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KUZNETSOV FORMULAS FOR GENERALIZED KLOOSTERMAN SUMS

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1. Introduction. Given two nonzero integers m and n and a positive integer c, the classical Kloosterman sum is defined as

$$K(m,n;c)=\sum_{\substack{1\le a\le c,\\(a,c)=1}} e^{2\pi i(am+\bar{a}n)/c}$$

where $a\bar{a}\equiv 1 \pmod c$. We can also define a generalized Kloosterman sum

$$K_k(m,n;c)=\sum_{\substack{1\le a\le c,\\(a,c)=1}} \varepsilon_a^{-\kappa}\left(\frac{c}{a}\right)e^{2\pi i(am+\bar{a}n)/c}$$

for odd κ with $\kappa=2k$, where $\varepsilon_a=1$ if $a\equiv 1 \pmod 4$ and $=i$ if $a\equiv 3 \pmod 4$. Here (c/a) is the extended Kronecker's symbol (Shimura [**16**] or see Iwaniec [**8**] and Sarnak [**14**]). In other words the generalized Kloosterman sum is the classical sum twisted by a character. It is known (Iwaniec [**8**]) that this generalized Kloosterman sum is essentially a Salié sum which is defined as

$$S(m,n;q)=\sum_{\substack{1\le a\le q,\\(a,q)=1}} \left(\frac{a}{q}\right)e^{2\pi i(am+\bar{a}n)/q}$$

for odd integer q.

The Linnik-Selberg conjecture (Linnik [**13**], Selberg [**15**]) predicts that there is considerable cancellation in a weighted sum of the classical Kloosterman sums:

$$\sum_{1\le c\le x}\frac{K(m,n;c)}{c}=O(x^{\varepsilon})$$

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for any $\varepsilon > 0$. The first nontrivial estimation in this direction was obtained by Kuznetsov [**12**]

$$\sum_{1\le c\le x} \frac{K(m,n;c)}{c} = O(x^{1/6}(\log x)^{1/3}). \tag{1}$$

The method used in [**12**] is a Kuznetsov formula proved by Kuznetsov and independently by Bruggeman in [**2**]. Briefly speaking, the Kuznetsov formula is a weighted sum of classical Kloosterman sums

$$\sum_{c=1}^{\infty} \frac{1}{c} K(m,n;c)\varphi\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{2}$$

which can be expressed essentially as a bilinear form of Fourier coefficients of Maass cusp forms plus a spectral integral of Fourier coefficients of Eisenstein series with coefficients given by certain Bessel transforms of the function φ.

For the generalized Kloosterman sum $K_k(m,n;c)$ estimation of its weighted sum is also of interest and has important applications in number theory. For instance, estimates of the sum

$$\sum_{\substack{c\le x,\\ c\equiv 0 \pmod{Q}}} \frac{K_k(n,n;c)}{\sqrt{c}} e^{4\pi i\nu n/c} \tag{3}$$

for some $-1 \le \nu \le 1$ and $Q \equiv 0 \pmod 8$, or equivalently, estimates of

$$\sum_{\substack{q\le x,\\ q\equiv 0 \pmod{N}}} \frac{S(n,n;q)}{\sqrt{q}} \tag{4}$$

for some N as pointed out by Sarnak [**14**], play a crucial role in Iwaniec [**8**]. These estimates in turn imply bounds for Fourier coefficients of modular forms of half-integral weights. The technique here is based on explicit evaluation of the generalized Kloosterman sum in order to control its oscillatory behavior. In the direction of the Linnik-Selberg conjecture for the sum of generalized Kloosterman sums, Goldfield and Sarnak [**7**] proved a result similar to the estimate in (1):

$$\sum_{1\le c\le x} \frac{K_k(m,n;c)}{c} = O_\varepsilon(x^{1/6+\varepsilon}) \tag{5}$$

for odd $\kappa = 2k$. Note that here x^ε actually represents a power of $\log x$ higher than $(\log x)^{1/3}$ (Goldfield [**6**]). This result is based on estimation of the Selberg's Kloosterman zeta function

$$Z(m, n; s) = \sum_{c \geq 1} \frac{K_k(m, n; c)}{c^{2s}}$$

as $t = \mathrm{Im}\,(s) \to \infty$.

In this paper we will first establish a Kuznetsov formula for generalized Kloosterman sums twisted by a real character in Section 2 (Theorem 2.1). Our Kuznetsov formula may be regarded as a "soft" formula, because we do not use Bessel transforms on the spectral side. We use the theory of automorphic irreducible representations of $GL(2)$ to give the spectral side of the Kuznetsov formula another interpretation (Section 3). It is possible to get an "explicit" formula following the arguments in Cogdell and Piatetski-Shapiro [**3**]. An advantage of this "soft" Kuznetsov formula is that it can be lifted to a new Kuznetsov formula over a quadratic number field. This will be done in Section 4 (Theorem 4.1). One possible application of our Kuznetsov formulas is that we might be able to estimate weighted sums of generalized Kloosterman sums as those in (3), (4) and (5), using Kuznetsov's approach. This hopefully would improve the results in [**7**] and [**8**], at least to smaller powers of logarithmic functions.

We want to point out that a similar Kloosterman sum or Salié sum appears in Jacquet [**9**]. The geometric kernel function $K_f(g, h)$ in [**9**] is similar to ours, but the relative trace formula there is based on different integration of $K_f(g, h)$ which leads to different conclusions.

2. A new Kuznetsov formula.

First let us choose an additive character $\psi = \psi_{\mathbf{R}} \cdot \prod_{p<\infty} \psi_p$ of $\mathbf{Q}_{\mathbf{A}}$ trivial on $\mathbf{Q}$ such that $\psi_{\mathbf{R}}(x) = e^{2\pi i x}$ and, at each finite place p, the order of the local character ψ_p is zero. Since $(am + \bar{a}n)/c \in \mathbf{Q}$ and $\psi((am + \bar{a}n)/c) = 1$ for $1 \leq a \leq c$ and $(a, c) = 1$, we can write

$$e^{2\pi i (am+\bar{a}n)/c} = \prod_{p<\infty} \bar{\psi}_p\left(\frac{am + \bar{a}n}{c}\right)$$

and the classical Kloosterman sum becomes

$$K(m,n;c)=\sum_{\substack{1\le a\le c,\\(a,c)=1}}\prod_{p<\infty}\bar{\psi}_p\left(\frac{am+\bar{a}n}{c}\right).$$

Since the order of ψ_p is zero, the product above is actually taken over prime factors of c. Thus

$$K(m,n;c)=\prod_{p|c}\sum_{x\in R_p^\times/(1+\varpi_p^{\operatorname{ord}_p(C)}R_p)}\bar{\psi}_p\left(\frac{mx+n/x}{c}\right)$$

where R_p is the ring of integers in $\mathbf{Q}_p$, ϖ_p is a prime element in R_p, and $R_p^\times$ is the group of units. Using integration, we get

$$K(m,n;c)=\phi(c)\prod_{p|c}\int_{R_p^\times}\bar{\psi}_p\left(\frac{mx+n/x}{c}\right)d^\times x,$$

where ϕ is the Euler function.

