The types of series that define the formal solution operators for the problems considered are closely tied to the ${}_1F_n$ hypergeometric functions. We will use the *simple quasi inner product* (qip), with binary symbol $\circ$, for constructing these integral representations from the formal solution operators. At the heart of this is a *factor switching property* of the qip which assists in obtaining bounds on solutions and specifying analyticity requirements on the $\phi_j(x)$ needed for the integral solution forms. This is particularly true when the $\phi_j(x)$ must be entire with growth to be determined. The qip is essentially a function theoretic tool (see [**1, 14**] for a general background).

Bounds on solutions obtained in [**6**] permitted developing expansion theorems for solutions of problems of type (1.1) when $n = 2$ and $p = 1$ in terms of solution sets corresponding to polynomial data. It would be useful to investigate expansion results for the cases $p \geq 2$ and $n \geq 3$ in order to extend work carried out, using other approaches, in [**9, 10, 18** and **19**]. Because of the numerous details required for such an expansion theory, we shall defer this study to a future paper.

A general approach for obtaining complex integral representations of solutions of problems of type (1.1) without the stated factorability assumption on $P(D)$ was given in [**3**]. It made use of the *generalized quasi inner product* (gqip), with binary operation denoted by ${}_r\circ_s$ where r and s are relatively prime integers with $r > 1$ (this gqip does not have a convenient factor switching property). The problem was first reduced to one of solving a set of "heat" problems of the form

$$(1.2)\qquad \begin{aligned} &\frac{\partial}{\partial t} H_j(x,t) = P(D)H_j(x,t), \quad t > 0; \\ &H_j(x,0) = \phi_j(x), \quad j = 0, 1, \dots, n-1, \end{aligned}$$

and then applying appropriate complex integral transformations to these $H_j(x,t)$. In solving the "heat" problems (1.2), the differential operator $P(D)$ was expressed as $\sum_{m=1}^{k} P_m(D)$ where each $P_m(D)$ was a constant times a product of powers of the D_j. The formal solution for the $H_j(x,t)$ was given by $H_j(x,t) = e^{t\sum_{m=1}^{k} P_m(D)}\phi_j(x)$ and the generalized quasi inner product was applied term by term to each of the operators $e^{tP_m(D)}$ acting on an appropriate data or subsequently constructed function to finally yield $H_j(x,t)$. *As a general rule, this method required that the data functions* $\phi_j(x)$ *be entire of growth* (ρ,τ) with $\rho < 1$ (*see Section* 2) *even when analyticity would have sufficed for restricted values of* t. *One need only examine the classical heat problem to see that the gqip method does not lead to optimal growth conditions on the data* [**19**]. *Nor does it lead to the standard d'Alembert type solution form for the classical wave problem.* Qips have been used in constructing ascent type formulas for solutions of partial differential equations [**4**].

The method of switching a pair of commutative "derivative" operators in exponential functions appearing in quasi inner products was used in [**6**] to solve several second order Cauchy problems, including the Yukawa and the Helmholtz problems. The approach used there suggests that we make the following factorability assumption on the differential operator $P(D)$ in (1.1), namely

$$P(D) = \prod_{m=1}^{l} (a_m^1 D_1 + a_m^2 D_2 + \cdots + a_m^p D_p + a_m^{p+1})^{r_m} \tag{1.3}$$

where the a_m^j are complex constants for $1 \le j \le p+1$, $1 \le m \le l$ and the r_m are positive integers (in most cases of interest, we have the $a_m^{p+1} = 0$). An example of one such problem with $n = 4$ and $P(D) = D_1^4$ was considered in section 8 of [**6**]. Depending upon the choice of n and the factors of $P(D)$, the problem (1.1) may be solvable or nonsolvable for the given data function(s). By replacing the data by analytic or entire functions, a previously nonsolvable problem can, nevertheless, have a solution. In those cases when (1.1) is solvable and has a fundamental solution, the derivative operators D_j may be replaced by the generators of commutative group operators in some Banach space along with the corresponding replacements of the data functions by elements in an appropriate dense subspace of that Banach space. The solution of (1.1) then can be used to infer a solution of this abstract problem.

A formal approach to solving (1.1) is to first regard $P(D)$ as a constant and then express the solution of the problem formally as a suitable linear combination of solution operator series that involve $P(D)$ acting upon the data functions $\phi_j(x)$. Various solution operator series associated with (1.1) can be expressed in terms of the hypergeometric function operators ${}_0F_{n-1}(\_;\lambda_1,\lambda_2,\dots,\lambda_{n-1};TP(D))$ in which T is a simple function of t (see Section 6) and where the λ_i are constants determined by parameters appearing in the partial differential equation in (1.1). For the general class of problems considered in this paper (and including (1.1)), the formal series solution operators can be expressed as ${}_1F_n$ type hypergeometric operator. In [**6**] we made use of a function $F_{b,c}$ of an operator. By making repeated applications of the qip, switching first order derivative operator factors in the exponentials appearing in these qips and replacing exponentials of higher powers of first order operators by a multiple integral involving the first power of that operator, we can reduce the problem of constructing solutions of (1.1) to successive applications of g_a and $F_{b,c}$ type operators to the data.

In Section 2, we recall those notions from quasi inner products, groups and entire functions that will be needed in the ensuing developments. We also summarize a number of results from [**3**] including the definitions of the functions g_a and $F_{b,c}$ along with their integral forms. Of particular importance will be the property alluded to above on moving derivative factors around in exponential functions appearing in quasi inner products. A number of reduction formulas for exponentials involving products and powers of operators will be obtained. Section 3 will be concerned with the applications of qips to lower order Cauchy problems. The examples will illustrate the switching of derivative factors in solving the classical heat equation in two space variables. We also illustrate how switching constants in a qip can alter the solution formula for a problem along with the data requirements. Finally, for a generalized wave problem of type (1.1), i.e., $n = 2$, we show how the solution obtained by qips can be constructed by means of a set of transmutations. A discussion of the ${}_1F_n$ type hypergeometric solution operators and their qip reductions to lower order operators will be carried out in Section 4. The results will play a key role in solving higher order problems. Some preliminaries on analyticity requirements on the data will be considered in Section 5. To simplify the writing of integral solution formulas, we introduce a vector notation for variables of integration. Finally, in Section 6, we construct integral representations for a pair of higher order Cauchy problems. The first of these has the form (1.1) while the second is a third order Euler Poisson Darboux type problem. Included in this treatment is the imposing of restrictions on the underlying data functions.

2. Basic background. For the convenience of the reader, we summarize the basic notions on quasi inner products, their properties, and notions related to them. The exponential of an operator is central to the qip method by (i) defining a complex translation on a function or (ii) defining the solution of some generalized initial value heat problem [**3**]. In case (ii), one must usually select the data on which this exponential operator acts to be entire of appropriate growth as was noted in the introduction. The growth bounds on the data determine bounds on the corresponding "heat" solution as well as possible limitations on t. We use this in applications in Sections 3 and 6. We also note some results on groups of operators associated with abstract versions of problem (1.1).

Let us first suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function of z in some region in the complex plane. We say that $f(z)$ is *entire of growth* (ρ, τ) *with* $\rho > 0$ *and* $\tau \geq 0$ if

$$\limsup_{n\to\infty} (n/e\rho)|a_n|^{\rho/n} = \tau$$

[**2**]. This implies the existence of a positive constant M such that $|f(z)| \leq Me^{\tau|z|^\rho}$ for all complex z. In the work to follow, we will usually be working with functions of several complex variables $z_1, z_2, \dots, z_p$. A function may be analytic in a region in a number of these variables and entire in the others. A simple example of such a function of two variables is given by $f(z_1, z_2) = I_0(z_1)/(4-z_2)^2$ in which $I_0(z_1)$ denotes a modified Bessel function. In this case we have $I_0(z_1) = e^{z_1/2} \circ e^{z_1/2}$ and $|I_0(z_1)| \leq e^{|z_1|}$. If we make the restriction $|z_2| < 4 - \varepsilon$ with $\varepsilon > 0$, then for all z_1 and the restricted values of z_2, we have $|f(z_1, z_2)| \leq e^{|z_1|}/\varepsilon^2$. A number of the data functions to be used later will be required to be entire in all of their arguments and of suitable limited growth in order that improper solution integrals associated with them converge. For most of these switching factors in the qip, it will permit requiring $|f(z_1, z_2, \dots, z_p)| \leq Me^{\tau \sum_{j=1}^{p} |z_j|^\rho}$ with $1 < \rho \leq 2$. The reader is referred to [**3**] for further details on qips and entire solution functions.

