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IDEAL BANACH CATEGORY THEOREMS

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ABSTRACT. Three abstract versions of the Banach category theorem are compared when an arbitrary ideal replaces the σ-ideal of meager sets. Every ideal is contained in a unique smallest ideal for which a given Banach category theorem holds. The behavior of these ideal extensions is also investigated.

1. Introduction. Throughout this paper (X,τ) denotes a topological space and $I \subseteq P(X)$ is an ideal of subsets of X. An ideal is a nonempty family of sets which is closed under finite union and a subset of its members. It is σ-ideal if it also is closed under countable union of its members. For any subset $A \subseteq X$, let $A^*(\tau, I)$ or simply A^* if τ and I are understood, be the adherence of A modulo I. In particular, $A^* = \{x \in X \mid x \in U \in \tau \Rightarrow U \cap A \notin I\}$. It may be noted that A^* is a closed subset of $\mathrm{cl}\,(A)$, the closure of A in X. For convenience, let $A^\circ(\tau, I)$ or simply A° if τ and I are understood, denote the set $A - A^*$, i.e., $A^\circ = \{x \in A \mid \text{there exists a } U \in \tau \text{ such that } x \in U \text{ and } U \cap A \in I\}$. By the terminology of A.H. Stone [**13**], et al., $A \cap A^*(\tau, I)$ is the kernel of the subspace $(A, \tau|A)$, relative to the ideal $I|A = I \cap P(A)$. This would make A° the cokernel of the subspace A. Note that $A^\circ(\tau, I) = A^\circ(\tau|A, I|A)$. In [**6**], some general forms of the Banach category theorem were found useful in the context of continuity apart from a meager set when the ideal of meager sets $M(\tau)$ was replaced by an arbitrary σ-ideal. In this paper comparison is made of three abstract versions of the Banach category theorem for an arbitrary ideal I. These are mutually equivalent when $I = M(\tau)$, and are referred to as properties B_1, B_2 and B_3 below.

Property B_1. For each subset $A \subseteq X$, $A \in I$, if for each nonempty open set U there is a nonempty open subset $V \subseteq U$ such that $V \cap A \in I$,

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i.e., V and A are "almost" disjoint.

Property B_2. The cokernel of each subspace A of X belongs to I, i.e., $A^\circ \in I$ for each $A \subseteq X$.

Property B_3. The union of any family of open sets belonging to I is a member of I, i.e., $\cup(I \cap \tau) \in I$.

Let us agree that a subset $A \subseteq X$ is locally in I if $A \subseteq A^\circ$, i.e., $A \cap A^* = \varnothing$. Then B_2 is easily equivalent to the statement that every set A which is locally in I belongs to I. This is in fact the form of the Banach category theorem first proved by Banach for metric spaces when $I = M(\tau)$ [**1**]. It was extended to arbitrary spaces in [**7**]. When $I = M(\tau)$, B_3 is the statement of the Banach category theorem in [**8**]. Theorem 1 below will demonstrate the equivalence of B_1 and B_2 when $I = M(\tau)$. In each B_i, we do not limit ourselves to σ-ideals since preservation of B_i under various ideal extensions including σ-extension is a fundamental consideration. Of course, space axioms can also influence these conditions. For example, if X is hereditarily Lindelöf, each σ-ideal I satisfies B_2 by Theorem 4.2 of [**3**]. In fact, the σ-ideal of countable sets I_{ω_1} satisfies B_2 if and only if X is hereditarily Lindelöf by Theorem 4.10 of [**3**]. For any infinite cardinal κ, let I_κ be the numerical ideal containing all subsets of cardinality less than κ. Then the ideal of finite sets is I_ω and it is also shown in [**3**] that this ideal satisfies B_2 if and only if X is hereditarily compact. Here we identify the ordinal ω with the cardinal $\aleph_0$ and the ordinal ω_1 with the cardinal $\aleph_1$. By a principal ideal is meant $P(A)$ for some $A \subseteq X$. Principal ideals are σ-ideals and in fact are closed under κ-union (union of subfamilies of cardinality not greater than κ) for any cardinal κ, and they always satisfy B_2. In the literature, any ideal I satisfying B_2 is called a (τ)-local ideal since it contains all subsets of X which locally belong to I. A set A locally belongs to I, or $A \subseteq A^\circ$, if it has an open cover $\Gamma \subseteq \tau$ such that $U \cap A \in I$ for each $U \in \Gamma$. The Banach category theorem in its historically original form asserts that the σ-ideal of meager subsets of X, $M(\tau)$, is τ-local, i.e., every locally meager set is meager. That the ideal of nowhere dense sets, $N(\tau)$,is always τ-local is well known and was shown in [**14, 3**] and [**11**]. A new and easy proof of this fact is given herein by observing that $N(\tau)$