Similarly we can consider Kloosterman sums twisted by a real character. Let $\eta=\eta_{\mathbf{R}}\prod_{p<\infty}\eta_p$ be a nontrivial real idele class character of $\mathbf{Q}_{\mathbf{A}}^\times$ trivial on $\mathbf{Q}^\times$. By the class field theory there is a unique quadratic number field $E=\mathbf{Q}(\sqrt{\tau})$ such that η is trivial on the norm group $N_{E/\mathbf{Q}}(E_{\mathbf{A}}^\times)$, where $\tau\in\mathbf{Z}$ is nonzero and square-free. In particular, for $p<\infty$, $n\in\mathbf{Z}$ and $x>0$ with $(x,p)=1$, we have

$$\eta_p(p^nx)=\begin{cases}1 & \text{if } p>2,\ p\nmid\tau,\ (\tau/p)=1,\\ & \text{or } p=2,\ \tau\equiv 1 \pmod 8;\\ (-1)^n & \text{if } p>2,\ p\nmid\tau,\ (\tau/p)=-1,\\ & \text{or } p=2,\ \tau\equiv 5 \pmod 8;\\ (-1/x) & \text{if } p=2,\ \tau\equiv 7 \pmod 8;\\ (-1)^n(-1/x) & \text{if } p=2,\ \tau\equiv 3 \pmod 8;\\ (2/x) & \text{if } p=2,\ \tau\equiv 2 \pmod{16};\\ (-1)^n(2/x) & \text{if } p=2,\ \tau\equiv 10 \pmod{16};\\ (-2/x) & \text{if } p=2,\ \tau\equiv 14 \pmod{16};\\ (-1)^n(-2/x) & \text{if } p=2,\ \tau\equiv 6 \pmod{16};\\ (-\tau_1/p)^n(x/p) & \text{if } p>2,\ p\mid\tau,\ \tau=p\tau_1.\end{cases}$$

Then the conductor exponent of ramified η_p is 2 when $p = 2$ and $\tau \equiv 3 \pmod 4$, is 3 when $p = 2$ and $\tau \equiv \pm 2 \pmod 8$, and is 1 when $p > 2$ and $p \mid \tau$. We have

$$\prod_{p|c} \eta_p(a) = \begin{cases} (-1/a)(a/(c,\tau)) & \text{if } 2 \mid c \text{ and } \tau \equiv 3 \pmod 4, \\ (a/(c,\tau)) & \text{otherwise} \end{cases}$$

for $(a, c) = 1$ where

$$\left(\frac{a}{(c,\tau)}\right)$$

is the Kronecker symbol. Now we define a Kloosterman sum twisted by the real character $\prod_{p|c} \eta_p$ as

$$K(m, n; c; \eta) = \sum_{\substack{1 \le a \le c, \\ (a,c)=1}} \left(\prod_{p|c} \eta_p(a)\right) e^{2\pi i(am+\bar{a}n)/c}.$$

Then from

$$\left(\prod_{p|c} \eta_p(a)\right) e^{2\pi i(am+\bar{a}m)/c} = \prod_{p|c} \eta_p(a)\bar{\psi}_p\left(\frac{am+\bar{a}n}{c}\right)$$

the twisted Kloosterman sum can be written as

$$K(m, n; c; \eta) = \sum_{\substack{1 \le a \le c, \\ (a,c)=1}} \prod_{p|c} \eta_p(a)\bar{\psi}_p\left(\frac{am+\bar{a}n}{c}\right).$$

By the same argument as above we get an integral expression

$$K(m, n; c; \eta) = \phi(c) \prod_{p|c} \int_{R_p^\times} \eta_p(x)\bar{\psi}_p\left(\frac{mx+n/x}{c}\right) d^\times x. \tag{6}$$

Although our generalized Kloosterman sum $K(m, n; c; \eta)$ is different from the sum $K_k(m, n; c)$, they both are essentially the Salié sum after removing a power of 2 from c. Indeed, following the proof of Lemma 2 in Iwaniec [**8**], we have

Lemma 2.1. *Let $c = qr$ with $4 \mid r$ and $(q, r) = 1$. Then*

$$K(m, n; c; \eta) = K(m\bar{q}, n\bar{q}; r; \eta)K(m\bar{r}, n\bar{r}; q; \eta)$$

where $\bar{q}$ and $\bar{r}$ are given by $q\bar{q} \equiv 1 \pmod{r}$ and $r\bar{r} \equiv 1 \pmod{q}$.

We note that, when q is odd,

$$K(m, n; q; \eta) = \sum_{\substack{1 \le a \le q, \\ (a,q)=1}} \left(\frac{a}{(q,\tau)}\right) e^{2\pi i(am+\bar{a}n)/q}$$

is the Salié sum $S(m, n; q)$ for suitable q and τ.

Let f be a smooth function of compact support on $GL(2, \mathbf{Q_A})$. We assume that $f = f_{\mathbf{R}} \cdot \prod_{p<\infty} f_p$ and that almost every local function f_p, $p < \infty$, is the characteristic function of $K_p = GL(2, R_p)$. Then the geometric kernel is defined as

$$K_f(g, h) = \int\limits_{Z(\mathbf{Q})\backslash Z(\mathbf{Q_A})} \sum_{\gamma \in GL(2,\mathbf{Q})} f(g^{-1}\gamma z h)\eta(z)d^{\times}z.$$

By the spectral decomposition it is equal to the sum of the corresponding cuspidal kernel, Eisenstein kernel and special kernel, cf., Gelbart and Jacquet [**4**]:

$$K_f(g, h) = K_f^{\mathrm{cusp}}(g, h) + K_f^{\mathrm{eis}}(g, h) + K_f^{\mathrm{sp}}(g, h).$$

We will come back to the actual expressions of these kernels in later sections. Now the Kuznetsov trace formula is given by the integral, cf., Goldfeld [**5**], Ye [**17**],

$$(7) \quad \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}\right) \psi(y + mx)\,dx\,dy$$
$$= \sum_{A \in \mathbf{Q}^{\times}} I_f(A)$$
$$+ \int\limits_{\mathbf{Q_A}} \int\limits_{Z(\mathbf{Q_A})} f\left(z \begin{pmatrix} -1/m & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \eta(z)\psi(x)d^{\times}z\,dx$$

where m is a nonzero integer, and

$$I_f(A) = \prod_{p\le\infty} \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p}\int_{Z(\mathbf{Q}_p)} f_p\left(z\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix} & A\\ 1 & \end{pmatrix}\begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)$$
$$\eta(z)\psi_p(y-mx)d^\times z\,dx\,dy.$$

Here the product is taken over $p=\mathbf{R}$ and $p<\infty$. We denote by $I_{f_p}(A)$ the p-adic local integral on the right side for $p<\infty$, and by $I_{f_\mathbf{R}}(A)$ the real local integral for $p=\mathbf{R}$.

We want to select functions $f_\mathbf{R}$ and f_p, $p<\infty$, so that the right side of (7) equals a weighted sum of generalized Kloosterman sums similar to (2). For this purpose, let us fix a positive integer n. If $n>1$ we write $n=p_1^{b_1}\cdots p_r^{b_r}$, where $p_1,\dots,p_r$ are distinct primes and $b_1,\dots,b_r>0$.

We now choose local functions f_p for $p<\infty$. First let us assume that $p\ne p_1,\dots,p_r$. We want the function f_p to be supported in K_p, left-invariant under the principal congruence subgroup K_3 consisting of $k\equiv I \pmod{\varpi_p^3 R_p}$, and bi-invariant under

$$N(R_p)=\left\{\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\;\middle|\; x\in R_p\right\}.$$

Then f_p is determined by its values at

$$\begin{pmatrix}a_1 & \\ & a_2\end{pmatrix}\quad\text{and}\quad\begin{pmatrix} & a_2\\ a_1 & \end{pmatrix}$$

for $a_1,a_2\in R_p^\times/(1+\varpi_p^3R_p)$. We set

$$f_p\begin{pmatrix}a_1 & \\ & a_2\end{pmatrix}=f_p\begin{pmatrix} & a_2\\ a_1 & \end{pmatrix}=\eta_p(a_1);$$

this can always be done because η_p is either unramified or ramified with its conductor exponent less than or equal to 3.