Next, let $f_j(z_j)$, $j = 1, 2, 3$, denote three analytic functions of the complex variables z_j in the disks D_j centered at the origin where

$f_j(z_j) = \sum_{n=0}^{\infty} a_n^j z_j^n$. For $z_j \in D_j$, $j = 1, 2$, we recall the definition of the *standard quasi inner product*, namely

$$f_1(z_1) \circ f_2(z_2) = (2\pi)^{-1} \int_0^{2\pi} f_1(z_1 e^{i\theta}) f_2(z_2 e^{-i\theta})\, d\theta \tag{2.1}$$
$$= \sum_{n=0}^{\infty} a_n^1 a_n^2 z_1^n z_2^n.$$

When one or both of the functions composed by $\circ$ depend upon two or more variables, we use underscores to indicate the variables being singled out in the functions in forming the qip. For example, we write

$$f_1(\underline{z}_1, z_2) \circ f_2(z_1, \underline{z}_3) = (2\pi)^{-1} \int_0^{2\pi} f_1(z_1 e^{i\theta}, z_2) f_2(z_1, z_3 e^{-i\theta})\, d\theta. \tag{2.2}$$

It follows from (2.1) that $f_1(z_1) \circ f_2(z_2) = f_2(z_2) \circ f_1(z_1)$ and $f_1(z_1) \circ [f_2(z_2) + f_3(z_2)] = f_1(z_1) \circ f_2(z_2) + f_1(z_1) \circ f_3(z_2)$ provided that $z_2 \in D_3$. Further, if z_j and $Z_j \in D_j$, $j = 1, 2$, or if $Z_1 = cz_1$ and $Z_2 = c^{-1} z_2$ for restricted choices of c, then

$$f_1(z_1) \circ f_2(z_2) = f_1(Z_1) \circ f_2(Z_2). \tag{2.3}$$

This property permits moving a complex factor from the argument in one of the functions to the argument in the other function to reformulate the quasi inner product. In particular, if the $f_j(z_j)$ are *entire functions* of the z_j, this switching formula can be expressed as

$$f_1(\alpha \underline{z}_1) \circ f_2(\beta \underline{z}_2) = f_1(\alpha\beta \underline{z}_1) \circ f_2(\underline{z}_2) = f_1(\underline{z}_1) \circ f_2(\alpha\beta \underline{z}_2) \tag{2.4}$$

where α and β are any pair of complex scalars. In applications of (2.4) to partial differential equations, the scalars α and β are usually replaced, respectively, by partial differential operators $P_1(D)$ and $P_2(D)$ that commute. For such entire functions, a combination of (2.1) and (2.4) leads to the operator identity

$$\sum_{n=0}^{\infty} a_n^1 a_n^2 P_1^n(D) P_2^n(D) t^{2n} \tag{2.5}$$
$$= f_1(P_1(D)\underline{t}) \circ f_2(P_2(D)\underline{t})$$
$$= (2\pi)^{-1} \int_0^{2\pi} f_1(P_1(D) t e^{i\theta}) f_2(P_2(D) t e^{-i\theta})\, d\theta$$
$$= (2\pi)^{-1} \int_0^{2\pi} f_1(P_1(D) P_2(D) t e^{i\theta}) f_2(t e^{-i\theta})\, d\theta.$$

Remark 1. For the Cauchy problems we discuss, the equation and initial conditions are described in terms of real variables. When we state that a data function $\phi(x)$ is analytic or entire of growth (ρ, τ), we understand this to mean that the extended function $\phi(z)$ in complex p space (where $z = (z_1, \dots, z_p)$) is analytic in a region that includes x or is entire of growth (ρ, τ) in some or all of the variables.

A. *Exponential operators.* In most of our applications involving (2.5), the f_j are exponential functions. It is therefore useful to note the essential properties of the operator $e^{tP(D)}$ in those cases when (a) $P(D) = \sum_{j=1}^{p} \alpha_j D_j + \alpha_{p+1}$ and when (b) $P(D) = (\sum_{j=1}^{p} \alpha_j D_j + \alpha_{p+1})^m$ where m is a positive integer ≥ 2. The formula (2.9) employed for part (b) can also be used to treat the case when the factors of $P(D)$ are distinct.

Case a. If t is a real number, the exponential operator e^{tD_1} defines a real translation on the function $\phi(x_1)$ that is defined by the formula

$$e^{tD_1}\phi(x_1) = \phi(x_1 + t), \qquad \phi(x_1) \in C^1. \tag{2.6}$$

Denoting this translated function by $u(x_1, t)$, we note that this function is a solution of the first order Cauchy problem $u_t(x_1, t) = u_{x_1}(x_1, t)$, $u(x_1, 0) = \phi(x_1)$. If the t in (2.6) is taken to a nonreal complex number, then that formula holds only if $\phi(x_1)$ is analytic in a convex region that contains x_1 and $x_1 + t$. Similarly, we have

$$e^{tP(D)}\phi(x_1, x_2, \dots, x_n) = e^{\alpha_{p+1}t}\phi(x_1 + \alpha_1 t, x_2 + \alpha_2 t, \dots, x_n + \alpha_n t). \tag{2.7}$$

If all of the α_j are real, then this formula holds if $\phi \in C^1$ in all of the x_j. If one or more of the α_j are nonreal complex numbers, then (2.7) holds only if ϕ is analytic in the corresponding x_j variables and C^1 in the remaining ones.

Case b. As we will see, the choices $m = 2$ and $m \geq 3$ lead to different integral type formulas of the exponential operator considered. These values of m along with the realness or nonrealness of the complex parameters α_j dictate the analyticity requirements on the data on which this operator acts. To handle this case, we must first write a reduction formula, on the powers of b, for the exponential function $e^{tb^{k+1}}$ where k is a positive integer. From Section 2 of [**3**], we recall that

$$e^{tab} = \int_0^\infty e^{-\zeta}(e^{t\underline{a}} \circ e^{\zeta \underline{b}})\, d\zeta \tag{2.8}$$

if $|b| < 1$. We make use of this when a and b are replaced by differential operators (in particular, see C of Section 3 for an example where $P(D)$ has two distinct factors). For the exponential $e^{tb^{k+1}}$, we get

$$\begin{aligned} e^{tb^{k+1}} = e^{tb\cdot b^k} &= \int_0^\infty e^{-\zeta}(e^{t\underline{b}} \circ e^{\zeta \underline{b}^k})\, d\zeta \\ &= \int_0^\infty e^{-\zeta}\left\{(2\pi)^{-1}\int_0^{2\pi} e^{tbe^{i\theta}} e^{\zeta b^k e^{-i\theta}}\, d\theta\right\} d\zeta. \end{aligned} \tag{2.9}$$

I. $m = 2$. Taking $k = 1$ in (2.9), we observe that the inner integral in this reduces to $e^{t\underline{b}} \circ e^{\zeta\underline{b}}$. By (2.4), this becomes $e^{\sqrt{t\zeta}\underline{b}} \circ e^{\sqrt{t\zeta}\underline{b}} = (2\pi)^{-1}\int_0^{2\pi} e^{2\sqrt{t\zeta}(\cos\theta)b}\, d\theta$. Inserting this into (2.9) with b replaced by $\sum_{j=1}^p \alpha_j D_j$, we have the *formal operator identity*

$$\begin{aligned} e^{tP(D)} = \int_0^\infty e^{-\zeta}\bigg\{&(2\pi)^{-1} \\ &\cdot \int_0^{2\pi} e^{\alpha_{p+1}\sqrt{t\zeta}\cos\theta} e^{2\sqrt{t\zeta}(\cos\theta)\sum_{j=1}^p \alpha_j D_j}\, d\theta\bigg\}\, d\zeta \end{aligned} \tag{2.10}$$

where $P(D) = (\sum_{j=1}^p \alpha_j D_j + \alpha_{p+1})^2$. We must, of course, justify the above replacement by restricting the data on which the two members of this operate.

