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A CLASS OF PRÜFER DOMAINS THAT ARE SIMILAR TO THE RING OF ENTIRE FUNCTIONS

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1. Introduction. Let R be the ring of entire functions, and let $\mathbf{K}$ be the field of complex numbers. Much is known concerning the algebraic properties of R. For example, Helmer proved [**5**, Theorem 9] that R is a Bezout domain. Henriksen [**6**] proved that R is infinite dimensional and completely characterized the prime ideals. Theorems concerning the algebraic structure of R tend to focus on the sets of zeros of functions in R. Let α be a complex number, and let M_α be the ideal of R generated by $z-\alpha$. Then an entire function $f(z)$ lies in M_α if and only if $f(\alpha)=0$. Hence, properties of the zeros of entire functions are largely embodied in the properties of the ideals M_α. Several additional facts are readily apparent concerning these ideals.

1. Each M_α is maximal in R.
2. R_{M_α} is a Noetherian valuation domain for each $\alpha \in \mathbf{K}$.
3. $R = \cap_{\alpha\in\mathbf{K}} R_{M_\alpha}$.

In this paper we consider a class of Prüfer domains which will be defined by intersecting Noetherian valuation domains in such a way that the centers of the defining valuation domains emulate the ideals M_α of R. These domains, which we call E-domains, have many properties in common with R. In Section 2 we consider some basic properties concerning the prime ideals of E-domains and investigate the structure of divisorial ideals. In each case we will draw comparisons with the structure of R. In Section 3 we show how E-domains can be constructed as overrings of Noetherian domains and investigate the relationship between the ideal structure of an E-domain constructed in this manner and the ideal structure of the underlying Noetherian domain. In Section 4 we consider some explicit examples of E-domains. We also use our knowledge of E-domains to construct an example of a

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Prüfer domain D with a prime ideal P such that P is divisorial and P^2 is not. Throughout the paper we will pose questions, many of which we will not resolve.

The term ideal will refer to an integrated ideal. When fractional ideals are being considered, they will be specified as such. Also, the term prime ideal will refer to a proper, nonzero, prime ideal.

2. E-domains.

To begin, we define non-D-ring as in [**8**, Definition 1] and [**9**, Definition 1.1].

Definition 2.1. A domain D will be called a *non-D-ring* provided there exists a nonconstant polynomial $f(x)$ in $D[x]$ such that $f(d)$ is a unit in D for each $d \in D$. The polynomial $f(x)$ will be called a *uv-polynomial* (for unit valued).

Now we are ready to define E-domain.

Definition 2.2. Let D be a domain with quotient field F. We call D an *E-domain* provided it satisfies the following conditions.

1. $W = \{V_\alpha \mid \alpha \in \Lambda\}$ is a collection of Noetherian valuation overrings of D.

2. For each $\alpha \in \Lambda$, v_α is the normed valuation on F corresponding to V_α, M_α is the maximal ideal of V_α and $P_\alpha = M_\alpha \cap D$.

3. $D = \cap_{\alpha\in\Lambda} V_\alpha$.

4. There exists a monic polynomial $f(x) \in D[x]$ of degree $n \geq 2$ which is a uv-polynomial for each V_α.

5. For each $\alpha \in \Lambda$ there exists an element $d_\alpha \in D$ such that $v_\alpha(d_\alpha) > 0$ and $v_\beta(d_\alpha) = 0$ whenever $\beta \in \Lambda$ and $\beta \neq \alpha$.

For the duration of Section 2 we assume the notation of Definition 2.2.

Our stated objective is to study E-domains along the same lines that R has been studied. Accordingly, we classify the ideals of E-domains using the terminology of [**6**].

Definition 2.3. Call W the set of *fixed valuation overrings* of D. (We show in Proposition 2.7 that W is uniquely determined.) We call the ideals $\{P_\alpha \mid \alpha \in \Lambda\}$, the *maximal fixed ideals* of D (this terminology will be justified in Proposition 2.8). If I is an ideal of D we call I *fixed* if $I \subseteq P_\alpha$ for some maximal fixed ideal P_α, and otherwise we say that I is *free*.

The conditions placed on D and W in Definition 2.2 are a stronger version of conditions considered in [**9**]. Hence, we can draw some immediate inferences.

Proposition 2.4. *D is a Prüfer non-D-ring with $f(x)$ serving as a uv-polynomial.*

Proof. This follows immediately from [**9**, Corollary 2.6]. □

Proposition 2.5. *Let I be a finitely generated ideal of D. Then I^{n^t} is principal for some nonnegative integer t.*

Proof. This follows immediately from [**9**], Theorem 2.5]. □

We also need some results of Gilmer and Heinzer [**4**] on Prüfer domains which can be expressed as irredundant intersections of valuation domains.

A representation of a domain T as an intersection of valuation domains (say $T = \cap_{\alpha\in\Omega} V_\alpha$) is *irredundant* if the omission of one valuation domain from the intersection changes the result, i.e., $T_\beta = \cap_{\alpha\in\Omega,\alpha\neq\beta} V_\alpha \neq T$ for each $\beta \in \Omega$. The following result then follows immediately from condition 5 of Definition 2.2.