When $p=p_i$, we want f_p to be supported in

$$K_p\begin{pmatrix}1 & \\ & \varpi_p^{b_i}\end{pmatrix}N(R_p),$$

left-invariant under K_3, and bi-invariant under $N(R_p)$. Then f_p is determined by its values at

$$\begin{pmatrix} a_1 & \\ & a_2\varpi_p^{b_i} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & a_2\varpi_p^{b_i} \\ a_1 & \end{pmatrix}$$

for $a_1, a_2 \in R_p^\times/(1+\varpi_p^3 R_p)$. We set

$$f_p \begin{pmatrix} a_1 & \\ & a_2\varpi_p^{b_i} \end{pmatrix} = f_p \begin{pmatrix} & a_2\varpi_p^{b_i} \\ a_1 & \end{pmatrix} = \eta_p(a_1).$$

To choose the real function $f_{\mathbf{R}}$ we use the Bruhat decomposition. We set $f_{\mathbf{R}} = 0$ on the parabolic subgroup $A(\mathbf{R})N(\mathbf{R})$, where

$$A(\mathbf{R}) = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \quad \text{and} \quad N(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

On the big cell

$$N(\mathbf{R})A(\mathbf{R}) \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} N(\mathbf{R})$$

we set

$$f_{\mathbf{R}} \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} & az \\ z & \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \right) = f_1(x) f_2(y) f_3(z) f_4(a)$$

for $x, y \in \mathbf{R}$, $a, z \in \mathbf{R}^\times$, where f_1 and f_2 are compactly supported smooth functions on $\mathbf{R}$ and f_3, f_4 are compactly supported smooth functions on $\mathbf{R}^\times$, such that

$$\int_{\mathbf{R}} f_1(x) e^{-2\pi i m x}\, dx = 1,$$

$$\int_{\mathbf{R}} f_2(y) e^{2\pi i y}\, dy = 1,$$

and

$$\int_{\mathbf{R}^\times} f_3(z) \frac{dz}{|z|_{\mathbf{R}}} = 1.$$

Since the support of $f_{\mathbf{R}}$ does not intersect a neighborhood of $A(\mathbf{R})N(\mathbf{R})$, the function $f_{\mathbf{R}}$ defined above is indeed a smooth function of compact support on $GL(2,\mathbf{R})$.

Using these local functions $f_{\mathbf{R}}$ and f_p, $p<\infty$, we can calculate the right side of (7). Since $f_{\mathbf{R}}$ vanishes on $A(\mathbf{R})N(\mathbf{R})$, the second term on the right side of (7) equals zero. Now to compute $I_f(A)$ we look at the local integrals $I_{f_p}(A)$, $p<\infty$, and $I_{f_{\mathbf{R}}}(A)$.

We will use an expansion formula for p-adic local orbital integrals proves in Ye [**18**]. By assuming $b_i\geq 0$ we only need to compute $I_{f_p}(A)$ when $p=p_i$. Since f_p is left-invariant under K_3, it can be written as a convolution $f_p=\mathrm{vol}\,(K_3)^{-1}f_0 * f_p$, where f_0 is the characteristic function on K_3. Substituting this convolution into the integral defining $I_{f_p}(A)$, we can rewrite $I_{f_p}(A)$ as

$$I_{f_p}(A)=\sum_{\substack{\lambda_1,\lambda_2\in\mathbf{Z},\\ k\in K_3\backslash K_p}}\Psi_{f_p}\left(k\begin{pmatrix}\varpi_p^{\lambda_1}&\\&\varpi_p^{\lambda_1+\lambda_2}\end{pmatrix}\right)$$
$$\cdot\int\limits_{\substack{x,y\in\mathbf{Q}_p,z\in Z(\mathbf{Q}_p),\\ z\begin{pmatrix}1&x\\&1\end{pmatrix}\begin{pmatrix}&A\\1&\end{pmatrix}\begin{pmatrix}\varpi_p^{-\lambda_1}&\\&\varpi_p^{-\lambda_1-\lambda_2}\end{pmatrix}\begin{pmatrix}1&y\\&1\end{pmatrix}\in K_3k}}\eta_p(z)\psi_p(y\varpi_p^{\lambda_2}-mx)\,dx\,dy\,d^\times z$$

where

$$\Psi_{f_p}(g)=\int\limits_{\mathbf{Q}_p}f_p\left(g\begin{pmatrix}1&x\\&1\end{pmatrix}\right)\psi_p(x)\,dx.$$

Since f_p is supported in

$$K_p\begin{pmatrix}1&\\&\varpi_p^{b_i}\end{pmatrix}N(R_p),$$

we know that

$$\Psi_{f_p}\left(k\begin{pmatrix}\varpi_p^{\lambda_1}&\\&\varpi_p^{\lambda_1+\lambda_2}\end{pmatrix}\right)$$

is nonzero only if $\lambda_1=0$ and $\lambda_2=b_i$. When this is the case

$$\Psi_{f_p}\left(k\begin{pmatrix}\varpi_p^{\lambda_1}&\\&\varpi_p^{\lambda_1+\lambda_2}\end{pmatrix}\right)=f_p\left(k\begin{pmatrix}1&\\&\varpi_p^{b_i}\end{pmatrix}\right).$$

Consequently

$$I_{f_p}(A) = \sum_{k \in K_3 \backslash K_p} f_p\left(k\begin{pmatrix}1 & \\ & \varpi_p^{b_i}\end{pmatrix}\right) \cdot \int_{\substack{x,y \in \mathbf{Q}_p, z \in Z(\mathbf{Q}_p), \\ z\begin{pmatrix}x & A\varpi_p^{-b_i}+xy \\ 1 & y\end{pmatrix} \in K_3 k}} \eta_p(z)\psi_p(y\varpi_p^{b_i} - mx)\, dx\, dy\, d^\times z.$$

Since the function f_p is bi-invariant under $N(R_p)$, we may take the sum above over $k \in K_3 N(R_p)\backslash K_p/N(R_p)$ and the integral over

$$z\begin{pmatrix}x & A\varpi_p^{-b_i}+xy \\ 1 & y\end{pmatrix} \in K_3 N(R_p) k N(R_p).$$

This last condition implies that $\operatorname{ord}_p(A) \le b_i$, $\operatorname{ord}_p(A) \equiv b_i \pmod 2$, and hence $z \in \varpi_p^{(b_i - \operatorname{ord}_p(A))/2} R_p^\times$.

If $\operatorname{ord}_p(A) = b_i$, then $z \in R_p^\times$, $x, y \in R_p$, and we can integrate over x and y to get

$$I_{f_p}(A) = \sum_{k \in K_3 N(R_p)\backslash K_p/N(R_p)} f_p\left(k\begin{pmatrix}1 & \\ & \varpi_p^{b_i}\end{pmatrix}\right) \cdot \int_{\substack{z \in Z(\mathbf{Q}_p), \\ z\begin{pmatrix} & A\varpi_p^{-b_i} \\ 1 & \end{pmatrix} \in K_3 N(R_p) k N(R_p)}} \eta_p(z)\, d^\times z.$$

Hence we can set

$$k = \begin{pmatrix} & a_2 \\ a_1 & \end{pmatrix}$$

with $a_1, a_2 \in R_p^\times/(1+\varpi_p^3 R_p)$ and get

$$I_{f_p}(A) = \sum_{a_1, a_2 \in R_p^\times/(1+\varpi_p^3 R_p)} f_p\begin{pmatrix} & a_2\varpi_p^{b_i} \\ a_1 & \end{pmatrix} \cdot \int_{\substack{z \in a_1(1+\varpi_p^3 R_p), \\ zA\varpi_p^{-b_i} \in a_2(1+\varpi_p^3 R_p)}} \eta_p(z)\, d^\times z.$$

Recall

$$f_p\begin{pmatrix} & a_2\varpi_p^{b_i} \\ a_1 & \end{pmatrix} = \eta_p(a_1).$$

Since $\eta_p(z)=\eta_p(a_1)$, we can integrate over z and sum over a_2 to get $I_{f_p}(A)=1$ when $\operatorname{ord}_p(A)=b_i$.