First, assume that all of the α_j in this are real. Let $\phi(x_1, \dots, x_p)$ be continuous and suppose that the derivatives $\partial\phi(x_1, \dots, x_p)/\partial x_j$ are continuous for $j = 1, \dots, p$. Applying the operators defined by the two members of (2.10) to this ϕ and making use of (2.7), we define the action of the above operator on ϕ by the formula

$$
\begin{aligned}
(2.11)\quad & e^{tP(D)}\phi(x_1,\dots,x_p)\\
&= \int_0^\infty e^{-\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{2\sqrt{t\zeta}\cos\theta(\sum_{j=1}^p \alpha_j D_j+\alpha_{p+1})}\phi(x_1,\dots,x_p)\,d\theta\right\}d\zeta\\
&= \int_0^\infty e^{-\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{2\sqrt{t\zeta}\alpha_{p+1}\cos\theta}\right.\\
&\qquad \left.\cdot\,\phi(x_1+2\alpha_1\sqrt{t\zeta}\cos\theta,\dots,x_p+2\alpha_p\sqrt{t\zeta}\cos\theta)\,d\theta\right\}d\zeta
\end{aligned}
$$

In order that this improper integral converge, we must require the data to satisfy a condition of the form $|\phi(x_1,\dots,x_p)|\le Me^{\tau\sum_{j=1}^p |x_j|^\rho}$ with $0<\rho\le 2$ where M and τ are positive constants. If $\rho<2$, the integral in the second member of (2.11) exists for all choices of the α_j. On the other hand, if $\rho=2$, the integral in the second member of (2.11) exists only for a restricted set of choices for t, τ and the α_j. For example, if we choose $\phi(x_1,\dots,x_p)=e^{\tau\sum_{j=1}^p x_j^2}$, then

$$
\begin{aligned}
& e^{tP(D)}\phi(x_1,\dots,x_p)\\
&\quad = \int_0^\infty e^{-\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{2\alpha_{p+1}\sqrt{t\zeta}\cos\theta}e^{\tau\sum_{j=1}^p x_j^2}\right.\\
&\qquad\qquad \left.\cdot\, e^{4\tau\sqrt{t\zeta}\cos\theta\sum_{j=1}^p a_j x_j}\cdot e^{4t\zeta\cos^2\theta\sum_{j=1}^p \alpha_j^2}\,d\theta\right\}d\zeta\\
&\quad \le e^{\tau\sum_{j=1}^p x_j^2}\int_0^\infty e^{-\zeta(1-4t\tau\sum_{j=1}^p \alpha_j^1)}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{2\alpha_{p+1}\sqrt{t\zeta}\cos\theta}\right.\\
&\qquad\qquad \left.\cdot\, e^{4\sqrt{t\zeta}\cos\theta\sum_{j=1}^p \alpha_j x_j}\,d\theta\right\}d\zeta.
\end{aligned}
$$

This last integral converges only if $1-4t\tau\sum_{j=1}^p \alpha_j^2>0$ or it $t<1/[4\tau\sum_{j=1}^p \alpha_j^2]$. A condition analogous to this appears in the study of expansions of solutions of the heat equation in terms of heat polynomials in one space variable [**18**].

When the α_j's are nonreal complex numbers, a formula of the form (2.11) is valid provided that $\phi(x_1,\dots,x_p)$ is selected to be entire in the variables x_j of growth (ρ,τ) where $0<\rho\le 2$, i.e., $|\phi(x_1,\dots,x_p)|\le Me^{\tau\sum_{j=1}^p |x_j|^\rho}$. If ϕ is in this class with $\rho<2$, then the integral corresponding to (2.11) converges for all α_j's. If $\rho = 2$, a requirement for the convergence of this integral is easily shown to be $1 - 4t\tau \sum_{j=1}^{p} \mathrm{Re}\,(\alpha_j^2) > 0$.

II. $m \geq 3$. The reduction procedure employed to deduce (2.9) can be reapplied to the term $e^{\zeta b^k e^{-i\theta}}$ appearing in the integrand in the last member of the formula (2.9). By repeating this, one eventually reaches the point where b appears only to the first power in the exponentials in the resulting integral. In carrying out this reduction, one should take into account the possibility of switching factors, including commutative derivative operators, among the exponentials in the successive qips. This switching property (2.3) is particularly important in constructing integral formulas for solution operators of Cauchy problems which should reasonably suggest that the initial conditions are satisfied. Moreover, that formula should be valid for the broadest possible class of data functions. In the following, we construct an integral formula for the special symbolic expression $e^{tD_1^3}\phi(x_1)$ to show how one can use this factor switching property to advantage.

By (2.9), we have

(2.12)
$$\begin{aligned} e^{tD_1^3} &= \int_0^\infty e^{-\zeta_1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{tD_1 e^{i\theta_1}} e^{\zeta_1 D_1^2 e^{-i\theta_1}} \, d\theta_1 \right\} d\zeta_1 \\ &= \int_0^\infty e^{-\zeta_1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{t^{1/3}\zeta_1^{2/3} e^{i\theta_1} D_1} e^{t^{2/3}\zeta_1^{1/3} e^{-i\theta_1} D_1^2} \, d\theta_1 \right\} d\zeta_1. \end{aligned}$$

Equality of these last integrals follows from the fact that the inside integral in each case defines the qip $e^{\underline{tD_1}} \circ e^{\underline{\zeta_1} D_1^2}$. It further follows that

$$\begin{aligned} e^{t^{2/3}\zeta_1^{1/3} e^{-i\theta_1} D_1^2} = \int_0^\infty e^{-\zeta_2} \bigg\{ \frac{1}{2\pi} \int_0^{2\pi} & e^{t^{1/3}\zeta_1^{1/3}\zeta_2^{1/3} e^{i(\theta_2-\theta_1)} D_1} \\ & \cdot e^{t^{1/3}\zeta_2^{2/3} e^{-i\theta_2} D_1} \, d\theta_2 \bigg\} d\zeta_2, \end{aligned}$$

by appealing to the case $m = 2$. Inserting this back into (2.12) and simplifying, we get

$$e^{tD_1^3} = \frac{1}{(2\pi)^2}\int_0^\infty \int_0^\infty e^{-(\zeta_1+\zeta_2)} \left\{\int_0^{2\pi}\int_0^{2\pi} e^{t^{1/3}[\zeta_1^{2/3}e^{i\theta_1}+\zeta_1^{1/3}\zeta_2^{1/3}e^{i(\theta_2-\theta_1)}+\zeta_2^{2/3}]D_1}\,d\theta_2\,d\theta_1\right\} d\zeta_2\,d\zeta_1.$$

Now formally apply this to an appropriate entire function $\phi(x_1)$ to obtain

$$e^{tD_1^3}\phi(x_1) = \frac{1}{(2\pi)^2}\int_0^\infty \int_0^\infty e^{-(\zeta_1+\zeta_2)}\Phi(x_1,t,\zeta_1,\zeta_2)\,d\zeta_2\,d\zeta_1 \tag{2.13}$$

where

$$\Phi = \left\{\int_0^{2\pi}\int_0^{2\pi} \phi(x_1 + t^{1/3}(\zeta_1^{2/3}e^{i\theta_1} + \zeta_1^{1/3}\zeta_2^{1/3}e^{i(\theta_2-\theta_1)} + \zeta_2^{2/3}e^{-i\theta_2}))\,d\theta_2\,d\theta_1\right\}.$$

It is not difficult to show, by using the definition of the growth of an entire function and the inequality $|x+y|^\lambda \le 2^{\lambda-1}(|x|^\lambda + |y|^\lambda)$ when $1 < \lambda < 2$, that the second member of (2.13) is well defined if $\phi(x_1)$ has growth (ρ,τ) with $\rho \le 3/2$ ($\rho = 3/2$ is the optimal possible growth of data to go with the operator $e^{tD_1^3}$). When $\rho = 3/2$, we must restrict the size of t. One should observe that there are other possible ways of switching factors in the qips, particularly the ζ_j's that lead to integral formulas that are different from (2.13) and which require different growth possibilities for $\phi(x_1)$.

From this example it is obvious that successive reductions of the above type quickly lead to complicated integral expressions. This will show up again in Section 4 when we examine other types of reductions for solutions operators. At the end of Section 5, we introduce a vector integral notation that will simplify the writing of solutions of Cauchy problems. It takes into account the various types of integration variables and the intervals of integration.

B. *The g_a function.* For immediate and later use, we recall the definition of the g_a function, namely, $g_a(t) = \sum_{j=0}^\infty t^j/(a)_j$ [**6**]. This reduces to e^t if $a=1$ and has the integral form $(a-1)\int_0^1 \sigma^{a-2}e^{t(1-\sigma)}\,d\sigma$ if $a>1$. If $P(D)$ is the required type of partial differential operator, we can write

$$g_a(tP(D))\phi(x)=(a-1)\int_0^1 \sigma^{a-2}e^{t(1-\sigma)P(D)}\phi(x)\,d\sigma. \tag{2.14}$$

The calculation of the integrand of this can now be carried out using the methods of A above.