Lemma 2.6. *$D = \cap_{\alpha\in\Lambda} V_\alpha$ is an irredundant representation of D.*

We now make direct application of several results from [**4**] to our present setting.

Proposition 2.7. *$D = \cap_{\alpha \in \Lambda} V_\alpha$ is the unique irredundant representation of D.*

Proof. See [**4**, Corollary 1.9]. □

Proposition 2.8. *The maximal fixed ideals, $\{P_\alpha \mid \alpha \in \Lambda\}$ are maximal ideals of D.*

Proof. See [**4**, Theorem 1.7 and Corollary 1.8]. □

Proposition 2.9. *If I is a finitely generated ideal of D which is contained in only finitely many maximal fixed ideals $P_1, P_2, \ldots, P_t$ of D, then $P_1, P_2, \ldots, P_t$ are the only maximal ideals of D which contain I.*

Proof. See [**4**, Corollary 1.11]. □

The maximal fixed ideals of R are all principal and the free ideals all require infinitely many generators. It seems natural then to ask if the fixed and free ideals of D have corresponding properties.

Proposition 2.10. *Every maximal fixed ideal of D is generated by two elements.*

Proof. Let P_α be a maximal fixed ideal of D. Use condition 5 of Definition 2.2 to find an element $d_\alpha \in P_\alpha$ such that P_α is the only maximal fixed ideal of D which contains d_α. Proposition 2.9 implies that P_α is the only maximal ideal of D which contains d_α. Since P_α extends to the principal ideal M_α in V_α, then [**3**, Lemma 37.3] implies immediately that P_α is finitely generated. The stronger statement that only two generators are required follows easily from examination of the proof of [**3**, Lemma 37.3]. □

The P_α-primary ideals can also be neatly characterized for each maximal fixed ideal P_α.

Proposition 2.11. *Suppose that P_α is a maximal fixed ideal of D and that I is a P_α-primary ideal. Then $I = P_\alpha^t$ for some positive integer t.*

Proof. The result follows from the fact that P_α is maximal and D_{P_α} is a Noetherian valuation domain. See the $a \to b$ argument for [**3**, Theorem 36.4]. □

Now we address the generation of free ideals.

Proposition 2.12. *No free ideal of D is finitely generated.*

Proof. Suppose that I is an ideal of D which is finitely generated. Proposition 2.5 implies that I^m is principal for some $m \in Z^+$. Suppose that $I^m = (d)$. Since d is a nonunit in D, then d must lie in some maximal fixed ideal. Hence, I lies in a maximal fixed ideal and so is itself fixed. □

In R, each principal ideal $(f(z))$ is completely determined by its set of zeros, each taken with the correct multiplicity. Hence, each principal ideal can be uniquely identified with a collection of powers of maximal fixed ideals. In R, it is also the case that the set of principal ideals is identical to the set of all divisorial ideals. (Recall that an ideal is divisorial if it can be expressed as an intersection of principal fractional ideals.) This inspires the following result.

Proposition 2.13. *Suppose that I is a divisorial ideal of D. Then I can be expressed in a unique way as an intersection* (*possibly infinite*) *of primary fixed ideals. Conversely, each nonzero intersection of primary fixed ideals of D is divisorial.*

Proof. We will prove the second part of the statement first. So, let $\{P_\alpha \mid \alpha \in \Lambda_1 \subseteq \Lambda\}$ be a collection of maximal fixed ideals of D, and let $\{e_\alpha \mid \alpha \in \Lambda_1\}$ be a collection of positive integers. Let $J = \cap_{\alpha \in \Lambda_1} P_\alpha^{e_\alpha}$ and suppose that J is nonzero. Proposition 2.10 states that each maximal fixed ideal of D is finitely generated. It is well known that any finitely generated ideal of a Prüfer domain is divisorial and that any nonzero intersection of divisorial ideals is divisorial. Hence, J is divisorial.

The proof of the first assertion of our result will be accomplished in four stages. First we show that any principal ideal can be expressed as an intersection of primary fixed ideals. Then we prove the same result for finitely generated ideals and then we extend to divisorial ideals. Finally, we show that the primary decomposition of a divisorial ideal of D using fixed primary ideals of D is unique.

First, assume that (d) is a nonzero principal ideal of D. Let $\{P_\alpha \mid \alpha \in \Lambda_2 \subseteq \Lambda\}$ be the set of all maximal fixed ideals in which d is a nonunit. Then for each $\alpha \in \Lambda_2$ set $e_\alpha = v_\alpha(d)$. Then let $I = \cap_{\alpha\in\Lambda_2} P_\alpha^{e_\alpha}$. We claim that $(d) = I$. $(d) \subseteq I$ is clear. Suppose that $r \in I$. Then $v_\alpha(r) \geq v_\alpha(d)$ for each $\alpha \in \Lambda_2$ and $v_\alpha(d) = 0$ for each $\alpha \in \Lambda - \Lambda_2$. Hence, $r/d \in D$ and so $I \subseteq (d)$.