Now we consider the case of $\operatorname{ord}_p(A) < b_i$ and $\operatorname{ord}_p(A)\equiv b_i \pmod 2$. This time $z\in\varpi_p^{(b_i-\operatorname{ord}_p(A))/2}R_p^\times$, $x\in\varpi_p^{(\operatorname{ord}_p(A)-b_i)/2}R_p^\times$, and $y=-A\varpi_p^{-b_i}/x+y_1$ with $y_1\in R_p$. Integrating over y_1, we get

$$I_{f_p}(A)=\sum_{k\in K_3N(R_p)\backslash K_p/N(R_p)} f_p\left(k\begin{pmatrix}1 & \\ & \varpi_p^{b_i}\end{pmatrix}\right) \cdot \int\limits_{\substack{x\in\varpi_p^{(\operatorname{ord}_p(A)-b_i)/2}R_p^\times,\\ z\in\varpi_p^{(b_i-\operatorname{ord}_p(A))/2}R_p^\times,\\ z\begin{pmatrix}x & \\ & -A\varpi_p^{-b_i}/x\end{pmatrix}\in K_3N(R_p)kN(R_p)}} \eta_p(z)\bar{\psi}_p\left(\frac{A}{x}+mx\right)dx\,d^\times z.$$

Setting

$$k=\begin{pmatrix}a_1 & \\ & a_2\end{pmatrix}$$

we get

$$I_{f_p}(A)=\sum_{a_1,a_2\in R_p^\times/(1+\varpi_p^3R_p)} f_p\begin{pmatrix}a_1 & \\ & a_2\varpi_p^{b_i}\end{pmatrix} \cdot \int\limits_{\substack{x\in\varpi_p^{(\operatorname{ord}_p(A)-b_i)/2}R_p^\times,\\ z\in\varpi_p^{(b_i-\operatorname{ord}_p(A))/2}R_p^\times,\\ xz\in a_1(1+\varpi_p^3R_p),\\ -zA\varpi_p^{-b_i}/x\in a_2(1+\varpi_p^3R_p)}} \eta_p(z)\bar{\psi}_p\left(\frac{A}{x}+mx\right)dx\,d^\times z.$$

Note that

$$f_p\begin{pmatrix}a_1 & \\ & a_2\varpi_p^{b_i}\end{pmatrix}=\eta_p(a_1)\quad\text{and}\quad \eta_p(z)=\eta_p(x)\eta_p(a_1).$$

If we integrate over $z \in (a_1/x)(1+\varpi_p^3 R_p)$ and sum over a_2, we get

$$I_{f_p}(A) = \int\limits_{\varpi_p^{(\operatorname{ord}_p(A)-b_i)/2} R_p^\times} \eta_p(x)\bar{\psi}_p\left(\frac{A}{x}+mx\right)dx.$$

We observe that $I_{f_p}(A)$ is nonzero only when $\operatorname{ord}_p(A) \le b_i$ and $\operatorname{ord}_p(A) \equiv b_i \pmod 2$. Thus in the sum $\sum_{A\in\mathbf{Q}^\times} I_f(A)$ we can set $A = \pm n/c^2$ with $c \in \mathbf{Z}_+^\times$. Changing variables from x to x/c when $\operatorname{ord}_p(c) \ne 0$, we get

$$I_{f_p}\left(\pm\frac{n}{c^2}\right) = (1-p^{-1})p^{\operatorname{ord}_p(c)}\eta_p(c)\int\limits_{R_p^\times}\eta_p(x)\bar{\psi}_p\left(\frac{mx \pm n/x}{c}\right)d^\times x.$$

Therefore the Kuznetsov trace formula in (7) can be written as

$$\begin{aligned}\sum_{A\in\mathbf{Q}^\times} I_f(A) = \sum_{c=1}^\infty & I_{f_\mathbf{R}}\left(\frac{n}{c^2}\right) \\ &\cdot\prod_{p|c}(1-p^{-1})p^{\operatorname{ord}_p(c)}\eta_p(c) \\ &\cdot\int\limits_{R_p^\times}\eta_p(x)\bar{\psi}_p\left(\frac{mx+n/x}{c}\right)d^\times x \\ +\sum_{c=1}^\infty & I_{f_\mathbf{R}}\left(-\frac{n}{c^2}\right) \\ &\cdot\prod_{p|c}(1-p^{-1})p^{\operatorname{ord}_p(c)}\eta_p(c) \\ &\cdot\int\limits_{R_p^\times}\eta_p(x)\bar{\psi}_p\left(\frac{mx-n/x}{c}\right)d^\times x.\end{aligned}$$

Using the twisted Kloosterman sum in (6), we get

$$\sum_{c=1}^\infty I_{f_\mathbf{R}}\left(\frac{n}{c^2}\right)K(m,n;c;\eta)\prod_{p|c}\eta_p(c) + \sum_{c=1}^\infty I_{f_\mathbf{R}}\left(-\frac{n}{c^2}\right)K(m,-n;c;\eta)\prod_{p|c}\eta_p(c).$$

Back to the definition of $f_{\mathbf{R}}$, we see that

$$I_{f_{\mathbf{R}}}(A) = \int\limits_{\mathbf{R}} f_1(x)e^{-2\pi imx}\,dx \int\limits_{\mathbf{R}} f_2(y)e^{2\pi iy}\,dy \int\limits_{\mathbf{R}^\times} f_3(z)\frac{dz}{|z|_{\mathbf{R}}} f_4(A) = f_4(A).$$

Therefore, the geometric side of the Kuznetsov trace formula is

$$\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy$$
$$= \sum_{c=1}^{\infty} f_4\left(\frac{n}{c^2}\right)K(m,n;c;\eta)\prod_{p|c}\eta_p(c) + \sum_{c=1}^{\infty} f_4\left(-\frac{n}{c^2}\right)K(m,-n;c;\eta)\prod_{p|c}\eta_p(c).$$

We note that $\prod_{p|c}\eta_p(c) = (-1)^{\Omega_{\mathrm{unr}}(c)}$ if all primes $p \mid c$ split or are inert unramified (see Section 4), where $\Omega_{\mathrm{unr}}(c)$ is the number of inert unramified prime divisors of c counting multiplicity. Using the spectral decomposition of the automorphic representations of $GL(2)$, we finally get a new Kuznetsov formula.

Theorem 2.1. *Let f_4 be a smooth function of compact support in $\mathbf{R}^\times$. For a positive integer n and a nonzero integer m, define functions $f_1, f_2, f_3, f_{\mathbf{R}}, f_p$, $p < \infty$, and f as above. Then*

$$\sum_{c=1}^{\infty} f_4\left(\frac{n}{c^2}\right)K(m,n;c;\eta)\prod_{p|c}\eta_p(c) + \sum_{c=1}^{\infty} f_4\left(-\frac{n}{c^2}\right)K(m,-n;c;\eta)\prod_{p|c}\eta_p(c)$$
$$= \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f^{\mathrm{cusp}}\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy + \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f^{\mathrm{eis}}\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy. \tag{8}$$

Here we used the fact that the integral of the special kernel equals zero. This is trivial, because

$$K_f^{\mathrm{sp}}(g,h) = \frac{1}{2}\sum_{\chi^2=\eta} \chi(\det g)\bar{\chi}(\det h) \cdot \int\limits_{Z(\mathbf{Q_A})\backslash GL(2,\mathbf{Q_A})} \chi(\det y)\,dy \cdot \int\limits_{Z(\mathbf{Q_A})} f(zy)\eta(z)\,d^\times z.$$

We remark that if we assume $f_4(A)$ to be zero for negative A, we only get the first sum on the left side of (8). If, instead, f_4 is compactly supported on $(-\infty, 0)$, we get the second sum only, which is a Kuznetsov formula with opposite signs when m is positive.