C. *The $F_{b,c}$ function.* Let $b\geq 1$ and $c\geq 1$, and let $P_1(D)$ and $P_2(D)$ be a pair of differential operators. The operator $F_{b,c}(tP_1(D)P_2(D))$ is defined by the infinite series

$$\sum_{n=1}^{\infty}\frac{t^nP_1^n(D)P_2^n(D)}{(b)_n(c)_n}\phi(x)$$

for a suitable choice of the function $\phi(x)$. In this we have $(b)_n=1$ if $n=0$ and $(b)_n=b(b+1)\cdots(b+n-1)$ if $n\geq 1$. From [**6**], we have, for γ and δ a pair of complex numbers, the following integral formulas for $F_{b,c}(t\delta\gamma)$:

(2.15)

$$(b-1)(c-1)\int_0^1\int_0^1 \sigma_2^{b-2}\sigma_2^{c-2} \left\{\frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{t}(1-\sigma_1)e^{i\theta}\gamma}e^{\sqrt{t}(1-\sigma_2)e^{-i\theta}\delta}\,d\theta\right\}d\sigma_1\,d\sigma_2,$$
$$b>1,\ c>1$$
$$(b-1)\int_0^1 \sigma^{b-2}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{t(1-\sigma)}e^{i\theta}\gamma}e^{\sqrt{t(1-\sigma)}e^{-i\theta}\delta}\,d\theta\right\}d\sigma,$$
$$b>1,\ c=1$$
$$\frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{t}e^{i\theta}\gamma}e^{\sqrt{t}e^{-i\theta}\delta}\,d\theta,\quad b=c=1.$$

The reader is referred to [**6**] for applications of these formulas to second order Cauchy problems.

3. Some lower order problems. Before discussing problems such as (1.1) for $n\geq 3$, it is useful to re-examine initial value heat and wave problems by using the quasi inner product approach. This will permit us to gain some facility with the method, will illustrate different approaches for deducing familiar results and will fill in some gaps not covered in [**3**] and [**6**]. At the same time, it will point out some of the connections between the qip method and the method of transmutations.

A. *Classical and abstract heat problem.* Let's first consider the initial value problem $u_t(x_1, x_2, t) = (D_1^2 + D_2^2)u(x_1, x_2, t),\ t > 0$; $u(x_1, x_2, 0) = \phi(x_1, x_2)$. Its solution can be expressed symbolically as

$$u(x_1, x_2, t) = e^{t(D_1^2+D_2^2)}\phi(x_1, x_2). \tag{3.1}$$

By a slight modification of formula (2.8), we have, for $t > 0$,

$$e^{tab} = \int_0^\infty e^{-\zeta}\left\{(2\pi)^{-1}\int_0^{2\pi} e^{(\sqrt{t\zeta}e^{i\theta})a}e^{(\sqrt{t\zeta}e^{-i\theta})b}\,d\theta\right\}d\zeta \tag{3.2}$$

if $|\sqrt{t}b| < 1$. Now we have the factorization $D_1^2+D_2^2 = (D_1+iD_2)(D_1-iD_2)$. Replace a in (3.2) by $D_1 + iD_2$ and b by $D_1 - iD_2$ and apply both sides of the resulting operator identity to the function $\phi(x_1, x_2)$. After a rearrangement of terms, we get

$$\begin{aligned} e^{t(D_1^2+D_2^2)}\phi(x_1, x_2) &= \int_0^\infty e^{-\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{(2\sqrt{t\zeta}\cos\theta)D_1}\right.\\ &\qquad\left.\cdot\, e^{(-2\sqrt{t\zeta}\sin\theta)D_2}\phi(x_1, x_2)\,d\theta\right\}d\zeta \\ &= \int_0^\infty e^{-\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi}\phi(x_1 + 2\sqrt{t\zeta}\cos\theta, x_2\right.\\ &\qquad\left. - 2\sqrt{t\zeta}\sin\theta)\,d\theta\right\}d\zeta. \end{aligned} \tag{3.3}$$

The integral in the third member of this converges for all $t > 0$ if $\phi(x_1, x_2) \in C^1$ in both variables and if $|\phi(x_1, x_2)| \le Me^{\tau(|x_1|^\rho+|x_2|^\rho)}$ with $0 < \rho < 2$ and $\tau \ge 0$. If $\rho = 2$, one can show that this integral converges only for restricted values of $t > 0$ (see the discussion following (2.11)). Assuming that ϕ satisfies these conditions, we leave it to the reader to show, making the changes of variables $\xi = x_1 + 2\sqrt{t\zeta}\cos\theta$

and $\eta = x_2 - 2\sqrt{t\zeta}\sin\theta$, that the last member of (3.3) can be reduced to the familiar classical solution formula

$$(4\pi t)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[(x_1-\xi)^2+(x_2-\eta)^2]/4t}\phi(\xi,\eta)\,d\eta\,d\xi.$$

Next, let A_1 and A_2 generate continuous groups in a Banach space X and assume that $A_1A_2\phi = A_2A_1\phi$ where $\phi \in D(A_1^2 \cap A_2^2)$ (see [**11**] and [**13**] for a general treatment of groups and semigroups of operators). An abstract generalization of the above two space variable heat problems can be written as

$$u_t(t) = (A_1^2 + A_2^2)u(t), \qquad t > 0; \qquad u(0) = \phi. \tag{3.4}$$

Let $G_{A_i}(t)$, $i = 1, 2$, denote the groups generated by the A_i. Then, using (2.14), one can also write the solution of (3.4) in the form

$$\begin{aligned} u &= T_{A_1}(t)\{T_{A_2}(t)\phi\} \\ &= \frac{1}{4\pi t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\xi_1^2+\xi_2^2)/4t}G_{A_1}(\xi_1)\{G_{A_2}(\xi_2)\phi\}\,d\xi_2\,d\xi_1 \end{aligned} \tag{3.5}$$

where $T_{A_i}(t)$ denotes the semigroup of operators generated by A_i^2, $i = 1, 2$.

B. *A pair of "fourth order" heat problems.* Let us consider the pair of initial value problems

$$\begin{aligned} &\text{(a)} \quad u_t(x_1,t) = D_1^4 u(x_1,t), \quad t > 0; \qquad u(x_1,0) = \phi(x_1) \\ &\text{(b)} \quad v_t(x_1,t) = -D_1^4 v(x_1,t), \quad t > 0; \qquad v(x_1,0) = \phi(x_1) \end{aligned} \tag{3.6}$$

in which $\phi(x_1) \in (\rho,\tau)$ with $\rho < 2$. The solutions of these are given formally by $u(x_1,t) = e^{tD_1^4}\phi(x_1)$ and $v(x_1,t) = e^{-tD_1^4}\phi(x_1)$. If we define $H(x_1 t) = e^{tD_1^2}\phi(x_1)$ to be a solution of the standard heat problem $H_t(x_1,t) = D_1^2 H(x_1,t)$, $t > 0$; $H(x_1,0) = \phi(x_1)$, it follows from the relation $e^{tD_1^4}\phi(x_1) = (2\pi)^{-1}\int_0^\infty\int_0^{2\pi} e^{-\zeta}e^{(2\sqrt{t\zeta}\cos\theta)D_1^2}\phi(x_1)\,d\theta\,d\zeta$ that

$$u(x_1,t) = (2\pi)^{-1}\int_0^\infty\int_0^{2\pi} e^{-\zeta}H(x_1, 2\sqrt{t\zeta}\cos\theta)\,d\theta\,d\zeta. \tag{3.7}$$

In a similar way, one can show that

$$v(x_1,t) = (2\pi)^{-1}\int_0^\infty \int_0^{2\pi} e^{-\zeta} H(x_1, 2i\sqrt{t\zeta}\cos\theta)\, d\theta\, d\zeta. \tag{3.8}$$

The entireness condition imposed on the data function ensures the existence of $H(x,T)$ for all complex T and that it has growth ρ in x_1. Hence, both of the functions u and v exist for all $t > 0$. With the familiar changes of variables, the integral form for u can be rewritten as

$$u(x_1,t) = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} e^{-\xi^2/4t} H(x_1,\xi)\, d\xi.$$

Now the problem (3.6b) has a fundamental solution (see [**17**]) while (3.6a) fails to do so in the usual sense of distributions. The growth properties of that fundamental solution show that the problem (3.6b) has a solution if $\phi(x_1)$ is continuous and $|\phi(x_1)| \le Me^{\tau|x_1|^{4/3}}$. Upon applying (2.11) to the functions H in (3.7) and (3.8), it is a straightforward calculation to show that both u and v are defined for all t by the qip method if $\phi(x_1) \in (\rho,\tau)$ with $\rho < 4/3$. The above computations further show that one cannot construct the fundamental solution of (3.6b) by a means of a simple integral transformation of a solution of the initial value heat problem. Viewed in another way, the operator $A = D_1^2$ generates a semigroup and there is no real transform that connects the exponential operator $e^{\pm tA^2}$ to this semigroup.