satisfies B_1 and that B_1 implies B_2. The nontriviality of the Banach category theorem indicates that generally preservation of B_2 or locality for I under σ-extension is nontrivial. Let $\Sigma(I)$ denote the σ-extension of I, i.e., the intersection of all σ-ideals containing I. More generally, let $\Sigma_\kappa(I)$ be the intersection of all ideals containing I which are closed under κ-union. R. Vaidyanathaswamy showed in [**14**] that $\Sigma_\kappa(N(\tau))$ satisfies B_2 for every (infinite) cardinal κ and thereby generalized the Banach category theorem since $\Sigma_\omega(N(\tau)) = \Sigma(N(\tau)) = M(\tau)$. From Theorems 3.3 and 4.5 of [**4**], it is easily deduced that $\Sigma(I)$ satisfies B_2 whenever I satisfies B_2 and $N(\tau) \subseteq I$ which also generalizes the Banach category theorem. In [**10**] it is shown that the role of $N(\tau)$ is inessential in each of these generalizations. In particular, it is shown that $\Sigma_\kappa(I)$ satisfies B_2 if I satisfies B_2 for any ideal I and for any cardinal κ. In this sense the Σ_κ operator preserves B_2. We will show that it also preserves B_1 by identifying the precise relationship between B_1 and B_2.

An ideal I is τ-codense if each member of I is codense, i.e., if $I \cap \tau = \{\varnothing\}$. Clearly, $N(\tau)$ is always τ-codense and $M(\tau)$ is τ-codense if and only if (X, τ) is a Baire space. Obviously, each codense ideal I satisfies B_3 for it is clear that any ideal I satisfies B_3 if and only if $\cup(I \cap \tau) \in I$. Also, even though (X, τ) may fail to be a Baire space, $M(\tau)$ always satisfies B_3 since, as will be shown, B_2 implies B_3.

2. Basic relationships. Since B_2 is an idealized version of the original form of the Banach category theorem and has already received attention in the literature [**14, 4**] and [**10**], we will relate B_1 and B_3 to B_2. Recall that a set $E \in N(\tau)$ if and only if, for each nonempty $U \in \tau$ there exists a nonempty $V \in \tau$ with $V \subseteq U$ and $V \cap E = \varnothing$. Since every ideal contains $\varnothing$ as a member, it is evident that $N(\tau) \subseteq I$ if I satisfies B_1. Moreover, we have the following characterization.

Theorem 1. *An ideal I satisfies B_1 if and only if $N(\tau) \subseteq I$ and I satisfies B_2.*

Proof. For the necessity, assume that I satisfies B_1. It remains only to show that I satisfies B_2. Suppose that A locally belongs to I. Let Γ be an open cover of A such that $U \cap A \in I$ for each $U \in \Gamma$. Now if

W is any nonempty open set, either $W \cap A = \varnothing$ or $V = W \cap U \neq \varnothing$ for some $U \in \Gamma$. In either case, there exists a nonempty open subset $V \subseteq W$ such that $V \cap A \in I$. Since I satisfies B_1, $A \in I$, so that I is local and hence satisfies B_2.