If we set η to be the trivial character, the discussion in this section also applies. Consequently, we can get a Kuznetsov formula for the usual classical Kloosterman sums. This is essentially the soft Kloosterman-spectral formula of Cogdell and Piatetski-Shapiro [**3**]:

$$\begin{aligned}\sum_{c=1}^{\infty} f_4\left(\frac{n}{c^2}\right)K(m,n;c) + \sum_{c=1}^{\infty} f_4\left(-\frac{n}{c^2}\right)K(m,-n;c) \\ = \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}}\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f'^{\mathrm{cusp}}\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy \\ + \int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}}\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f'^{\mathrm{eis}}\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}, \begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy\end{aligned}$$

where $K_f'^{\mathrm{cusp}}$ and $K_f'^{\mathrm{eis}}$ are the cuspidal and Eisensteinian components of the kernel

$$K'_f(g,h) = \int\limits_{Z(\mathbf{Q})\backslash Z(\mathbf{Q_A})} \sum_{\gamma\in GL(2\mathbf{Q})} f(g^{-1}\gamma z h)\,d^\times z.$$

3. The spectral decomposition. Now we explain the meaning of the spectral side of our Kuznetsov formula.

Let σ be an automorphic irreducible cuspidal representation of $GL(2,\mathbf{Q_A})$ with central character η containing the unit representation of K^S. Here S is a finite set of places of $\mathbf{Q}$ containing $\mathbf{R}, p_1, \dots, p_r$ (from n), and all ramified places, and $K^S = \prod_{p\notin S} K(\mathbf{Q}_p)$. We denote by $V_S(\sigma)$ the subspace of the space of σ consisting of the forms invariant under K^S. Let $\{\phi_j\}_{j\in J}$ be an orthonormal basis of $V_S(\sigma)$, and let σ_S be the corresponding representation of $GL(2,\mathbf{Q}_S)$ on $V_S(\sigma)$. Then, for the function f chosen above, the cuspidal kernel is given by (Jacquet and Lai [**10**])

$$K_f^{\mathrm{cusp}}(g,h) = \sum_\sigma \sum_{j\in J} \sigma_S(\tilde f_S)\phi_j(g)\bar\phi_j(h)$$

where $\tilde f(g) = \int_{Z(\mathbf{Q_A})} f(zg)\eta(z)\,d^\times z$. Hence the cuspidal integral can be written as

$$\begin{aligned}\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}}\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} & K_f^{\mathrm{cusp}}\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix},\begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy \\ &= \sum_\sigma\sum_{j\in J}\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}} (\sigma_S(\tilde f_S)\phi_j)\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\psi(mx)\,dx \\ &\quad\cdot\int\limits_{\mathbf{Q}\backslash\mathbf{Q_A}}\bar\phi_j\begin{pmatrix}1 & y\\ & 1\end{pmatrix}\psi(y)\,dy.\end{aligned}$$

To define the Eisenstein kernel, we first introduce a Hilbert space $H(s,\mu,\nu)$ for $s\in\mathbf{C}$ consisting of functions ϕ on $GL(2,\mathbf{Q_A})$ which satisfy conditions

$$\phi\left(\begin{pmatrix}a & x\\ & b\end{pmatrix}g\right) = \mu(a)\nu(b)\left|\frac{a}{b}\right|_{\mathbf{Q_A}}^{s+1/2}\phi(g)$$

for $a,b\in\mathbf{Q_A^\times}$, $x\in\mathbf{Q_A}$ and $g\in GL(2,\mathbf{Q_A})$, and

$$\int\limits_{K(\mathbf{Q_A})}|\phi(k)|^2\,dk < +\infty.$$

Here μ and ν are unitary characters of $\mathbf{Q_A^\times}$ trivial on $\mathbf{Q}^\times$ with $\mu\nu=\eta$. The spaces $H(s,\mu,\nu)$ form a trivial holomorphic fiber bundle of base

$\mathbf{C}$; hence, we may identify $H(s,\mu,\nu)$ with $H(\mu,\nu) = H(0,\mu,\nu)$ so that each $\phi \in H(\mu,\nu)$ determines a section $\phi(s,\mu,\nu)$ whose restriction to $K(\mathbf{Q_A})$ is independent of s. Let $\{\phi_\alpha\}$, $\alpha \in I$, be an orthonormal basis of $H(\mu,\nu)$, and let $\pi(s,\mu,\nu)$ be the representation of $GL(2,\mathbf{Q_A})$ on $H(s,\mu,\nu)$ by right translations. This representation $\pi(s,\mu,\nu)$ is unitary if s is purely imaginary. The Eisenstein series is defined as

$$E(g,\phi,s,\mu,\nu) = \sum_{\gamma\in A(\mathbf{Q})N(\mathbf{Q})\backslash GL(2,\mathbf{Q})} \phi(\gamma g, s)$$

for $\operatorname{Re}(s) > 1/2$ and is continued analytically to the whole complex lane, where ϕ is a section of the bundle $H(s,\mu,\nu)$. Then, as in Arthur [**1**] and Gelbart and Jacquet [**4**], the Eisenstein kernel is

$$(9)\quad K_f^{\mathrm{eis}}(g,h) = \frac{1}{4\pi}\sum_{\mu\nu=\eta}\int_{-\infty}^{\infty}\sum_{\alpha,\beta\in I}(\pi(it,\mu,\nu)(\tilde{f})\phi_\beta,\phi_\alpha) \cdot E(g,\phi_\alpha,it,\mu,\nu)\overline{E}(h,\phi_\beta,it,\mu,\nu)\,dt.$$

This expression of Eisenstein kernel is actually valid for smooth $K(\mathbf{Q_A})$-finite functions f. Since $f = f_{\mathbf{R}} \cdot \prod_{p<\infty} f_p$ with f_p being locally constant and compactly supported, it is equivalent to say that (9) is valid for those f with $K(\mathbf{R})$-finite smooth functions $f_{\mathbf{R}}$; in this case, the two sums in (9) are indeed reduced to finite sums for each fixed f. Arthur pointed out in [**1**] that the expression in (9) is a continuous functional on the space of $K(\mathbf{Q_A})$-finite compactly supported smooth functions. Since this space is dense in $C_c^\infty(GL(2,\mathbf{Q_A}))$, we can then extend the definition of K_f^{eis} to a continuous functional on $C_c^\infty(GL(2,\mathbf{Q_A}))$.

More precisely, let

$$r(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Since $f_{\mathbf{R}}(r(\varphi_1)gr(\varphi_2))$ is a smooth 2π-periodic function of φ_1 and φ_2, we can expand $f_{\mathbf{R}}(g)$ into a Fourier series

$$(10)\quad f_{\mathbf{R}}(g) = \frac{1}{4\pi^2}\sum_{l,n\in\mathbf{Z}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{\mathbf{R}}(r(\varphi_1)gr(\varphi_2)) \cdot e^{-in\varphi_1}e^{-il\varphi_2}\,d\varphi_1\,d\varphi_2.$$

Since $f_{\mathbf{R}}$ is smooth and compactly supported, the double series in (10) is uniformly convergent for $g \in GL(2, \mathbf{R})$. We also observe that any finite partial sum of the right side of (10) is a smooth $K(\mathbf{R})$-finite function; in particular, the integral in (10) is $K(\mathbf{R})$-finite for any $l, n \in \mathbf{Z}$. Therefore, for any compactly supported smooth function f, the Eisenstein kernel can be written as

$$\begin{aligned} &K_f^{\mathrm{eis}}(g,h) \\ &= \frac{1}{16\pi^3} \sum_{l,n\in\mathbf{Z}} \sum_{\mu\nu=\eta} \int_{-\infty}^{+\infty} \sum_{\alpha,\beta\in I} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\pi(it,\mu,\nu)(\tilde{f}_{\varphi_1,\varphi_2})\phi_\beta, \phi_\alpha) \\ &\qquad\qquad \cdot e^{-in\varphi_1} e^{-il\varphi_2}\, d\varphi_1\, d\varphi_2 \\ &\qquad\qquad \cdot E(g,\phi_\alpha,it,\mu,\nu)\overline{E}(h,\phi_\beta,it,\mu,\nu)\, dt \end{aligned}$$

where

$$f_{\varphi_1,\varphi_2}(g) = f_{\mathbf{R}}(r(\varphi_1) g_{\mathbf{R}} r(\varphi_2)) \prod_{p<\infty} f_p(g_p)$$

for $g = g_{\mathbf{R}} \prod_{p<\infty} g_p \in GL(2, \mathbf{Q_A})$. Note that the two inner sums here are indeed finite sums for fixed f and l, n.