C. *Altering a solution integral.* Consider the Cauchy problem given by $u_t = (D_1 + D_2)(D_1 + 2D_2)u$, $u(x_1,x_2,0) = \phi(x_1,x_2)$. A solution of it can be written symbolically as $u = e^{t(D_1+D_2)(D_1+2D_2)}\phi(x_1,x_2)$. Employing (2.8) with a replaced by D_1+D_2 and b replaced by D_1+2D_2, and letting $\alpha\beta = 1$, $\alpha > 0$, we find

$$\begin{aligned} u &= \frac{1}{2\pi}\int_0^\infty e^{-\zeta}\bigg\{\int_0^{2\pi} e^{\sqrt{t\zeta}e^{i\theta}(D_1+D_2)} \\ &\qquad\qquad \cdot e^{\sqrt{t\zeta}e^{-i\theta}(D_1+2D_2)}\phi(x_1,x_2)\,d\theta\bigg\}\,d\zeta \\ &= \frac{1}{2\pi}\int_0^\infty e^{-\zeta}\bigg\{\int_0^{2\pi} e^{\alpha\sqrt{t\zeta}e^{i\theta}(D_1+D_2)} \\ &\qquad\qquad \cdot e^{\beta\sqrt{t\zeta}e^{-i\theta}(D_1+2D_2)}\phi(x_1,x_2)\,d\theta\bigg\}\,d\zeta \\ &= \frac{1}{2\pi}\int_0^\infty e^{-\zeta}\int_0^{2\pi}\phi(x_1+\sqrt{t\zeta}(\alpha e^{i\theta}+\beta e^{-i\theta}), x_2 \\ &\qquad\qquad + \sqrt{t\zeta}(\alpha e^{i\theta}+2\beta e^{-i\theta}))\,d\theta\,d\zeta. \end{aligned}$$

If we select $\alpha = \beta = 1$ in the last member of this, the first argument in ϕ in the integrand is real while the second argument is a nonreal complex number. This last integral exists if $\phi(x_1, x_2)$ is C^1 and bounded by $Me^{\lambda x_1^2}$ in x_1 and entire of growth (ρ, τ) in x_2 with $\rho \le 2$. With the choices $\alpha = \sqrt{2}$ and $\beta = 1/\sqrt{2}$, the analyticity conditions on the variables in the data ϕ must be interchanged. This shows that a switching of scalars in the above qip may alter the solution formula to fit in with the data.

D. *Connections with transmutations.* The method of transmutations relates the solution of one initial or boundary value problem in linear partial differential equations to another such problem by means of a real integral transformation [**7, 8**]. One can often relate the solution of one problem to another by a sequence of transmutations. Here we show how the integral solution of an abstract wave type problem obtained by the method of qips can be decomposed into two successive transmutations acting on the solution of a simple first order Cauchy problem. For this purpose, let X be a Banach space, and let A be the generator of a continuous group $G_A(t)$ with a dense domain $D(A^2) \subseteq X$. Then consider, for $m > -2$, the problem

$$\begin{gathered} w_{tt}(t) - t^m A^2 w(t) = 0, \quad t > 0; \\ w(0) = 0, \quad w_t(0) = \phi \quad \text{where} \quad \phi \in D(A^2). \end{gathered} \tag{3.9}$$

Using the methods of [**6**] for wave type problems, it is not difficult to show that

$$w(t) = \frac{t}{m+2}\int_0^1 \sigma^{-(m+1)/(m+2)} \cdot \left\{\frac{1}{2\pi}\int_0^{2\pi} G_A(2t^{(m+2)/2}\sqrt{1-\sigma}\cos\theta/(m+2))\phi\, d\theta\right\} d\sigma. \tag{3.10}$$

To see how this solution function can be expressed through a set of transmutations, let $u(t)$ be a solution of the problem $u_t(t) = Au(t)$, $u(0) = \phi$. The solution of this is given by $u(t) = G_A(t)\phi$ as was noted in Section 2. Next, let $v(t) = (2\pi)^{-1}\int_0^{2\pi} u(t\cos\theta)\, d\theta$. From Section 8 of [**3**], it follows that $v(t)$ is a solution of the Euler Poisson Darboux problem $v_n(t) + t^{-1}v_t(t) = A^2v(t)$, $v(0) = \phi$, $v_t(0) = \phi$. A comparison of the formula (3.12) with the definitions of $u(t)$ and $v(t)$ shows that

$$w(t) = \frac{t}{m+2}\int_0^1 \sigma^{-(m+1)/(m+2)} \cdot v(2t^{(m+2)/2}\sqrt{1-\sigma}/(m+2))\, d\sigma.$$

Thus, we solve the first order problem for $u(t)$ and then successively construct from this, by elementary integral formulas, the functions $v(t)$ and $w(t)$.

4. Reduction formulas for formal solution operators. Given a generalized wave problem such as $w_{tt}(x,t) = P(D)w(x,t)$, $w(x,0) = \phi_0(x)$, $w_t(x,0) = \phi_1(x)$, we can write its solution symbolically in the form

$$w(x,t) = \cosh(t\sqrt{P(D)})\phi_0(x) + \frac{\sinh(t\sqrt{P(D)})}{\sqrt{P(D)}}\phi_1(x)$$

(by viewing $P(D)$ a constant and solving it as a problem in ordinary differential equations). The basic question then becomes one of assigning a precise meaning to this symbolic solution as was done for special choices of $P(D)$ in [**6**]. The pair of *formal solution operators* $\cosh(t\sqrt{P(D)})$ and $\sinh(t\sqrt{P(D)})/\sqrt{P(D)}$ have the following respective series expansions (after rewriting certain factorial symbols): $\sum_{j=0}^{\infty}(t^2P(D)/4)^j/[j!\cdot(1/2)_j]$ and $t\sum_{j=0}^{\infty}(t^2P(D)/4)^j/[j!\cdot(3/2)_j]$. Using the $F_{b,c}$ notation, we can finally rewrite these solution operators as $\cosh(t\sqrt{P(D)}) = F_{1/2,1}(t^2P(D)/4)$ and $\sinh(t\sqrt{P(D)})/\sqrt{P(D)} = t \cdot F_{1,3/2}(t^2P(D)/4)$. We have already noted that a solution of the problem $u_t(x,t) = P(D)u(x,t)$, $u(x,0) = \phi(x)$ is given symbolically by $u(x,t) = e^{tP(D)}\phi(x)$ with $e^{tP(D)}$ serving as the formal solution operator.

In Section 6, we will consider Cauchy problems, including (1.1), in which the formal solution operators are related to series having the forms

$$O(TP(D), \lambda_1, \dots, \lambda_n) = \sum_{j=0}^{\infty} \frac{(T \cdot P(D))^j}{(\lambda_1)_j(\lambda_2)_j \cdots (\lambda_n)_j}. \tag{4.1}$$

This can also be written as ${}_1F_n(1; \lambda_1, \dots, \lambda_n; TP(D))$, a generalized hypergeometric type function of $TP(D)$ [**16**]. In this, T is a parameter associated with t and the λ_i are real but none of them is a nonpositive integer. Closely associated with these excluded values is a class of *exceptional Cauchy problems* which must be treated by other methods [**5**].

Suppose we replace the term $TP(D)$ in (4.1) by a parameter S. We wish to develop two types of reduction relations for expressing the function $O(S, \lambda_1, \dots, \lambda_n)$, through repeated applications of qips, in terms of component functions g_a and $F_{b,c}$. The choice of which of these to use depends upon one's viewpoint on how best to solve a Cauchy problem. We also provide a reduction formula when $R < n$ where $R = \sum r_m$, see (1.3). For obtaining these, it suffices to assume that $\min_i \lambda_i \geq 1$. For, if $-m < \min_i \lambda_i < -m+1$ where m is a nonnegative integer, we can rewrite $O(S, \lambda_1, \dots, \lambda_n)$ in the form

$$\begin{aligned} O(S, \lambda_1, \dots, \lambda_n) = \sum_{j=0}^{m} &\frac{S^j}{(\lambda_1)_j \cdots (\lambda_n)_j} \\ &+ \frac{S^{m+1}}{(\lambda_1)_{m+1} \cdots (\lambda_n)_{m+1}} \\ &\qquad \cdot O(S, \lambda_1 + m + 1, \dots, \lambda_n + m + 1). \end{aligned} \tag{4.2}$$

It is easy to check that $\lambda_j + m + 1 \geq 1$ for all j in the last term in this. We shall call upon this formula later for solving Cauchy problems when one or more of the λ_j in the solution operator of the form (4.1) are negative.