For the sufficiency, suppose that $N(\tau) \subseteq I$ and that I satisfies B_2. Let A be a subset of X such that, for every nonempty open set U, there exists a nonempty open subset $V \subseteq U$ with $V \cap A \in I$. Let $\Gamma = \{V \in \tau \mid V \cap A \in I\}$, and let $W = \cup\Gamma$. Then $A \cap W$ locally belongs to I so that $A \cap W \in I$. If $G = \operatorname{int}((\operatorname{cl} A) - W) \neq \varnothing$, then $G \subseteq (\operatorname{int}(\operatorname{cl} A)) - W$ and there exists a nonempty open subset $H \subseteq G$ such that $H \cap A \in I$. Thus, $H \in \Gamma$ so that $H \subseteq W - W = \varnothing$. This contradiction shows that $(\operatorname{cl} A) - W \in N(\tau) \subseteq I$ so that $A - W \in I$. Therefore, $A = (A \cap W) \cup (A - W) \in I$ so that I satisfies B_1. □

It follows immediately that $N(\tau)(M(\tau))$ satisfies B_1 if and only if it satisfies B_2.

Corollary 2. *For each space (X, τ), $N(\tau)$ is τ-local.*

Proof. It is enough to show that $N(\tau)$ satisfies B_1. Let A be a subset of X such that, for every nonempty open set U, there exists a nonempty open subset $V \subseteq U$ with $V \cap A \in N(\tau)$. Then let W be a nonempty open subset of V such that $W \cap (V \cap A) = \varnothing$. Then $\varnothing \neq W \subseteq U$ and $W \cap A = (W \cap V) \cap A = \varnothing$ implies that $A \in N(\tau)$ so that $N(\tau)$ satisfies B_1. □

Corollary 3. *For each topology τ, $M(\tau)$ satisfies B_1.*

Of course, $M(\tau) = \Sigma(N(\tau))$ and in the next section it will be shown that B_1 is preserved by the Σ_κ operator for any (infinite) cardinal κ.

Caution must be used when relativizing topology-dependent ideals such as $N(\tau)$ or $M(\tau)$ to an arbitrary subspace $(A, \tau|A)$. Generally, $N(\tau|A) \subseteq N(\tau)|A$. Theorem 4.1 of [**5**] states that equality here holds if and only if A is almost locally dense, meaning that $A \subseteq \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))$. From [**14**] we have that, for any A, $\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A))) = A^*(\tau, N(\tau))$. Thus, when $A \subseteq A^*(\tau, N(\tau))$, we also have $M(\tau|A) = M(\tau)|A$. Let us agree that an ideal I satisfies a condition hereditarily if, for each $A \subseteq X$, $I|A$ satisfies the condition as an ideal of subsets of the subspace $(A, \tau|A)$. For example, if (X, τ) is any dense-in-itself T_1 space, $N(\tau)$ is codense but not hereditarily codense since each singleton subset $A = \{x\}$ is a nonempty nowhere dense set. Thus, $(N(\tau)|A) \cap (\tau|A) \neq \{\varnothing\}$. Let us say that I has property HB_i if I has property B_i hereditarily for $i = 1, 2$, or 3. Note that an ideal I satisfies B_2 if and only if, for each $A \subseteq X$, $A^\circ \in I$, whereas I satisfies B_3 if and only if $X^\circ \in I$. This suggests the following.

Theorem 4. *An ideal I satisfies B_2 if and only if it satisfies HB_3.*

Proof. Recall that, for each $A \subseteq X$, $A^\circ(\tau, I) = A^\circ(\tau|A, I|A)$. □

Corollary 5. *An ideal I satisfies B_2 if and only if it satisfies HB_2.*

Corollary 6. *An ideal I satisfies B_1 if and only if it satisfies HB_1.*

Proof. Only the necessity is necessary (pun intended). Suppose that I satisfies B_1. Then $N(\tau) \subseteq I$ and I satisfies B_2 hereditarily by Theorem 1 and Corollary 5. Since, for each $A \subseteq X$, $N(\tau|A) \subseteq N(\tau)|A \subseteq I|A$, we have again by Theorem 1 that $I|A$ satisfies B_1 on the subspace $(A, \tau|A)$ so that I satisfies B_1 hereditarily. □

It is clear from Theorems 1 and 4 that, for any ideal I, B_1 implies B_2, and B_2 implies B_3. These theorems also strongly suggest that these implications are not reversible, even for σ-ideals.

Example 7. Property B_2 does not imply B_1. Let (X, τ) be any nonpartition space, i.e., $N(\tau) \neq \{\varnothing\}$. For $\varnothing \neq E \in N(\tau)$, if $A = X - E$, the principal ideal $P(A)$ is a local σ-ideal (actually closed under κ-union for any cardinal κ) but not satisfying B_1 since $E \notin P(A)$.