According to Arthur [**1**], the Eisenstein kernel is bounded by

$$|K_f^{\mathrm{eis}}(g,h)| \le \|f\|_0 \cdot \|g\|^L \|h\|^L \tag{11}$$

for any compactly supported smooth function f, not necessarily $K(\mathbf{Q_A})$-finite, where $\|\cdot\|_0$ is a continuous seminorm on $C_c^\infty(GL(2, \mathbf{Q_A}))$ and L is a constant; here $\|\cdot\|$ is a norm function on $GL(2, \mathbf{Q_A})$. Since the Eisenstein series are left-invariant under $GL(2, \mathbf{Q})$, the integral of K_f^{eis} in (8) can be taken over compact subsets of $\mathbf{Q_A}$. By the bounds given in (11), we obtain as in Ye [**17**], that

$$\int_{\mathbf{Q}\backslash\mathbf{Q_A}}\int_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f^{\mathrm{eis}}\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix},\begin{pmatrix}1 & y\\ & 1\end{pmatrix}\right)\psi(y+mx)\,dx\,dy$$

$$= \frac{1}{16\pi^3}\sum_{l,n\in\mathbf{Z}}\sum_{\mu,\nu=\eta}\int_{-\infty}^{+\infty} dt$$

$$\cdot \sum_{\alpha,\beta\in I}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{-in\varphi_1}e^{il\varphi_2}\,d\varphi_1\,d\varphi_2$$

$$\cdot \int_{K(\mathbf{Q_A})}\int_{K(\mathbf{Q_A})} \phi_\beta(k'r(\varphi_2))\bar{\phi}_\alpha(kr(\varphi_1))\,dk\,dk'$$

$$\cdot \int_{\mathbf{Q}_S^\times}\int_{\mathbf{Q}_S} \tilde{f}_S\left(k_S^{-1}\begin{pmatrix}a & ax\\ & 1\end{pmatrix}k'_S\right)\mu_S(a)|a|_S^{it+1/2}\,dx\,d^\times a$$

$$\cdot \int_{\mathbf{Q}\backslash\mathbf{Q_A}} E\left(\begin{pmatrix}1 & v\\ & 1\end{pmatrix},\phi_\alpha,it,\mu,\nu\right)\psi(mv)\,dv$$

$$\cdot \int_{\mathbf{Q}\backslash\mathbf{Q_A}} \overline{E}\left(\begin{pmatrix}1 & w\\ & 1\end{pmatrix},\phi_\beta,it,\mu,\nu\right)\psi(w)\,dw$$

where S is the same as in the cuspidal kernel, $f_S = \prod_{p\in S} f_p$, $\mu_S = \prod_{p\in S}\mu_p$, $k_S = \prod_{p\in S} k_p$, $kr(\varphi_1) = k_{\mathbf{R}}r(\varphi_1)\cdot\prod_{p<\infty} k_p$, etc.

4. A Kuznetsov formula over a quadratic number field. The quadratic idele class character η in Theorem 2.1 is associated uniquely with a quadratic number field $E = \mathbf{Q}(\sqrt{\tau})$ with a nonzero square-free integer τ. To simplify the matter, we will assume that $m = n$ in this section; this is indeed the situation when one estimates (3) and (4).

We will derive our results by local arguments. We note that p splits in E if $p = 2$, $\tau \equiv 1 \pmod 8$, or if $p > 2$, $p \nmid \tau$, $(\tau/p) = 1$; p is inert unramified in E if $p = 2$, $\tau \equiv 5 \pmod 8$, or if $p > 2$, $p\nmid\tau$, $(\tau/p) = -1$; p is inert ramified in E if $p = 2$, $\tau \equiv 2, 3 \pmod 4$, or if $p > 2$, $p \mid \tau$. When p is ramified, the different exponent d of E_p over $\mathbf{Q}_p$ is 1 if $p > 2$, $p \mid \tau$, is 2 if $p = 2$, $\tau \equiv 3 \pmod 4$, and is 3 if $p = 2$, $\tau \equiv 2 \pmod 4$.

We will use a lemma proved in Ye [**19**]. A formula of this kind was first proved by Zagier [**20**].

Lemma 4.1. *Suppose that p is inert in $\mathbf{Q}(\sqrt{\tau})$, $b \in R_p^\times$, $c \in \varpi_p R_p$ if p is unramified, and $c \in \varpi_p^d R_p$ if p is ramified. Then*

$$\int_{(1/c)R_p^\times} \eta_p(x)\bar{\psi}_p\left(x + \frac{b}{c^2 x}\right) dx = p^{-d/2}\lambda_{E_p/\mathbf{Q}_p}(\bar{\psi}_p) \sum_{\substack{\alpha \in R_{E_p}^\times/(1+cR_{E_p}),\\ \alpha\bar{\alpha} \in b+\varepsilon c R_p}} \bar{\psi}_p \circ \mathrm{tr}\left(\frac{\alpha}{c}\right)$$

where $\varepsilon = 2$ if $p = 2$ and $\tau \equiv 2, 3 \pmod 4$, and $\varepsilon = 1$ otherwise.

Here the local λ factor is defined as in Jacquet and Langlands [**11**]. That is,

$$\lambda_{E_p/\mathbf{Q}_p}(\bar{\psi}_p) = \begin{cases} 1 & \text{if } p = 2,\ \tau \equiv 1 \pmod 4 \\ & \text{or } p > 2,\ p \nmid \tau; \\ i & \text{if } p = 2,\ \tau \equiv 3 \pmod 4; \\ 1 & \text{if } p = 2,\ \tau \equiv 2 \pmod{16}; \\ -1 & \text{if } p = 2,\ \tau \equiv 10 \pmod{16}; \\ i & \text{if } p = 2,\ \tau \equiv 14 \pmod{16}; \\ -i & \text{if } p = 2,\ \tau \equiv 6 \pmod{16}; \\ \left(\dfrac{-\tau_1}{p}\right) \dfrac{\sum_{1 \le x < p} (x/p) e^{2\pi i x/p}}{\left|\sum_{1 \le x < p} (x/p) e^{2\pi i x/p}\right|} & \text{if } p > 2,\ p \mid \tau \text{ with} \\ & \tau = p\tau_1. \end{cases}$$

For the local orbital integral

$$I_{f_p}\left(\pm \frac{n}{c^2}\right) = \int_{(1/c)R_p^\times} \eta_p(x)\bar{\psi}_p\left(mx \pm \frac{n}{c^2 x}\right) dx$$

when $p \mid c$, we have the following results.