Reductions in terms of the g_a functions. From the definition of g_a in Section 2, it follows by (2.1) that if $S = S_1 S_2$, then

(4.3)
$$O(S,\lambda_1,\dots,\lambda_n) = g_{\lambda_1}(\underline{S}_1)\circ O(\underline{S}_2,\lambda_2,\dots,\lambda_n) = \frac{1}{2\pi}\int_0^{2\pi} g_{\lambda_1}(S_1 e^{i\theta_1}) O(S_2 e^{-i\theta_1},\lambda_2,\dots,\lambda_n)\,d\theta.$$

A repetition of this shows that, in general,

$$(4.4)\quad O(S_{2k-1}S_{2k},\lambda_k,\dots,\lambda_n) = \frac{1}{2\pi}\int_0^{2\pi} g_{\lambda_k}(S_{2k-1}e^{i\theta_k}) O(S_{2k}e^{-i\theta_k},\lambda_{k+1},\dots,\lambda_n)\,d\theta$$

for $k = 1,2,\dots,n-1$ where $S_{2k-1}S_{2k} = S_{2k-2}e^{-i\theta_{k-1}}$ with $S_0 = S$ and $\theta_0 = 0$. One has some flexibility in choosing the S_k's to satisfy these relationships. This will permit writing solutions of Cauchy problems in a variety of integral forms.

Reductions in terms of the $F_{b,c}$ functions. In view of the formulas (2.15) for the $F_{b,c}$ type functions, it is useful to express the function $O(S,\lambda_1,\dots,\lambda_n)$ directly in terms of them. This is particularly so when n is even in order to view the separate components as solutions operators for second order equations in a t-like variable. When n is odd, we will find it necessary to bring in one of the g_a functions.

Case 1. First assume that $n = 2q$ with $q \geq 2$. Given the forms of the series for $O(S,\lambda_1,\dots,\lambda_{2q})$ and $F_{b,c}$, it follows from the definition of the qip $\circ$ that

$$(4.5)\quad O(S,\lambda_1,\dots,\lambda_{2q}) = \frac{1}{2\pi}\int_0^{2\pi} O(S_1 e^{i\theta_1},\lambda_1,\dots,\lambda_{2q-2}) \times F_{\lambda_{2q-1},\lambda_{2q}}(S_2 e^{-i\theta_1})\,d\theta_1$$

where $S_1 S_2 = S$. If $q = 2$, then the reduction is complete and we can then replace $O(S_1 e^{i\theta_1},\lambda_1,\dots,\lambda_{2q-2})$ by $F_{\lambda_1,\lambda_2}(S_1 e^{i\theta_1})$. If $q > 2$, we can repeat the reduction process on the function $O(S_1 e^{i\theta_1},\lambda_1,\dots,\lambda_{2q-2})$. For $j = 2,3,\dots,q-1$, we obtain the following set of formulas:

$$(4.6)\quad O(S_{2j-3}e^{i\theta_{j-1}},\lambda_1,\dots,\lambda_{2q-2j+2}) = \frac{1}{2\pi}\int_0^{2\pi} O(S_{2j-1}e^{i\theta_j},\lambda_1,\dots,\lambda_{2q-2j}) F_{\lambda_{2q-2j+1},\lambda_{2q-2j+2}}(S_{2j}e^{-i\theta_j})\,d\theta_j$$

in which S_{2j-1} and S_{2j} satisfy the relation $S_{2j-1}S_{2j} = S_{2j-3}e^{i\theta_{j-1}}$.

Case 2. Now suppose that $n = 2q+1$. From the definition of the function $g_a(x)$, we can now write the following first reduction formula, namely,

$$(4.7)\quad O(S, \lambda_1, \dots, \lambda_{2q+1}) = \frac{1}{2\pi}\int_0^{2\pi} O(S_1e^{i\theta_1}, \lambda_1, \dots, \lambda_{2q})\, g_{\lambda_{2q+1}}(S_2e^{-i\theta_1})\, d\theta_1$$

where, once again, $S_1S_2 = S$. A complete reduction can now be carried out by the process employed in Case 1.

A reduction formula when $R < n$. Finally, let us note in applications of (4.1), the number $R = \sum r_m$ of factors of $P(D)$ may be less than the number n of parameters λ_j. If so, we can apply (4.1) to a data function $\phi(x)$ to obtain the quasi inner product formula

$$(4.8)\quad \begin{aligned} O(TP(D), \lambda_1, \lambda_2, \dots, \lambda_n)\phi(x) &= \frac{1}{2\pi}\int_0^{2\pi} O(\sqrt{T}e^{i\theta}, \lambda_1, \dots, \lambda_{n-R}) \\ &\qquad \cdot \{O(\sqrt{T}P(D), \lambda_{n-R+1}, \dots, \lambda_n)\phi(x)\}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} {}_1F_{n-R}(1; \lambda_1, \dots, \lambda_{n-R}; \sqrt{T}e^{i\theta})u(x, \sqrt{T}e^{-i\theta})\, d\theta \end{aligned}$$

where $u(x,t) = O(tP(D), \lambda_{n-R+1}, \dots, \lambda_n)\phi(x)$ and ${}_1F_{n-1}$ denotes a hypergeometric function.

The reduction formulas above as well as those for exponentials in Section 2 lead us to the following:

Theorem 4.1. *Let the differential operator $P(D)$ be given by* (1.3). *Then the function defined formally by $O(tP(D), \lambda_1, \dots, \lambda_n)\phi(x)$ can be expressed as a multiple integral of complex translations of $\phi(x)$ provided that $\phi(x)$ is analytic in its variables or entire of appropriate exponential growth in certain of the variables and analytic in the others.*

As noted before, one must determine restrictions on the size of t. In the section to follow, we provided more details about these data functions and conditions under which the integrals in the multi-integral representation can be interchanged.

5. Conditions on data functions. Before treating various higher order Cauchy problems, it is useful to make some general observations about symbolic operators acting on data and the corresponding conditions required on that data in the worst possible cases. In constructing solutions of problems such as (1.1), one must typically assign evaluations to symbolic expressions such as $O(tP(D), \lambda_1, \dots, \lambda_n)\phi(x)$ with this O operator as given in the previous section. To do this, one needs to have sufficient analyticity on the function $\phi(x)$. Using the reduction formulas of Section 4 and the results from Section 2, the evaluation of that symbolic expression can be written as a multiple complex integral of complex translations of the function $\phi(x)$. The individual integrals in this involve (i) integrations in variables θ_j over $[0, 2\pi]$, (ii) integrations in variables σ_j over $[0, 1]$, and, possibly, (iii) integrations in variables ζ_j over $[0, \infty)$. If only the first two types of integrals appear, it suffices to select $\phi(x)$ to be analytic in all of its variables in some appropriate region in p space. If the third type of integral appears, then it is necessary to choose $\phi(x)$ to be entire of appropriate growth in those variables associated with the improper integration. Whether $\phi(x)$ needs to be analytic in all variables or analytic in some and entire in the others is dictated by the number of factors, including repetitions, of $P(D)$ of the type considered in the introduction. If $P(D)$ has n or fewer such factors, the switching property for qips permits these to be moved to distinct exponential functions in the reduction of $O(t(P(D), \lambda_1, \dots, \lambda_n)\phi(x)$ to a multiple integral form. In summary,

Theorem 5.1. *If the number of factors of $P(D)$ of the type given in (1.3) is $\leq n$, then the function $O(tP(D), \lambda_1, \dots, \lambda_n)\phi(x)$ is defined for restricted t if $\phi(x)$ is analytic in a region of p space. If $\phi(x)$ is entire of exponential growth in all of the x_j, then $O(tP(D), \lambda_1, \dots, \lambda_n)\phi(x)$ is defined for all t.*

For special choices of the operator O, the analytic requirement on $\phi(x)$ can be replaced by $\phi(x) \in C^1$ or possibly $\phi(x) \in C$. This is clearly the case for classical wave type problems. Thus we see that the theorem gives sufficient conditions on $\phi(x)$ in a worst case scenario. In view of the fact that the individual integrations are over finite intervals in this case, the analyticity of $\phi(x)$ suffices for interchanges in the orders of integration.