For an example oriented toward real analysis, if (X, τ) is the usual space of reals and $I = I_{\omega_1}$ is the ideal of countable subsets, I satisfies B_2 since (X, τ) is hereditarily Lindelöf, yet if C is the standard Cantor set,

$C \in N(\tau) - I$ so that I does not satisfy B_1. Now, in this same space, if J is the ideal of bounded countable sets, J is codense since (X, τ) has uncountable dispersion character (least cardinal of a nonempty open set), and thus J satisfies B_3. However, the unbounded set of rationals, Q, is locally bounded and countable so that J does not satisfy B_2. For a σ-ideal example we have the following.

Example 8. Property B_3 does not imply B_2. Let (X, τ) be the Tychonoff cube K^c where $K = [0,1]$ is the usual unit interval and c is the cardinality of K. Since X contains a discrete subspace D of cardinality c, X is not hereditarily Lindelöf and hence the ideal $I = I_{\omega_1}$ of countable subsets of X does not satisfy B_2. Yet I does satisfy B_3 being τ-codense since X has uncountable dispersion character.

Relationships between properties B_1, B_2, B_3 and their hereditary versions is shown in the diagram of Figure 1 below where it is understood that the uni-directional implications are irreversible.

$$\begin{array}{ccccc} B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \\ \updownarrow & & \updownarrow & & \uparrow \\ HB_1 & \longrightarrow & HB_2 & \longleftrightarrow & HB_3 \end{array}$$

FIGURE 1.

In light of Theorem 4 and the fact mentioned above that every σ-ideal of subsets of any hereditarily Lindelöf space satisfies B_2, one might speculate that each σ-ideal of subsets of any Lindelöf space must satisfy B_3. The next example disproves this conjecture.

Example 9. Let $I = I_{\omega_1}$ be the σ-ideal of countable subsets of the compact (and thus Lindelöf) ordinal space $\omega_1 + 1 = [0, \omega_1]$. Every open set containing ω_1 is uncountable whereas each $\alpha < \omega_1$ is contained in the countable open set $\alpha + 1 = [0, \alpha]$. Thus, the union of all countable open sets is $\omega_1 = [0, \omega_1)$, an uncountable set. So I does not satisfy B_3.

We will conclude this section with an argument showing why B_3 implies B_2 and thus also B_1 when $I = M(\tau)$.

Theorem 10. *For any space (X,τ), B_1, B_2 and B_3 are pairwise equivalent for the ideal $M(\tau)$.*

Proof. As remarked, it is sufficient to show that B_3 implies B_2 since $N(\tau) \subseteq M(\tau)$. Suppose that $A \subseteq X$ locally belongs to $M(\tau)$, and assume that $M(\sigma)$ satisfies B_3 in every space (Y,σ). Certainly $A = (A - \text{int}\,(\text{cl}\,(A))) \cup (A \cap \text{int}\,(\text{cl}\,(A)))$, and $A - \text{int}\,(\text{cl}\,(A)) \in N(\tau)$. But $B = A \cap \text{int}\,(\text{cl}\,(A))$ is locally dense in the sense that $B \subseteq \text{int}\,(\text{cl}\,(B))$ and hence B is almost locally dense so that, $M(\tau)|B = M(\tau|B)$, an ideal that satisfies B_3 for the space $(B,\tau|B)$. Hence, $B \in M(\tau|B) \subseteq M(\tau)$. Thus, $A \in M(\tau)$, showing that $M(\tau)$ satisfies B_2. □

Of course, this same argument works for $I = N(\tau)$ as well. One might wonder if B_3 implies B_2 for any ideal intermediate to $N(\tau)$ and $M(\tau)$ or perhaps for any ideal I containing $N(\tau)$ thereby obtaining another generalization of the Banach category theorem. A counterexample to these possibilities along with further contrasts between B_3 and properties B_1 and B_2 will be given in the next section.

3. Ideal extensions. In this section we first observe that every ideal I is contained in a unique smallest ideal satisfying B_i for each i. This ideal will be called the B_i-extension of I and will be denoted I^i for $i = 1, 2$ or 3. There should be no confusion since we will not be considering any set-theoretic powers of I. We will also be concerned with preservation of B_i under various ideal extensions such as σ-extension or extension by the operator Σ_κ for any infinite cardinal κ.