Second, assume that I is a finitely generated ideal of D. Proposition 2.5 implies that I^m is principal for some $m \in Z^+$. Let $(d) = I^m$. Let $\{P_\alpha \mid \alpha \in \Lambda_3 \subseteq \Lambda\}$ be the set of all maximal fixed ideals of D which contain I. (Note that this is also the set of all maximal fixed ideals of D which contain d.) Then for each $\alpha \in \Lambda_3$ set $e_\alpha = \min\{v_\alpha(r) \mid r \in I\}$. It follows easily that $me_\alpha = v_\alpha(d)$ for each $\alpha \in \Lambda_3$. Let $J = \cap_{\alpha\in\Lambda_3} P_\alpha^{e_\alpha}$. We claim that $I = J$. $I \subseteq J$ is clear. Suppose that $r \in J$. It is clear that $(I, r)^m \subseteq (d) = I^m$. Since I and (I, r) are finitely generated, this implies that $I = (I, r)$ and hence, $r \in I$ and so $J \subseteq I$.

Third, assume that I is a divisorial ideal of D. Then I can be expressed as an intersection of principal fractional ideals of D. Let (d) be a principal fractional ideal of D. Since D is Prüfer, $D \cap (d)$ is finitely generated. Hence, I can be expressed as an intersection of finitely generated ideals of D. Since each finitely generated ideal can be expressed as an intersection of primary fixed ideals and I can be expressed as an intersection of finitely generated ideals, it follows that I can be expressed as an intersection of primary fixed ideals.

Finally, assume that I is a divisorial ideal. We show that the representation of I as an intersection of primary fixed ideals is unique. Suppose not. Without loss of generality, we can suppose that there exist two subsets S_1 and S_2 of Λ such that $S_1 \subseteq S_2$ and that $I = \cap_{\alpha\in S_1} P_\alpha^{e_\alpha} = \cap_{\alpha\in S_2} P_\alpha^{f_\alpha}$ with $e_\alpha \leq f_\alpha$ for all $\alpha \in S_1$, and either $S_1 \neq S_2$ or $e_\alpha < f_\alpha$

for some $\alpha \in S_1$. Choose $\alpha \in S_2$ such that either $\alpha \notin S_1$ or $e_\alpha \leq f_\alpha$. In either case consideration of the two intersection expressions and the fact that maximal fixed ideals of D are all maximal ideals leads to the conclusion that $P_\alpha I = I$. This implies that if $d \in I$, then $v_\alpha(d) \geq m$ for each positive integer m. This is a contradiction. Hence the primary decomposition representation of I using primary fixed ideals is unique. □

A free ideal was defined to be a nonzero ideal which was not fixed and a fixed ideal was defined to be a nonzero ideal which was contained in a maximal fixed ideal. The following corollary then follows immediately.

Corollary 2.14. *Suppose that I is a free ideal of D. Then I is not divisorial.*

In R, each divisorial ideal is principal. It is not apparent that an arbitrary divisorial ideal of D is principal, or even invertible. We make two comments in this regard and pose two questions.

First we note that if I is an arbitrary fixed ideal of R which is not divisorial, then I_v (the intersection of all principal fractional ideals containing I) is divisorial and hence principal. This leads to a very natural factorization of I into a product of a divisorial ideal (I_v) and a free ideal $(I)(I_v)^{-1}$.

Question 1. Let I be a fixed, nondivisorial ideal of an E-domain D. Can I be factored into a product of a fixed divisorial ideal and a free ideal? If so, is the representation unique?

Also, we note that if D is Bezout, then each maximal fixed ideal is principal. In this case the primary decomposition of a divisorial ideal I of D can be associated in a very natural way with an infinite collection of powers of generators of maximal fixed ideals. This is suggestive of a formal version of the Weierstrass factorization theorem [**11**, Theorem 15.10] which represents any given entire function as an infinite product of exponential functions (units) and polynomials.

Question 2. Let D be a Bezout E-domain. Does there exist an extension of D (comparable to the extension from polynomials to entire functions) which is an E-domain, has maximal fixed ideals corresponding to those of D, and which has each divisorial ideal being principal?

We close this section by observing that no indication has been given that free ideals should exist. If the intersection, $D = \cap_{\alpha\in\Lambda} V_\alpha$, is locally finite, then D is both Prüfer and Krull and hence must be Dedekind. In this case, every ideal of D is fixed. Conversely, if the intersection $D = \cap_{\alpha\in\Lambda} V_\alpha$ is not locally finite, then some nonzero element of D must lie in an infinite number of prime ideals of D. This implies that D is not Dedekind, and since D is Prüfer it cannot be Noetherian. Then since D is not Noetherian, D must contain a prime ideal which is not finitely generated. Hence, D must contain a free prime ideal since the only fixed prime ideals are the maximal fixed ideals, which are all finitely generated. We summarize this in the following proposition.

Proposition 2.15. *The E-domain, D, contains a nontrivial free ideal if and only if the intersection $D = \cap_{\alpha\in\Lambda} V_\alpha$ is not locally finite.*

3. E-overrings of Noetherian domains.

In Section 2 no indication was given that nontrivial examples of E-domains exist. The construction of explicit examples will be deferred until Section 4. In this section, however, we lay the groundwork for some of the examples which will be presented in Section 4.