On the other hand, if $P(D)$ has more than n factors of the type indicated, the switching property will place a product of two or more of these factors in a single exponential function in the integrand of the resulting multiple integral solution. In the case of two such differential factors, denoted here by A and B, we can write

$$e^{tAB} = \int_0^\infty e^{-\sigma} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{te^{i\theta A}} e^{\sigma e^{-i\theta B}} \, d\theta \right\} d\sigma$$

(see Example C of Section 3). The integral form of this has each of the operators appearing in a different exponential function. In the case of three or more such factors, this reduction process can be repeated. From this and a careful analysis, we obtain

Theorem 5.2. *If the number of factors m of $P(D)$ of the type given in* (1.3) *is greater than n, then the function $O(tP(D), \lambda_1, \dots, \lambda_n)\phi(x)$ exists if $\phi(x)$ is entire of appropriate exponential growth in one or more of its variables. In fact, if $m = ns + r$ with r and s integers and with $1 \le r < n-1$, then $\phi(x)$ must be of growth (ρ, τ) with $0 < \rho < (s+1)/s$ in at least one of its variables.*

An examination of the classical initial value heat problem again shows that Theorem 5.2 suffices to handle the worst possible cases ($\rho = 2$ for this one). For the operator $e^{tD_1^3}$, recall that $\rho = 3/2$. The precise variables on which entireness is required depend upon the particular ways in which the derivative operators D_j appear in the various factors of $P(D)$. The examples in the following section will provide illustrations of this. Even though improper integrals appear in these integral representations, the growth of the data in those variables associated with the improper integrals permits interchanging orders of integration.

To simplify the writing of multiple integral formulas for solutions of Cauchy problems in the next section, it is useful to introduce a vector notation for the integration variables. As was noted earlier, the various solution operator reduction formulas can lead to a number of integrations in variables θ_j over $[0, 2\pi]$, a number of integrations in variables σ_j over $[0, 1]$, and a number of integrations in variables ζ_j over $[0, \infty)$. If q of the θ_j variables are needed in an integration, we set $\Theta_q = (\theta_1, \dots, \theta_q)$ and $d\Theta_q = d\theta_1 \cdots d\theta_q$. Similarly, if r of the variables σ_j are needed and s of the variables ζ_j, we write $\Sigma_r = (\sigma_1, \dots, \sigma_r)$, $d\Sigma_r = d\sigma_1 \cdots d\sigma_r$, $Z_s = (\zeta_1, \dots, \zeta_s)$, and $dZ_s = d\zeta_1 \cdots d\zeta_s$. We use the symbol $\int_{[0,2\pi]}(\ )\, d\Theta_q$ to denote the repeated integration $\int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} (\ )\, d\theta_1 \cdots d\theta_q$ with similar understandings for the symbols $\int_{[0,1]}(\ )\, d\Sigma_r$ and $\int_{[0,\infty)}(\ )\, dZ_s$. With this notation and the above comments on the interchange of orders of integration, we will usually write the multiple integral representations for the solutions u of Cauchy problems in the abbreviated forms

$$\int_{[0,\infty)} \int_{[0,1]} \int_{[0,2\pi]} F(\Theta_q, \Sigma_r, Z_s, x, t)\, d\Theta_q\, d\Sigma_r\, dZ_s$$

for proper choices of the function F. Thus, for example, the generalization of (2.13) to the case of $e^{tD_1^n}\phi(x_1)$ can be expressed in this notation as

$$\begin{aligned} e^{tD_1^n}\phi(x_1) = &\frac{1}{(2\pi)^{n-1}} \int_{[0,\infty)} \int_{[0,2\pi]} e^{-\sum_{j=1}^{n-1} \zeta_j} \\ &\cdot \phi(x_1 + t^{1/n}\{\zeta_1^{(n-1)/n} e^{i\theta_1} + \zeta_1^{1/n} \zeta_2^{(n-2)/n} e^{i(\theta_2 - \theta_1)} \\ &\qquad + \zeta_2^{2/n} \zeta_3^{(n-3)/n} e^{i(\theta_3 - \theta_2)} + \cdots \\ &\qquad + \zeta_{n-1}^{(n-1)/n} e^{-i\theta_{n-1}}\})\, d\Theta_{n-1}\, dZ_{n-1}. \end{aligned} \tag{5.1}$$

This is valid if $\phi(x_1)$ is entire of growth $\rho \le n/(n-1)$.

6. Higher order Cauchy problems. We shall now construct solutions for a pair of Cauchy problems in which the underlying equation contains time derivatives of order ≥ 3. These will serve to illustrate the use of the reduction formulas of Sections 2 and 4 for setting up integral forms for symbolic solution operators. This formalism is validated when applied to suitable data functions. As noted in Section 5, the precise conditions on these are determined by the form of the solution operator obtained. The second problem has an underlying equation with a regular singular point. We use ordinary differential equation methods to write its formal solution operator.

Example I. We first consider the Cauchy problem

$$\begin{gathered}\partial^4 u(x,t)/\partial t^4 = t^m D_1^3(D_1^2+D_2^2)u(x,t),\\ u(x,0)=u_t(x,0)=u_{tt}(x,0)=0, \quad u_{tt}(x,0)=\phi(x_1,x_2).\end{gathered} \tag{6.1}$$

Using series methods from ordinary differential equations, it is not difficult to show that the function u is defined symbolically by

$$u(x,t) = \frac{t^3}{6}\cdot {}_0F_3(\_;1+1/(m+4),1+2/(m+4),1+3/(m+4); T^4P(D))\phi(x_1,x_2) \tag{6.2}$$

with $T = t^{(m+4)/4}/(m+4)$ and $P(D) = D_1^3(D_1^2+D_2^2)$. Theorem 5.2 shows that it is sufficient that $\phi(x_1,x_2)$ be analytic in x_2 and entire of growth $\rho = 2$ in x_1. Using the reduction formulas from Section 4, we can write the solution operator O in (6.2) as

$$O = \frac{t^3}{12\pi}\int_0^{2\pi} F_{1,1+1/(m+4)}(T^2D_1^3e^{i\theta_1}) \cdot F_{1+2/(m+4),1+3/(m+4)}(T^2(D_1^2+D_2^2)e^{-i\theta_1})\,d\theta_1. \tag{6.3}$$

By taking $t=1$, $\gamma = T(D_1+iD_2)e^{-i\theta_1/2}$ and $\delta = T(D_1-iD_2)e^{-i\theta_1/2}$ in (2.15), we find

$$\begin{aligned} &F_{1+2/(m+4),1+3/(m+4)}(T^2(D_1^2+D_2^2)e^{-i\theta_1})\\ &\quad= \frac{6}{2\pi(m+4)^2}\int_0^1\int_0^1 \sigma_2^{-(m+2)/(m+4)}\sigma_3^{-(m+1)/(m+2)}\\ &\qquad\cdot\left\{\int_0^{2\pi} e^{T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}e^{i\theta_3}(D_1+iD_2)}\right.\\ &\qquad\quad\left.\cdot\, e^{T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}e^{-i\theta_3}(D_1-iD_2)}\,d\theta_3\right\}d\sigma_2\,d\sigma_3\\ &\quad= \frac{6}{2\pi(m+4)^2}\int_0^1\int_0^1 \sigma_2^{-(m+2)/(m+4)}\sigma_3^{-(m+1)/(m+2)}\\ &\qquad\cdot\left\{\int_0^{2\pi} e^{(2T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\cos\theta_3)D_1}\right.\\ &\qquad\quad\left.\cdot\, e^{(-2T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\sin\theta_3)D_2}\,d\theta_3\right\}d\sigma_2\,d\sigma_3. \end{aligned} \tag{6.4}$$

On the other hand, we can write

(6.5)

$$F_{1,1+1/(m+4)}(T^2 D_1^3 e^{i\theta_1}) = \frac{1}{2\pi(m+4)} \int_0^1 \sigma_1^{-(m+3)/(m+4)} \int_0^{2\pi} e^{T^{2/3} D_1 e^{i\theta_1/3} e^{i\theta_2}} e^{\sigma_1 T^{4/3} D_1^2 e^{2i\theta_1/3} e^{-i\theta_2}} \, d\theta_2 \, d\sigma_1$$

by taking $t = 1$, $\gamma = T^{2/3} D_1 e^{i\theta_1/3}$ and $\delta = T^{4/3} D_1^2 e^{2i\theta_1/3}$ in (2.15). Using (2.8) along with the switching property, we have further that

$$e^{\sigma_1 T^{4/3} D_1^2 e^{2i\theta_1/3} e^{-i\theta_2}} = \frac{1}{2\pi} \int_0^\infty e^{-\zeta_1} \int_0^{2\pi} e^{(2\sqrt{\sigma_1\zeta_1} T^{2/3} e^{i\theta_1/3} e^{-i\theta_2/2} \cos\theta_4) D_1} \, d\theta_4 \, d\zeta_1. \tag{6.6}$$