For any ideal I, let $I^i = \cap\{J \mid J \text{ is an ideal satisfying } B_i \text{ with } I \subseteq J\}$ for each i. It is understood that all ideals are contained in $P(X)$ for some topological space (X,τ). Clearly, $P(X)$ satisfies B_i for each i so that each I^i is an ideal extension of I contained in $P(X)$. The next theorem justifies calling each I^i the B_i-extension of I.

Theorem 11. *For any ideal I, I^i satisfies B_i for each i.*

Proof. It was shown in [**10**] that I^2 satisfies B_2 and there I^2 was called the local extension of I. Now suppose that I is an ideal and $A \subseteq X$

is such that, for any nonempty open set U, there exists a nonempty open subset $V \subseteq U$ with $V \cap A \in I^1$. If J is any ideal extension of I satisfying B_1, $I^1 \subseteq J$ so that $V \cap A \in J$. Since J satisfies B_1, $A \in J$. Thus, $A \in I^1$ showing that I^1 satisfies B_1. Finally, let I be any ideal and let J be any ideal extension of I satisfying B_3. Then $I^3 \subseteq J$ so that $\cup(I^3 \cap \tau) \subseteq \cup(J \cap \tau) \in J$. Thus, $\cup(I^3 \cap \tau) \in I^3$ showing that I^3 satisfies B_3. □

First note that the B_i-extension operator is monotonic. For, if I and J are any ideals with $I \subseteq J$, then, for each i, $I^i \subseteq J^i$. Further, since B_1 implies B_2 and B_2 implies B_3, for any ideal I we have $I \subseteq I^3 \subseteq I^2 \subseteq I^1$. Of course, for each i, an ideal I satisfies B_i if and only if $I = I^i$. Thus, for each i, the B_i-extension operator is idempotent, i.e., for each ideal I, $(I^i)^i = I^i$. Moreover, no change occurs if a B_j-extension operator is applied to the B_i-extension of an ideal I for $j > i$. For example, $(I^1)^2 = I^1$ for any ideal I. Also, it can be shown that $(I^2)^1 = I^1$ for any ideal I.

In [**4**], an ideal extension of an ideal I via an ideal J was defined by $I * J = \{A \subseteq X \mid A^*(\tau, I) \in J\}$. Of course, for each $A \in I$, $A^*(\tau, I) = \varnothing \in J$ for any ideal J so that $I \subseteq I * J$. That $I * J$ is an ideal is an easy consequence of the facts that the adherence operator modulo I is monotonic, it distributes over finite union, and J is an ideal. It is also clear that $I * K \subseteq J * K$ if $I \subseteq J$. In the very important case $J = N(\tau)$, $I * N(\tau)$ was denoted $\tilde{I}$ and it was shown that the extension $\tilde{I}$ satisfies B_2 and that $N(\tau) \subseteq \tilde{I}$ for any ideal I. Easily, $N(\tau) \subseteq \tilde{I}$ for if $E \in N(\tau)$, $E^*(\tau, I) \subseteq \mathrm{cl}\,(E) \in N(\tau) \Rightarrow E \in \tilde{I}$. By Theorem 1, $\tilde{I}$ satisfies B_1 for each ideal I. It was also shown in [**4**] that $\tilde{I} = I \vee N(\tau)$ if I satisfies B_2 where, for any ideals I and J, $I \vee J$ is the join of I and J, i.e., the smallest ideal containing $I \cup J$. Further, it was shown in [**10**] that the B_2-extension operator distributes over finite join. So, in particular, for any ideal I, $\tilde{I} \subseteq \tilde{I}^2 = I^2 \vee N(\tau) = (I \vee N(\tau))^2 \subseteq \tilde{I}$ showing that, for any ideal I, $\tilde{I}$ is the B_2-extension of the join $I \vee N(\tau)$. Thus, by the following theorem, for any ideal I, I^1 is the B_2-extension of $I \vee N(\tau)$.