We begin by recalling some necessary terminology and results regarding non-D-rings from [**7, 8**] and [**9**].

Definition 3.1. See [**7**, Definition 1.5]. Let T be a domain, $P \subseteq T$ a prime ideal and $f(x) \in T[x]$ a nonconstant polynomial. We say that P is an *f-non-D-ideal* of T provided $f(x)$ is a uv-polynomial for T_P.

Proposition 3.2. See [**7**, Proposition 1.12]. *Let T be a domain, a and b nonzero elements of T and $f(x) \in T[x]$ a monic polynomial of degree $n \geq 2$. If $P \subseteq T$ is an f-non-D-ideal of T, then $b^n f(a/b) \in P$ if and only if $a, b \in P$.*

Corollary 3.3. See [**7**, Theorem 2.11]. *Let T be a Noetherian domain, $f(x) \in T[x]$ a monic polynomial of degree $n \geq 2$, and $P \subseteq R$ an f-non-D-ideal of T. Then there exists an element $a \in P$ such that if P_1 is an f-non-D-ideal of T and $a \in P_1$, then $P \subseteq P_1$.*

Proposition 3.4. See [**8**, Proposition 3]. *Let T be a Noetherian domain, $f(x) \in T[x]$ a nonconstant polynomial and V a Noetherian valuation overring of T with maximal ideal Q_1. Suppose that Q_1 is an f-non-D-ideal of V. Then $P_1 = Q_1 \cap T$ is an f-non-D-ideal of T.*

Proposition 3.5 [**9**, Proposition 2.1]. *Let T be a Prüfer non-D-ring with monic uv-polynomial $f(x) \in T[x]$. Then every prime ideal of T is an f-non-D-ideal of T.*

Proposition 3.6 [**9**, Corollary 2.3]. *Let V be a valuation domain with quotient field F and with $f(x) \in V[x]$ a monic uv-polynomial for V of degree $n \geq 2$. Also, let v be a valuation on F corresponding to V. If $a, b \in V - \{0\}$, then $v(b^n f(a/b) = \min\{nv(a), nv(b)\}$.*

The following result is new, but is in the same spirit as the above results concerning non-D-rings.

Proposition 3.7. *Let T be a domain, and let $f(x) \in T[x]$ be a monic polynomial of degree $n \geq 2$. Also let $\{J_\alpha \mid \alpha \in \Omega\}$ be a collection of f-non-D-ideals of T such that $J = \cap_{\alpha \in \Omega} J_\alpha$ is a prime ideal of T. Then J is an f-non-D-ideal of T.*

Proof. Let $d_1, d_2 \in T$ such that $d_2 \notin J$. We need to show that $f(x)$ is a uv-polynomial for T_J. Hence it will suffice to show that $f(d_1/d_2)$ is a unit in T_J. Since $d_2 \notin J$, then $d_2 \notin J_\alpha$ for some $\alpha \in \Omega$. Hence, $d_1/d_2 \in T_{J_\alpha}$ and so $f(d_1/d_2)$ is a unit in T_{J_α}. It follows that $f(d_1/d_2)$ is a unit in T_J. □

Now we proceed to consideration of E-overrings of Noetherian domains. We begin with some terminology related to collections of Noetherian valuation overrings.

Definition 3.8. Suppose that T is a Noetherian domain and that $\{V_\alpha \mid \alpha \in \Lambda\}$ is a collection of Noetherian valuation overrings of T. Let $D = \cap_{\alpha\in\Lambda} V_\alpha$ and for each $\alpha \in \Lambda$, let M_α be the maximal ideal of V_α. We say that D is a *T-weakly-locally finite intersection* provided, for any $\beta \in \Lambda$, the set $\{M_\alpha \mid M_\beta \cap T \subseteq M_\alpha \cap T\}$ is finite. If it is clear what the relevant Noetherian underring T is, we will simply say that D is a weakly locally finite intersection.

The following is a list of assumptions that will hold for the remainder of Section 3.

Assumptions. 1. *T is a Noetherian domain with quotient field F.*

2. *$f(x)$ is a monic polynomial in $T[x]$ of degree $n \geq 2$.*

3. *$W = \{V_\alpha \mid \alpha \in \Lambda\}$ is a collection of Noetherian valuation overrings of T such that $f(x)$ is a uv-polynomial for each V_α.*

4. $D = \cap_{\alpha\in\Lambda} V_\alpha$.

5. *For each $\alpha \in \Lambda$ we say M_α is the maximal ideal of V_α,*

$$P_\alpha = M_\alpha \cap D, \qquad J_\alpha = M_\alpha \cap T$$

and v_α is the normed valuation on F corresponding to V_α.

6. *The intersection $D = \cap_{\alpha\in\Lambda} V_\alpha$ is weakly locally finite.*

Next we recall a result from [**8**, Proposition 5]. The hypotheses there required that the intersection of Noetherian valuation domains be locally finite. However, the assumption of local finiteness was used only to obtain weak local finiteness. Hence the proof from [**8**] is valid in this setting.