Lemma 4.2. *Suppose $m = n$ and $p \mid c$.*

(i) *When p is inert unramified with $c \in n\varpi_p R_p$, or when p is ramified with $c \in n\varpi_p^d R_p$,*

$$I_{f_p}\left(\pm\frac{n}{c^2}\right) = \frac{\eta_p(n)}{|n|_p} p^{-d/2} \lambda_{E_p/\mathbf{Q}_p}(\bar{\psi}_p) \cdot \sum_{\substack{\alpha \in R_{E_p}^\times/(1+(c/n)R_{E_p}),\\ \alpha\bar{\alpha} \in \pm 1 + (\varepsilon c/n)R_p}} \bar{\psi}_p \circ \mathrm{tr}\left(\frac{\alpha n}{c}\right).$$

(ii) *When p splits into E_1 and E_2 with $c \in n\varpi_p R_p$,*

$$I_{f_p}\left(\pm\frac{n}{c^2}\right) = |n|_p^{-1} \sum_{\substack{x \in R_{E_1}^\times/(1+(c/n)R_{E_1}),\\ y \in R_{E_2}^\times/(1+(c/n)R_{E_2}),\\ xy \in \pm 1 + (c/n)R_p}} \bar{\psi}_p\left(\frac{xn}{c} + \frac{yn}{c}\right).$$

(iii) *When p splits or is inert unramified with $c \notin n\varpi_p R_p$, we have*

$$I_{f_p}\left(\pm\frac{n}{c^2}\right) = \frac{\eta_p(c)}{|c|_p}(1 - p^{-1}).$$

(iv) *$I_{f_p}(\pm n/c^2)$ vanishes otherwise.*

Proof. Part (i) follows from Lemma 4.1. When p splits we have

$$I_{f_p}\left(\pm\frac{n}{c^2}\right) = |n|_p^{-1} \sum_{x \in R_p^\times/(1+(c/n)R_p)} \bar{\psi}_p\left(\frac{xn}{c} \pm \frac{n}{cx}\right).$$

Setting $y = \pm 1/x$ we get (ii). The last two parts are from direct computation.

By Lemma 4.2, we can rewrite the Kuznetsov trace formula in (7) as

$$\begin{aligned}
\sum_{A\in\mathbf{Q}^\times} I_f(A) = \sum_{c=1}^{\infty} I_{f_{\mathbf{R}}}\left(\frac{n}{c^2}\right)
&\cdot \prod_{\substack{p\mid(c/(c,n)) \text{ if } p \text{ is}\\ \text{inert unramified,}\\ p^d\mid(c/(c,n)) \text{ if } p \text{ is}\\ \text{inert ramified}}} \sum_{\substack{\alpha\in R_{E_p}^\times/(1+(c/n)R_{E_p}),\\ \alpha\bar{\alpha}\in 1+(\varepsilon c/n)R_p}} \bar{\psi}_p\circ\mathrm{tr}\left(\frac{\alpha n}{c}\right)\\
&\cdot \prod_{\substack{p\mid(c/(c,n))\\ \text{if } p \text{ splits}}} \sum_{\substack{x\in R_{E_1}^\times/(1+(c/n)R_{E_1}),\\ y\in R_{E_2}^\times/(1+(c/n)R_{E_2}),\\ xy\in 1+(c/n)R_p}} \bar{\psi}_p\left(\frac{xn}{c}+\frac{yn}{c}\right)\\
&\cdot \prod_{p\mid c} e_p(n,c;\tau)\\
+\sum_{c=1}^{\infty} I_{f_{\mathbf{R}}}\left(-\frac{n}{c^2}\right)
&\cdot \prod_{\substack{p\mid(c/(c,n)) \text{ if } p \text{ is}\\ \text{inert unramified,}\\ p^d\mid(c/(c,n))\text{if } p \text{ is}\\ \text{inert ramified}}} \sum_{\substack{\alpha\in R_{E_p}^\times/(1+(c/n)R_{E_p}),\\ \alpha\bar{\alpha}\in -1+(\varepsilon c/n)R_p}} \bar{\psi}_p\circ\mathrm{tr}\left(\frac{\alpha n}{c}\right)\\
&\cdot \prod_{\substack{p\mid(c/(c,n))\\ \text{if } p \text{ splits}}} \sum_{\substack{x\in R_{E_1}^\times/(1+(c/n)R_{E_1}),\\ y\in R_{E_2}^\times/(1+(c/n)R_{E_2}),\\ xy\in -1+(c/n)R_p}} \bar{\psi}_p\left(\frac{xn}{c}+\frac{yn}{c}\right)\\
&\cdot \prod_{p\mid c} e_p(n,c;\tau)
\end{aligned}$$

where

$$e_p(n,c;\tau)=\begin{cases}\eta_p(n)|n|_p^{-1} & \text{if } p \text{ is inert unramified or splits with } p|(c/(c,n)),\\ \eta_p(c)|c|_p^{-1}(1-p^{-1}) & \text{if } p \text{ is inert unramified or splits with } p\nmid(c/(c,n)),\\ \eta_p(n)|n|_p^{-1}p^{-d/2}\lambda_{E_p/\mathbf{Q}_p}(\bar{\psi}_p) & \text{if } p \text{ is inert ramified with } p^d|(c/(c,n)),\\ 0 & \text{if } p \text{ is inert ramified with } p^d\nmid(c/(c,n)).\end{cases}$$

If we set $\beta=\alpha n/(c,n)$, $x_1=xn/(c,n)$ and $y_1=yn/(c,n)$, we can rewrite the sums

$$\sum_{\substack{\alpha\in R_{E_p}^\times/(1+(c/n)R_{E_p}),\\ \alpha\bar{\alpha}\in\pm1+(\varepsilon c/n)R_p}}\bar{\psi}_p\circ\mathrm{tr}\left(\frac{\alpha n}{c}\right)=\sum_{\substack{\beta\in R_{E_p}^\times/(1+(c/(c,n))R_{E_p}),\\ \beta\bar{\beta}\in\pm n^2/(c,n)^2+(\varepsilon c/(c,n))R_p}}\bar{\psi}_p\circ\mathrm{tr}\left(\frac{\beta}{c/(c,n)}\right)\tag{12}$$

and

$$\sum_{\substack{x\in R_{E_1}^\times/(1+(c/n)R_{E_1}),\\ y\in R_{E_2}^\times/(1+(c/n)R_{E_2}),\\ xy\in\pm1+(c/n)R_p}}\bar{\psi}_p\left(\frac{xn}{c}+\frac{yn}{c}\right)=\sum_{\substack{x_1\in R_{E_1}^\times/(1+(c/(c,n))R_{E_1}),\\ y_1\in R_{E_2}^\times/(1+(c/(c,n))R_{E_2}),\\ x_1y_1\in\pm n^2/(c,n)^2+(c/(c,n))R_p}}\bar{\psi}_p\left(\frac{x_1}{c/(c,n)}+\frac{y_1}{c/(c,n)}\right).\tag{13}$$

Here we note that $p|(c/(c,n))$ implies that $p\nmid(n/(c,n))$ and $n/(c,n)\in R_p^\times$. If we denote $\beta=a+b\sqrt{\tau}$, then the conditions

$$\beta\in R_{E_p}^\times/\left(1+\frac{c}{(c,n)}R_{E_p}\right)\quad\text{and}\quad\beta\bar{\beta}\in\pm\frac{n^2}{(c,n)^2}+\frac{\varepsilon c}{(c,n)}R_p$$

are equivalent to

$$a, b \in R_p \Big/ \frac{c}{(c,n)} R_p \quad \text{and} \quad a^2 - \tau b^2 \in \pm n^2/(c,n)^2 + \frac{\varepsilon c}{(c,n)} R_p$$

for all local cases, except when $p = 2$ and $\tau \equiv 5 \pmod 8$. In this exceptional case the conditions on β are equivalent to $a = a_1/2$, $b = b_1/2$,

$$a_1, b_1 \in R_p \Big/ \frac{2c}{(c,n)} R_p, \quad \text{and} \quad a_1^2 - \tau b_1^2 \in \pm \frac{4n^2}{(c,n)^2} + \frac{4c}{(c,n)} R_p.$$

Similarly we may set $x_1 = a + b\sqrt{\tau}$ and $y_1 = a - b\sqrt{\tau}$ when p splits. Then the conditions on x_1 and y_1 attached to the above sum are equivalent to

$$a, b \in R_p \Big/ \frac{c}{(c,n)} R_p \quad \text{and} \quad a^2 - \tau b^2 \in \pm \frac{n^2}{(c,n)^2} + \frac{c}{(c,n)} R_p$$

if $p > 2$, $p \nmid \tau$, $(\tau/p) = 1$, and are equivalent to

$$a = a_1/2, \qquad b = b_1/2, \quad a_1, b_1 \in R_p \Big/ \frac{2c}{(c,n)} R_p,$$

and

$$a_1^2 - \tau b_1^2 \in \pm \frac{4n^2}{(c,n)^2} + \frac{4c}{(c,n)} R_p$$

if $p = 2$ and $\tau \equiv 1 \pmod 8$.