Theorem 12. *For any ideal I, $\tilde{I} = I^1 = I^2 \vee N(\tau) = \{A \subseteq X \mid A^* \in N(\tau)\}$.*

Proof. Since $\tilde{I}$ is an extension of I satisfying B_1, $I^1 \subseteq \tilde{I}$. On the other hand, $I \vee N(\tau) \subseteq I^1$ so that $\tilde{I} = (I \vee N(\tau))^2 \subseteq (I^1)^2 = I^1$. □

Theorem 12 not only identifies $\tilde{I}$ as the B_1-extension of I, but also provides a simple algorithm to find I^1: find the B_2-extension and join with $N(\tau)$. Since, for any ideal I, $I \subseteq I^3 \subseteq I^2 \subseteq I^1 = \tilde{I}$ and it was shown in [**12**] that I is codense if and only if $\tilde{I}$ is codense, we have the following

Corollary 13. *For any ideal I, the following are equivalent.*

(a) *The ideal I is codense.*

(b) *The ideal I^3 is codense.*

(c) *The ideal I^2 is codense.*

(d) *The ideal I^1 is codense.*

Corollary 14. *For any ideal I, $I^1 = (I^2)^1 = (I^1)^2$.*

Proof. Note only that $(I^2)^1 = I^2 \vee N(\tau) = I^1 = (I^1)^2$. □

Corollary 15. *For any two ideals I and J, $(I \vee J)^1 = I^1 \vee J^1$.*

Proof. By Theorem 12, $(I \vee J)^1 = (I \vee J)^2 \vee N(\tau) = (I^2 \vee N(\tau)) \vee (J^2 \vee N(\tau)) = I^1 \vee J^1$. □

Corollary 15 asserts that the B_1-extension operator distributes over finite join. We will improve this later and show by example that it does not distribute over infinite join.

For any ideal I of subsets of a space (X, τ), let $\tau[I]$ be the smallest topology on X containing τ for which members of I are closed. As a consequence of Corollary 2.5 of [**10**], we have for each infinite cardinal κ, $(I_\kappa)^2 = S(\tau[I_\kappa])$ where, for any T_1 topology τ, $S(\tau)$ is the (τ-local) ideal of scattered subsets of (X, τ). For example, when $\kappa = \omega$ and (X, τ) is a T_1 space, $(I_\omega)^2 = S(\tau)$. Thus, in this case, $(I_\omega)^1 = (S(\tau))^1 = S(\tau) \vee N(\tau)$ [**4**]. More generally, we have the following.

Corollary 16. *For any space (X, τ) and any infinite cardinal κ, $(I_\kappa)^1 = S(\tau[I_\kappa]) \vee N(\tau)$.*

Proof. Note that $(I_\kappa)^1 = (I_\kappa)^2 \vee N(\tau) = S(\tau[I_\kappa]) \vee N(\tau)$. For $(X, \tau[I_\kappa])$ is a T_1 space since $I_\omega \subseteq I_\kappa$ so that $S(\tau[I_\kappa])$ is a $\tau[I_\kappa]$-local ideal. Therefore, it is a τ-local ideal since $\tau \subseteq \tau[I_\kappa]$. Hence, it is the B_2-extension of I_κ relative to the space (X, τ). All extension operators here are understood to be relative to (X, τ). □

Corollary 17. *If (X, τ) is any dense-in-itself T_1 space, $(I_\omega)^1 = N(\tau)$.*

Proof. Since (X, τ) is dense-in-itself, $I_\omega \subseteq N(\tau)$, a τ-local ideal. Thus, since (X, τ) is a T_1 space, $(I_\omega)^2 = S(\tau) \subseteq N(\tau)$ so that $(I_\omega)^1 = S(\tau) \vee N(\tau) = N(\tau)$. □