Proposition 3.9. *For each $\alpha \in \Lambda$ there exists an element $d_\alpha \in D$ such that $v_\alpha(d_\alpha) > 0$ and $v_\beta(d_\alpha) = 0$ whenever $\beta \neq \alpha$.*

Corollary 3.10. *D is an E-domain with $\{P_\alpha \mid \alpha \in \Lambda\}$ serving as the maximal fixed ideals.*

Proof. Assumptions 1–5 together with Proposition 3.9 are essentially a restatement of the definition of E-domain. □

Recall that Proposition 2.15 states that D contains free ideals provided the intersection $D = \cap_{\alpha\in\Lambda} V_\alpha$ is not locally finite. Now we examine the relationship of the prime free ideals of D to the prime ideals of T.

Lemma 3.11. *Let Q be a prime free ideal of D. Then $Q \cap T \subseteq J_\alpha$ for some $\alpha \in \Lambda$.*

Proof. Suppose that $Q \cap T \not\subseteq J_\alpha$ for all $\alpha \in \Lambda$. Proposition 3.5 implies that Q is an f-non-D-ideal of D and then Proposition 3.4 implies that $Q \cap T$ is an f-non-D-ideal of T. Use Corollary 3.3 to choose an element $r_\alpha \in Q \cap T$ such that if Q_1 is an f-non-D-ideal of T and $r_\alpha \in Q_1$, then $Q \subseteq Q_1$. Propositions 3.4 and 3.5 also imply that J_α is an f-non-D-ideal for each $\alpha \in \Lambda$. Then our supposition forces $r_\alpha \notin J_\alpha$ for all $\alpha \in \Lambda$. This implies that r_α is not contained in any maximal fixed ideal of D, and so must be a unit in D. This is a contradiction. □

Lemma 3.12. *Let Q be a prime free ideal of D. Then $Q \cap T \neq J_\alpha$ for all $a \in \Lambda$.*

Proof. Suppose that $Q \cap T = J_\alpha$ for some $\alpha \in \Lambda$. Again, use Corollary 3.3 to choose an element $r_\alpha \in J_\alpha$ such that if Q_1 is an f-non-D-ideal of T and $r_\alpha \in Q_1$, then $J_\alpha \subseteq Q_1$. Since the intersection $D = \cap_{\alpha\in V} V_\alpha$ is weakly locally finite, $J_\alpha \subseteq M_\beta \cap T$ for only finitely many $\beta \in \Lambda$. It follows that $r_\alpha \in M_\beta \cap T$ for only finitely many $\beta \in \Lambda$. Let $\{M_1, M_2, \dots, M_t\}$ be this finite set. Let $P_i = M_i \cap D$ for $1 \le i \le t$. Proposition 2.9 implies that $\{P_1, P_2, \dots, P_t\}$ is precisely the collection of all maximal ideals of D which contain r_α. Then $r_\alpha \in Q$ implies that $Q \subseteq P_i$ for some $1 \le i \le t$. This is a contradiction since each P_i is a maximal fixed ideal and, hence, by definition, cannot contain a free ideal. □

The previous two lemmas can be summarized into the following result.

Proposition 3.13. *Suppose that Q is a prime free ideal of D. Then the contraction of Q to T is properly contained in the contraction to T of some maximal fixed ideal of D.*

Corollary 3.14. *Suppose that T has dimension 2. Then D is an almost Dedekind domain.*

Proof. The proof is accomplished by showing that D_P is a Noetherian valuation domain for every prime ideal P in D. This is already known when P is a maximal fixed ideal. Suppose that P is a free prime ideal of D. Proposition 3.13 implies that $P \cap T$ must be a height one prime of T. Then D_P is an overring of the one-dimensional Noetherian domain $T_{P \cap T}$, and hence must be a Noetherian valuation domain. □

Proposition 3.13 places a strong restriction on which primes of T can lie under free primes of D. We now question which primes of T actually do appear as the contractions of free prime ideals of D.

Proposition 3.15. *Suppose that Λ^* is an infinite subset of Λ such that $J = \cap_{\alpha \in \Lambda^*} J_\alpha$ is a prime ideal of T. Then there exists a prime free ideal P of D such that $P \cap T = J$.*