Insert (6.6) into (6.5), then insert (6.5) and (6.4) into (6.3) and apply the resultant multiple integral operator to $\phi(x_1, x_2)$. We finally obtain the following solution of the altered problem:

(6.7)

$$u(x,t) = \frac{t^3}{(2\pi)^4 (m+4)^3} \times \int_{[0,\infty)} \int_{[0,1]} \int_{[0,2\pi]} e^{-\zeta_1} \sigma_1^{-(m+3)/(m+3)} \sigma_2^{-(m+2)/(m+3)} \sigma_3^{-(m+1)/(m+3)} \times \Phi(x, t, \Theta_4, \Sigma_3, Z_1) \, d\Theta_4 \, d\Sigma_3 \, dZ_1$$

with $\Phi = \phi(X_1, X_2)$ where X_1 and X_2 are given by

$$X_1 = x_1 + T^{2/3} e^{i\theta_1/3} (e^{i\theta_2} + 2\sqrt{\sigma_1\zeta_1} e^{-i\theta_2/2}) + 2T\sqrt{(1-\sigma_2)(1-\sigma_3)} e^{-i\theta_1/2} \cos\theta_3$$

$$X_2 = x_2 - 2T\sqrt{(1-\sigma_2)(1-\sigma_3)} e^{-i\theta_1/2} \sin\theta_3.$$

Now X_1 involves the term $\sqrt{\zeta_1}$ but X_2 is independent of ζ_1. This makes it clear that we must choose $\phi(x_1, x_2)$ to be analytic in x_2 and entire of growth (ρ, τ) in x_1 with $0 < \rho \le 2$. If $\rho = 2$, then it is necessary to require that $T^{4/3} < 1/(4\tau)$ when $\tau > 0$. For a specific case, suppose we select $\phi(x_1, x_2) = f(x_1)/(a - x_2)$ where $f(x_1)$ is entire of growth $(2, \tau)$, $\tau > 0$ with $a > 0$. Let us restrict the choices of x_2 in this problem so that $|x_2| < b < a$ and then determine conditions under which the integrand in (6.7) is well defined. We have the inequality

$$\begin{aligned}|2T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\sin\theta_3 &\\ &\le |2T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\sin\theta_3 - x_2| + |x_2| \\ &\le a+b.\end{aligned}$$

Since this must hold for all possible values of the variables $\sigma_2, \sigma_3, \theta_1$ and θ_3, it follows that we must require $T \le (a+b)/2$. But from the entireness argument on the x_1 variable above, we must also require that $T < 1/(4\tau)^{3/4}$. From this, we see that (6.7) is a well defined solution function if we take T less than the minimum of these bounds. Had we selected the growth ρ of $f(x_1)$ to be less than 2, then (6.7) is valid for $T \le (a+b)/2$. These bounds on T define bounds on t.

Example 2. Finally, we consider the following third order time derivative generalization of the Euler Poisson Darboux (EPD) problem

$$\begin{aligned} &u_{ttt}(x,t) - \frac{a}{t^2}u_t(x,t) = P(D)u(x,t), \quad t > 0 \\ &u(x,0) = \phi(x), \qquad u_t(x,0) = u_{tt}(x,0) = 0 \end{aligned} \tag{6.8}$$

where $P(D) = (D_1^2 + D_2^2)D_3$ (see [**12**] for a detailed study of the classical abstract EPD problem). Closely associated with this problem is the ordinary differential equation problem $t^2y'''(t) - ay'(t) = \lambda t^2 y(t)$, $y(0) = K$, $y'(0) = y''(0) = 0$. The equation in this has a regular singular point at $t = 0$ with indices 0, $(3+\sqrt{1+4a})/2$ and $(3-\sqrt{1+4a})/2$. Assuming that these last two indices are not integers, we can show that the solution of this associated problem is given by

$$y(t) = \bigg(\sum_{m=0}^{\infty} \frac{(\lambda t^3)^m}{3^{3m}m!(1-\gamma_1/3)_m(1-\gamma_2/3)_m}\bigg)\cdot K.$$

Upon replacing the parameter λ by $P(D)$ and K by $\phi(x)$, the symbolic solution of (6.8) is given by the formula

$$u(x,t) = {}_0F_2(\_;1-\gamma_1/3, 1-\gamma_2/3; t^3P(D)/27)\phi(x). \tag{6.9}$$

With a little work, it can be shown that neither $1-\gamma_1/3$ nor $1-\gamma_2/3$ are negative integers or 0 if $2T_{3k-2} < a < 2T_{3k+1}$, $k = 1, 2, \dots,$ where

T_j denotes the jth triangular number. If we choose a in one of these intervals, it follows that $-k < 1 - \gamma_1/3 < -k+1$. We then appeal to (4.2) to rewrite the formal solution (6.9) in the form

$$u(x,t) = \sum_{m=0}^{k} \frac{t^{3m} P(D)^m \phi(x)}{3^{3m} m!(1-\gamma_1/3)_m (1-\gamma_2/3)_m} + \frac{t^{3k+3}}{3^{3k+3}(k+1)!(1-\gamma_1/3)_{k+1}(1-\gamma_2/3)_{k+1}} \times P(D)^{k+1} O(tP(D), k+2, \Gamma_1, \Gamma_2)\phi(x) \tag{6.10}$$

where $\Gamma_j = k+2-\gamma_j/3$, $j = 1, 2$, and where the operator O is given by the series

$$O = \sum_{n=0}^{\infty} \frac{(t^3 P(D)/27)^n}{(k+2)_n (\Gamma_1)_n (\Gamma_2)_n}. \tag{6.11}$$

Theorem 5.1 shows that it suffices that $\phi(x) = \phi(x_1, x_2, x_3)$ be analytic in the three variables x_1, x_2 and x_3. Our task now is to express this operator O as a multiple integral and then apply it to the data function. But, by (4.7), we can write this operator in the form

$$O = \frac{1}{2\pi} \int_0^{2\pi} g_{k+2}(Te^{i\theta_1} D_3) F_{\Gamma_1,\Gamma_2}(T^2 e^{-i\theta_1}(D_1^2 + D_2^2))\, d\theta_1 \tag{6.12}$$

where $T = t/3$. From Section 2, we have

$$g_{k+2}(Te^{i\theta_1} D_1) = (k+1) \int_0^1 \sigma_1^k e^{T(1-\sigma_1)e^{i\theta_1} D_3}\, d\sigma_1$$

and

$$\begin{aligned} &F_{\Gamma_1,\Gamma_2}(T^2 e^{-i\theta_1}(D_1^2 + D_2^2)) \\ &\quad = (\Gamma_1 - 1)(\Gamma_2 - 1) \int_0^1 \int_0^1 \sigma_2^{\Gamma_1 - 2} \sigma_3^{\Gamma_2 - 2} \\ &\qquad \times \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{(2T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\cos\theta_2) D_1} \right. \\ &\qquad\qquad \left. \times e^{(2iT\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\sin\theta_2) D_2}\, d\theta_2 \right\} d\sigma_3\, d\sigma_2. \end{aligned}$$

Inserting these into (6.12) and applying the resulting operator to $\phi(x_1, x_2, x_3)$, we find

(6.13)

$$O\phi(x_1,x_2,x_3) = \frac{(k+1)(\Gamma_1-1)(\Gamma_2-1)}{(2\pi)^2} \times \int_{[0,1]}\int_{[0,2\pi]} \sigma_1^k \sigma_2^{\Gamma_1-2}\sigma_3^{\Gamma_2-2}\Phi(x,t,\Theta_2,\Sigma_3)\,d\Theta_2\,d\Sigma_3$$

where

$$\begin{aligned}\Phi = \phi(&x_1 + 2T\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\cos\theta_2,\\ &x_2 + 2iT\sqrt{(1-\sigma_2)(1-\sigma_3)}e^{-i\theta_1/2}\sin\theta_2,\\ &x_3 + T(1-\sigma_1)e^{i\theta_1}).\end{aligned}$$

The solution of (6.8) now follows by inserting (6.13) back into (6.10). We must, of course, restrict the value of t so that each of the three coordinates in ϕ that define the function Φ lie within the region of analyticity of ϕ. For example, if we choose $\phi(x) = x_1/(100 - x_2 - x_3)$, then we see from the definition of Φ that we must restrict T so that $100 - x_2 - x_3 - 3T$ is bounded away from 0. If ϕ is entire in all three variables, the solution function (6.10) is valid for all t.

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