Let $\Lambda = \{I_\alpha \mid \alpha < \gamma\}$ be a nonempty ordinally indexed family of ideals of subsets of a space (X, τ). The join of Λ, denoted $\vee\Lambda$, is the intersection of all ideals of subsets of X which contain $\cup\Lambda$. So $\vee\Lambda$ is an ideal and it may be verified that $\vee\Lambda = \{\cup_{\alpha<\gamma} A_\alpha \mid A_\alpha \in I_\alpha$ and $|\{\alpha \mid A_\alpha \neq \varnothing\}| < \omega\}$. The box join of Λ, introduced in [**10**] and denoted $\sqcup\Lambda$, is an ideal extension of the join $\vee\Lambda$ defined by $\sqcup\Lambda = \{\cup_{\alpha<\gamma} A_\alpha \mid A_\alpha \in I_\alpha\}$, i.e., the members of $\sqcup\Lambda$ are unions of choice sets for Λ. Moreover, the box join of Λ is independent of the well-ordering of its members. We will show that $\sqcup\Lambda$ satisfies B_1 if each $I_\alpha \in \Lambda$ satisfies B_2 and at least one ideal in Λ satisfies B_1. But, first, an example shows that $\vee\Lambda$ may fail to satisfy B_1 even if each $I_\alpha \in \Lambda$ satisfies B_1.

Example 18. Let $X_n = \omega$ have the indiscrete topology τ_n for each $n < \omega$, and let (X, τ) be the free topological sum of the spaces (X_n, τ_n). Then (X, τ) is a partition space so that $N(\tau) = \{\varnothing\}$. Consequently, for any ideal I of subsets of X, B_1 holds if and only if B_2 holds. Let $I_n = P(X_n)$ for each $n < \omega$, and let $I = \vee\{I_n \mid n < \omega\}$. Being principal, each I_n satisfies B_2 and hence also B_1. Yet I satisfies neither B_1 nor B_2 since $I^2 \neq I$. Note that any choice set $A = \{a_n \mid n < \omega\}$ for the family $\{X_n \mid n < \omega\}$ is locally in I since, for each $n < \omega$,

$X_n \cap A = \{a_n\} \in I_n \subseteq I$. But each $B \in I$ has the property that $X_n \cap B = \varnothing$ for all but finitely many $n < \omega$. So $A \in I^2 - I$.

Theorem 19. *Let $\Lambda = \{I_\alpha \mid \alpha < \gamma\}$ be a nonempty ordinally indexed family of ideals satisfying B_2, and suppose that at least one ideal in Λ satisfies B_1. Then $\sqcup\Lambda$ satisfies B_1.*

Proof. Since each I_α satisfies B_2, we have from [**10**] that $\sqcup\Lambda$ satisfies B_2. Also, if $I_\beta \in \Lambda$ satisfies B_1, $N(\tau) \subseteq I_\beta \subseteq \sqcup\Lambda \Rightarrow \sqcup\Lambda$ satisfies B_1. □

It is evident that $\vee\Lambda = \sqcup\Lambda$ when Λ is finite. Hence, if ideals I and J both satisfy B_1, then $I \vee J$ satisfies B_1 since $I \vee J = I \sqcup J = \sqcup\{I, J\}$. This is equivalent to Corollary 15.

Corollary 20. *For any ideal I and any infinite cardinal κ, $\Sigma_\kappa(I)$ satisfies B_1 if I satisfies B_1.*

Proof. Let $\kappa = \gamma$ as an initial ordinal, and let $I_\alpha = I$ for each $\alpha < \gamma$. Then, if $\Lambda = \{I_\alpha \mid \alpha < \gamma\}$, $\Sigma_\kappa(I) = \sqcup\Lambda$. □

This corollary can be equivalently stated as follows. For any ideal I and infinite cardinal κ, $(\Sigma_\kappa(I))^1 \subseteq \Sigma_\kappa(I^1)$.

Corollary 21. *If I is an ideal satisfying B_1, then $\Sigma(I)$ satisfies B_1.*

Given an ideal I, is there a way to construct I^1 from I? Theorem 12 provides one way. Also, from [**10**], an "internal construction" of I^2 from I was found. This construction could be applied to the ideal $I \vee N(\tau)$ to obtain I^1 according to Corollary 14. Following the technique of [**10**], a more direct construction is possible yielding I^1 as the top (or union) of a chain of ideals containing I. Let $D^0(I) = I$ and for each ideal J, let $D(J)$ be the smallest ideal containing each subset $A \subseteq X$ such that for each nonempty open set U, there exists a nonempty open subset $V \subseteq U$ with $V \cap A \in J$. Clearly, for any ideal J, $J \subseteq D(J)$ and equality holds if and only if J satisfies B_1. For each ordinal α, let $D^{\alpha+1}(I) = D(D^\alpha(I))$