Proof. First note that if P is a prime ideal of D such that $P \cap T = J$, then P must be free because of the assumption of weak local finiteness. Let $G^* = \{J_\alpha \mid \alpha \in \Lambda^*\}$. Note that Propositions 3.4 and 3.5 imply that each $J_\alpha \in G^*$ is an f-non-D-ideal of T. Then Proposition 3.7 implies that J is an f-non-D-ideal of T. Expand G^* to the set $G^{**} = \{J_\beta \mid \beta \in \Omega\}$ consisting of all f-non-D-ideals of T which properly contain J. For each $\beta \in \Omega$, apply Corollary 3.3 to obtain an element $r_\beta \in J_\beta$ such that any f-non-D-ideal of T which contains r_β must contain J_β. Let $S_1 = \{r_\beta \mid \beta \in \Omega\}$, let S_1^* be the multiplicative set in D generated by S_1, and let $D_1 = (S_1^*)^{-1}D$ be the localization of D at S_1^*. We want to show that $JD_1 \neq D_1$. Suppose $JD_1 = D_1$. Then there exists a finite subset S_2 of S_1 such that if S_2^* is the multiplicative set generated by S_2 and $D_2 = (S_2^*)^{-1}D$, then $JD_2 = D_2$. This is impossible. To see this, let $S_2 = \{r_1, r_2, \dots, r_t\}$, and let $r = \prod_{i=1}^t r_i$. Since J is an f-non-D-ideal of T and since we applied Corollary 3.3

in choosing the elements r_β, then $r_i \notin J$ for $1 \leq i \leq t$, and so $r \notin J$. Hence, there must exist $\alpha \in \Lambda^*$ such that r is a unit in V_α. It follows that $J_\alpha D_2 \neq D_2$ and since $J \subseteq J_\alpha$, then $JD_2 \neq D_2$. So $JD_1 \neq D_1$. JD_1 is hence a proper ideal of D_1 and so must be contained in a maximal ideal M of D_1. Proposition 3.5 implies that M is an f-non-D-ideal of D_1 and Proposition 3.4 implies that M is centered on an f-non-D-ideal of T. $J \subseteq M \cap T$, but our choice of S_1 then forces $J = M \cap T$. Let $P = M \cap D$. □

Proposition 3.15 provides a means of identifying free prime ideals of E-domains. It leaves many questions open, however. We close this section with several such questions.

Question 3. Are the free prime ideals described in Proposition 3.15 all maximal free ideals?

Question 4. Are all of the maximal free ideals of D of the type described in Proposition 3.15?

Question 5. Can the set of maximal free ideals of D be characterized in terms of the ideal structure of T?

Question 6. Can there be free prime ideals of D which are not of the type described in Proposition 3.15?

4. Examples. We begin this section by describing a class of valuation domains which we will use in our constructions. These valuation domains are actually the valuation overrings of Int (Z), the ring of integer-valued polynomials over Z. The facts listed in Proposition 4.1 and Proposition 4.2 are all either well known, or are contained in or are easy consequences of [**1**, Proposition 2.2].

Terminology. 1. For a given prime number $p \in Z$, let $\hat{Z}_p$ be the p-adic completion of Z.

2. For a given prime number $p \in Z$ and a given p-adic integer $\alpha \in \hat{Z}_p$, let $V_{\alpha,p} = \{f(x)/g(x) \mid f(x), g(x) \in Z[x] \text{ and } f(\alpha)/g(\alpha) \in \hat{Z}_p\}$, and let

$M_{\alpha,p} = \{f(x)/g(x) \mid f(x), g(x) \in Z[x] \text{ and } f(\alpha)/g(\alpha) \in p\hat{Z}_p\}$.

3. For a given nonconstant, irreducible polynomial $f(x) \in Z[x]$, let $V_f = \{h(x)/g(x) \mid h(x), g(x) \in Z[x] \text{ and } f(x) \nmid g(x)\}$, and let $M_f = \{(h(x)f(x))/g(x) \mid h(x), g(x) \in Z[x] \text{ and } f(x) \nmid g(x)\}$.

Now we list the properties of $V_{\alpha,p}$ and V_f which we will need.

Proposition 4.1. *Let $p \in Z^+$ be prime, and let $\alpha \in \hat{Z}_p$. Then*

a) *$V_{\alpha,p}$ is a discrete valuation overring of $Z[x]$ with maximal ideal $M_{\alpha,p}$.*

b) *$V_{\alpha,p}$ has dimension one if α is transcendental over $\mathbf{Q}$ and has dimension two if α is algebraic over $\mathbf{Q}$.*

c) *The residue field of $V_{\alpha,p}$ is the field of p elements.*

d) *Suppose that α is transcendental over $\mathbf{Q}$, and let $v_{\alpha,p}$ be the normed valuation on $\mathbf{Q}(x)$ associated with $V_{\alpha,p}$. If $f(x)/g(x) \in V_{\alpha,p} - \{0\}$, then $v_{\alpha,p}(f(x)/g(x))$ is equal to the p-adic value of $f(\alpha)/g(\alpha)$. In particular, $v_{\alpha,p}(p) = 1$.*

e) *If $m \in Z$ and $\alpha - m \in p\hat{Z}_p$, then $V_{\alpha,p}$ is centered on the ideal $(x - m, p)$ of $Z[x]$.*

f) *If α is algebraic over $\mathbf{Q}$, then the unique one-dimensional valuation overring of $V_{\alpha,p}$ has the form V_f for some $f(x) \in Z[x]$. In particular, $f(x)$ is an irreducible polynomial which has α as a root.*

Proposition 4.2. *Let $f(x) \in Z[x]$ be a nonconstant irreducible polynomial. Then V_f is a one-dimensional Noetherian valuation overring of $Z[x]$ with maximal ideal M_f.*

Now we use the above valuation domains to construct some examples of E-domains. As noted above, the valuation domains of the form $V_{\alpha,p}$ and V_f are valuation overrings of $\operatorname{Int}(Z)$. In fact, every valuation overring of $\operatorname{Int}(Z)$ has one of these two forms [**1**, Proposition 2.2]. This facilitates the identification of prime ideals of E-domains constructed by intersecting these valuation domains.