Consequently, assuming $2 \nmid (c/(c,n))$ when $\tau \equiv 1 \pmod 4$, we can take the global product of the sums in (12) and (13) over $p|(c/(c,n))$ and write it as

$$\sum_{\substack{1 \le a \le c/(c,n), \\ 1 \le b \le c/(c,n), \\ a^2 - \tau b^2 \equiv \pm n^2/(c,n)^2 \pmod{\varepsilon c/(c,n)}}} e^{4\pi i a (c,n)/c}$$

except when $2|(c/(c,n))$, $2^d \nmid (c/(c,n))$ and $\tau \equiv 2$ or $3 \pmod 4$. Since in the last cases we have $\prod_{p|c} e_p(n, c; \tau) = 0$, these exceptional cases do not matter.

Now we consider the global product of the sums in (12) and (13) taken over $p|(c/(c,n))$ when $2|(c/(c,n))$ and $\tau \equiv 1$ or $5 \pmod 8$. In either case we have to use a_1 and b_1 to rewrite the sum in (12) or (13) as

$$\frac{1}{2} \sum_{\substack{a_1,b_1 \in R_p^\times/(1+(2c/(c,n))R_p), \\ a_1^2-\tau b_1^2 \in \pm 4n^2/(c,n)^2+(4c/(c,n))R_p}} \bar{\psi}_p\left(\frac{a_1+b_1\sqrt{\tau}}{2c/(c,n)} + \frac{a_1-b_1\sqrt{\tau}}{2c/(c,n)}\right).$$

Here we have the coefficient $1/2$ because the new sum covers the original sum twice. Therefore the corresponding global product is equal to

$$\frac{1}{2} \sum_{\substack{1 \le a \le 2c/(c,n), \\ 1 \le b \le 2c/(c,n), \\ a^2-\tau b^2 \equiv \pm 4n^2/(c,n)^2 \pmod{4c/(c,n)}}} e^{2\pi i a(c,n)/c}.$$

Substituting the above results into the Kuznetsov trace formula and using our computation of $I_{f_{\mathbf{R}}}(A)$ in Section 2, we get

$$\begin{aligned}\sum_{A \in \mathbf{Q}^\times} I_f(A) = & \frac{1}{2} \sum_{c=1}^{\infty} f_4\left(\frac{n}{c^2}\right) \\ & \cdot \sum_{\substack{1 \le a \le 2c/(c,n), \\ 1 \le b \le 2c/(c,n), \\ a^2-\tau b^2 \equiv 4n^2/(c,n)^2 \pmod{4c/(c,n)}}} e^{2\pi i a(c,n)/c} \cdot \prod_{p|c} e_p(n,c;\tau) \\ & + \frac{1}{2} \sum_{c=1}^{\infty} f_4\left(-\frac{n}{c^2}\right) \\ & \cdot \sum_{\substack{1 \le a \le 2c/(c,n), \\ 1 \le b \le 2c/(c,n), \\ a^2-\tau b^2 \equiv -4n^2/(c,n)^2 \pmod{4c/(c,n)}}} e^{2\pi i a(c,n)/c} \cdot \prod_{p|c} e_p(n,c;\tau)\end{aligned}$$

when $2|(c/(c,n))$ and $\tau \equiv 1 \pmod 4$, and

$$\sum_{A\in\mathbf{Q}^\times} I_f(A) = \sum_{c=1}^{\infty} f_4\Big(\frac{n}{c^2}\Big) \cdot \sum_{\substack{1\le a\le c/(c,n),\\ 1\le b\le c/(c,n),\\ a^2-\tau b^2\equiv n^2/(c,n)^2 \pmod{\varepsilon c/(c,n)}}} e^{4\pi i a(c,n)/c}\cdot\prod_{p|c} e_p(n,c;\tau) + \sum_{c=1}^{\infty} f_4\Big(-\frac{n}{c^2}\Big) \cdot \sum_{\substack{1\le a\le c/(c,n),\\ 1\le b\le c/(c,n),\\ a^2-\tau b^2\equiv -n^2/(c,n)^2 \pmod{\varepsilon c/(c,n)}}} e^{4\pi i a(c,n)/c}\cdot\prod_{p|c} e_p(n,c;\tau)$$

otherwise. We note that the exponential sums above can be regarded as the lifting of the Kloosterman sums $K(n,\pm n;c;\eta)$ over the quadratic number field $E=\mathbf{Q}(\sqrt{\tau})$ (Ye [**19**]). We will denote them as

$$Kh_\pm(n,c;\tau) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\substack{1\le a\le 2c/(c,n),\\ 1\le b\le 2c/(c,n),\\ a^2-\tau b^2\equiv \pm 4n^2/(c,n)^2 \pmod{4c/(c,n)}}} e^{2\pi i a(c,n)/c} & \text{if } 2|(c/(c,n)) \text{and } \tau\equiv 1 \pmod 4,\\ \displaystyle\sum_{\substack{1\le a\le c/(c,n),\\ 1\le b\le c/(c,n),\\ a^2-\tau b^2\equiv \pm n^2/(c,n)^2 \pmod{2c/(c,n)}}} e^{4\pi i a(c,n)/c} & \text{if } 2|(c/(c,n)) \text{and } \tau\equiv 2,3 \pmod 4,\\ \displaystyle\sum_{\substack{1\le a\le c/(c,n),\\ 1\le b\le c/(c,n),\\ a^2-\tau b^2\equiv \pm n^2/(c,n)^2 \pmod{c/(c,n)}}} e^{4\pi i a(c,n)/c} & \text{otherwise.} \end{cases}$$

We also note that the product $\prod_{p|c} e_p(n,c;\tau)$ vanishes when either (i) $2|(c/(c,n))$, $4\nmid (c/(c,n))$, and $\tau\equiv 3 \pmod 4$, or (ii) $2|(c/(c,n))$,

$8 \nmid (c/(c,n))$ and $\tau \equiv 2 \pmod 4$. Now we can finally establish a Kuznetsov formula over the quadratic number field $E = \mathbf{Q}(\sqrt{\tau})$:

Theorem 4.1. *Let f_4 be a smooth function of compact support in $\mathbf{R}^\times$. For a positive integer $m = n$ we define functions f_1, f_2, f_3, $f_{\mathbf{R}}$, f_p, $p < \infty$, and f as in Section 2. Then*

$$\sum_{c=1}^{\infty} f_4\left(\frac{n}{c^2}\right) Kh_+(n,c;\tau) \prod_{p|c} e_p(n,c;\tau) + \sum_{c=1}^{\infty} f_4\left(-\frac{n}{c^2}\right) Kh_-(n,c;\tau) \prod_{p|c} e_p(n,c;\tau)$$
$$= \int_{\mathbf{Q}\backslash\mathbf{Q_A}} \int_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f^{\mathrm{cusp}} \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \right) \psi(y+nx)\, dx\, dy + \int_{\mathbf{Q}\backslash\mathbf{Q_A}} \int_{\mathbf{Q}\backslash\mathbf{Q_A}} K_f^{\mathrm{eis}} \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \right) \psi(y+nx)\, dx\, dy.$$

The spectral side of this formula can also be lifted to E according to the relative trace formula in Ye [**17**].

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