Example 4.3. Let $\{p_1, p_2, p_3, \dots\}$ be a collection of distinct prime numbers. Also, let $f(x) \in Z[x]$ be a monic, nonconstant, irreducible polynomial which does not have a root in Z modulo any of the primes p_i. Then, for each $i \in Z^+$, choose a finite collection $\{\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,m_i}\}$ of p_i-adic integers (each transcendental over $\mathbf{Q}$) from Z_{p_i}. Let $D = \cap_{i=0}^{\infty}(\cap_{j=0}^{m_i} V_{\alpha_{i,j},p_i})$. Then D satisfies assumptions 1–6 following Example 3.8 with $f(x)$ serving as the uv-polynomial, and so Corollary 3.10 implies that D is an E-domain with $\{P_{\alpha_{i,j},p_i} = M_{\alpha_{i,j},p_i} \cap D \mid 1 \le i \le \infty, 1 \le j \le m_i\}$ serving as the maximal fixed ideals. Since $Z[x]$ has dimension two, D is an almost Dedekind domain by Corollary 3.14. Moreover, D contains nontrivial free ideals, i.e., is not Dedekind, provided the given intersection of valuation domains is not locally finite.

This construction can be accomplished in many ways. We focus on one specific simple example which we will then utilize in the next example. Let $\{p_1, p_2, p_3, \dots\}$ be the collection of all prime numbers which are congruent to 3 modulo 4 except for 3 itself. That is, $\{p_1, p_2, \dots\} = \{7, 11, 19, 23, 31, \dots\}$. For each p_i, choose one p_i-adic integer α_i which is transcendental over $\mathbf{Q}$ so that $M_{\alpha_i,p_i} \cap Z[x] = (x, p_i)$. Let $D = \cap_{i=0}^{\infty} V_{\alpha_i,p_i}$. As above, D is both an E-domain and an almost Dedekind domain. For each i, let $V_i = V_{\alpha_i,p_i}$, $M_i = M_{\alpha_i,p_i}$ and $P_i = M_i \cap D$. Then the set $\{P_i \mid i \in Z^+\}$ constitutes the set of all maximal fixed ideals of D. Also, for each i, let v_i be the normed valuation on $\mathbf{Q}(x)$ corresponding to V_i. Then, for each i, $v_i(p_i) > 0$ and $v_i(p_j) = 0$ whenever $i \ne j$. Also, since $\cap_{i=0}^{\infty}(x, p_i) = (x)$, then Proposition 3.15 implies that D contains a free prime ideal P such that $P \cap Z[x] = (x)$. Since D is almost Dedekind, P is a maximal free ideal. As noted above, D is an intersection of overrings of $\mathrm{Int}\,(Z)$, and so is itself an overring of $\mathrm{Int}\,(Z)$. It follows from our classification of valuation overrings of $\mathrm{Int}\,(Z)$ that the free prime ideal P is the contraction to D of the maximal ideal M_g of the valuation domain V_g where $g(x) = x$.

Before constructing another E-domain, we consider a particular subring of the domain D from Example 4.3 to illustrate the utility of the E-domain concept. In [**2**], Fontana, Huckaba, and Papick study nonmaximal prime ideals of Prüfer domains with the particular view of deciding whether or not a given nonmaximal prime ideal is divisorial or not. One question they ask in this context is whether or not it is possible to find a nonmaximal prime ideal M in a Prüfer domain D such that M is divisorial, but M^2 is not. They give one example of such an ideal. The example they give is a nonmaximal prime ideal M in a pull-back of the ring R of entire functions [**2**, Example 5]. In particular, M is the pull-back of a maximal free ideal of R. In Example 4.4, we use a similar construction to that of Fontana, et al., to obtain an example of the same phenomenon using D and P of Example 4.3.

Example 4.4. Assume the notation of Example 4.3. Let $V^* = V_{0,3}$. V^* is a two-dimensional valuation domain since 0 is algebraic over $\mathbf{Q}$. Also, let $g(x) = x$. Then 0 is a root of $g(x)$ and so V_g is the unique one-dimensional valuation overring of V^*. Also, $f(x) = x^2 + 1$ is a uv-polynomial for V^* since the residue field is the field of three elements. Let $D^* = D \cap V^*$. D^* is not an E-domain. Nevertheless, some of the results we have proven concerning E-domains remain true.

i) It follows from [**9**, Corollary 2.6] that D^* is a Prüfer non-D-ring with $f(x) = x^2 + 1$ serving as a uv-polynomial.

ii) The intersection $(\cap_{i=1}^{\infty} V_i) \cap V^*$ is irredundant. This follows immediately from the observations that each p_i is a unit in V^* and that 3 is a unit in each V_i. (Since irredundant representations are unique [**4**, Corollary 1.9] and v^* has dimension two, D^* cannot be an E-domain.)

Let M^* be the maximal ideal of V^*, let $P^* = M^* \cap D^*$, and let v^* be the valuation on $\mathbf{Q}(x)$ corresponding to V^*. Since $v_i(3) = 0$ for each i, it follows that P^* is the only maximal ideal of D^* which contains 3 [**4**, Corollary 1.11]. The definition of V^* implies that M^* is a principal ideal generated by 3. Hence, P^* is a principal ideal of D^* generated by 3. Now, recall from Example 4.3 that $D_P = V_g$. Let $P^c = P \cap D^*$. Since V_g is the unique valuation overring of V^*, it follows that P^* is a height two prime of D^* and that $P^c \subset P^*$. Then $P^c = \cap_{i=1}^{\infty} (3)^i$ and so P^c is a divisorial nonmaximal prime of D^*. However, $(P^c)^2$ is not divisorial. This follows from the following observations.

Let d be a nonzero element of $\mathbf{Q}(x)$. Then

i) Suppose that $v_i(d) > 0$ for some $i \in Z^+$. We observed in Example 4.3 that P was a free prime ideal. Hence, P is not contained in any maximal fixed ideal of D. Hence, $v_i(r) = 0$ for some $r \in P^2$.

Since D^* is Prüfer it follows that $v_i(r) = 0$ for some $r \in (P^c)^2$. Hence, the D^*-fractional ideal (d) does not contain $(P^c)^2$.

ii) Suppose that $v_i(d) \leq 0$ for all $i \in Z^+$ and $v^*(d) > 0$. Then, as noted above, the D^* fractional ideal (d) contains P^c.

iii) Suppose that $v_i(d) \leq 0$ for all $i \in Z^+$ and $v(d) \leq 0$. Then the D^*-fractional ideal (d) contains D^*.

These observations imply that $(P^c)^2$ is divisorial only if P^c is idempotent. It follows immediately from the definition of V_g that P^c is not idempotent.

The previous two examples involved intersections of Noetherian valuation overrings of $Z[x]$ which were centered on distinct maximal ideals of $Z[x]$. Our final example, a slight variant of which also appears in [**10**, Example 30], involves the intersection of an infinite collection of Noetherian valuation overrings of $Z[x]$, all of which are centered on a single maximal ideal of $Z[x]$. Before we present the example, we recall two results from [**10**].

Lemma 4.5. See [**10**, Lemma 25]. *Let $p \in Z$ be a prime number. Suppose that $A = \{\alpha_j \mid j \in \Lambda\} \subseteq \hat{Z}_p$ is a collection of p-adic integers. Let $D = \cap_{j\in\Lambda} V_{\alpha_j,p}$. Then every maximal ideal of D is the contraction to D of the maximal ideal of $V_{\beta,p}$ for some $\beta \in \hat{Z}_p$.*

Lemma 4.6. See [**10**, Lemma 26]. *Let p, A and D be as in the statement of Lemma* 4.5. *Also, let $\beta \in \hat{Z}_p$ and let C be the closure of A under the p-adic topology. Then $V_{\beta,p}$ is an overring of D if and only if $\beta \in C$.*

Now we proceed to our example. It is varied slightly from [**10**, Example 30], but the arguments given there for the deductions we make here are valid.

Example 4.7. See [**10**, Example 30]. Let $\alpha \in \hat{Z}_p$ be transcendental over **Q**. Then, for each $i \in Z^+$, let $\alpha_i = p^i\alpha$. Let $D = \cap_{i=0}^{\infty} V_{\alpha_i,p}$. Lemmas 4.5 and 4.6 imply that the intersection is irredundant. For each i, let $P_i = M_{\alpha_i,p} \cap D$. The argument in [**10**] shows that each P_i is principal. Hence, D is an E-domain and the collection $\{P_i \mid i \in Z^+\}$ is the collection of maximal fixed ideals of D. Moreover, the sequence $\{\alpha_n\}$ converges to 0 and so it follows from Lemmas 4.5 and 4.6 that $P_0 = M_{0,p} \cap D$ is the only maximal free ideal of D. Since 0 is algebraic over $\mathbf{Q}$, it follows that $V_{0,p}$ is a height two valuation domain and so P_0 is a height two prime free ideal.

We conclude by posing several more questions.

Question 7. Suppose that T is an integral domain with an infinite number of Noetherian valuation overrings. Does there exist an E-overring of T with an infinite number of maximal fixed ideals? Can such a domain be non-Noetherian, i.e., containing nontrivial free ideals?

Question 8. Assume that a non-Noetherian E-domain as described in Question 7 can be constructed. Can such a domain have exactly one maximal free ideal, e.g., Example 4.7? If so, is it possible to classify the Noetherian valuation overrings of T which can correspond to the one maximal free ideal?

Question 9. Is there a natural generalization of the concept of E-domain which would include all of the domains considered in this paper as well as similar domains, such as the ring R of entire functions and Dedekind domains which are not non-D-rings?

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