and if β is a limit ordinal, let $D^\beta(I) = \cup\{D^\alpha(I) \mid \alpha < \beta\}$. Then, by transfinite induction, $I \subseteq D^\alpha(I)$ for each α. If $\alpha < \beta$, let δ be the ordinal which is order-isomorphic to $\beta - \alpha$. Then β is the ordinal sum $\alpha + \delta$ and, by transfinite induction on δ, $D^\beta(I) = D^\delta(D^\alpha(I))$. Thus, $D^\alpha(I) \subseteq D^\beta(I)$ if $\alpha < \beta$. Since each $D^\alpha(I) \subseteq P(X)$ and $|P(X)| = 2^{|X|}$, there exists an ordinal γ so large that $D^\gamma(I) = D^{\gamma+1}(I)$. Since $D^\gamma(I)$ contains I and satisfies B_1, $I^1 \subseteq D^\gamma(I)$. This is half of the following construction theorem.

Theorem 22. *If (X, τ) is any topological space and $I \subseteq P(X)$ is any ideal, there exists an ordinal γ such that $I^1 = D^\gamma(I)$.*

Proof. As before, suppose that γ is such that $D^\gamma(I) = D^{\gamma+1}(I)$. It remains only to show that $D^\gamma(I) \subseteq I^1$. Easily, $D(I) \subseteq I^1$ and, by transfinite induction, $D^\alpha(I) \subseteq I^1$ for all α. For, if $D^\alpha(I) \subseteq I^1$ for all $\alpha < \beta$, then if $\beta = \delta + 1$ is a successor ordinal, $D^\beta(I) = D(D^\delta(I)) \subseteq I^1$ since $D^\delta(I) \subseteq I^1$. And, if β is a limit ordinal, then $D^\beta(I) \subseteq I^1$ since $D^\alpha(I) \subseteq I^1$ for each $\alpha < \beta$. Thus, $D^\gamma(I) \subseteq I^1$. □

Even as I^1 and I^2 are representable as maximum elements in an increasing ordinally indexed tower of subideals stacked on I, a similar representation for I^3 will be given below. On the other hand, the behavior of B_3 seems to be quite different from the afore noted similar behaviors of properties B_1 and B_2, particularly with respect to preservation under certain operations. Firstly, B_3 is not generally preserved by a finite join of ideals. Let R be the usual space of real numbers, let $I = 2^Q \cap B(R)$ be the ideal of bounded subsets of the set Q of rational numbers, and let $J = 2^P \cap B(R)$ be the ideal of bounded subsets of the set P of irrational numbers. Then I and J both have B_3 being codense. Clearly, $I \vee J \subseteq B(R)$ so that $R \notin I \vee J$ being unbounded. Thus, $I \vee J$ does not have B_3 since R is a union of bounded open sets and each bounded open set belongs to $I \vee J$. A similar example can be concocted where I and J are σ-ideals.

Example 23. Let $X = \omega_1$ be the first uncountable ordinal equipped with a subtopology of the usual order topology whose nonempty basic open sets are of the form $[0, \beta]$ where $\beta \in \omega_1$ is a limit ordinal. Clearly, every nonempty open set contains a limit ordinal and nonlimit ordinals. Let $L \subseteq \omega_1$ be the subset of limit ordinals, and let $B(\omega_1)$ be the collection of all bounded subsets of ω_1. Then $I = \{A \in B(\omega_1) \mid A \cap L = \varnothing\}$ and $J = \{B \in B(\omega_1) \mid B \subseteq L\}$ are codense ideals in X with $I \vee J = B(\omega_1)$. Evidently, I and J each have property B_3 and $I \vee J$ does not have B_3 since ω_1 is an unbounded union of bounded open sets. Further, I and J are σ-ideals since a countable union of countable sets is countable, and the countable subsets of ω_1 are bounded.

The next example shows that B_3 is not preserved by σ-extension.

Example 24. Let $X = \{\alpha | \alpha < \omega_1\} \times (Q \cap [0,1]))$ be a subspace of the long line where Q is the set of rational numbers, i.e., X has the lexicographic order topology. Then I_ω, the ideal of finite subsets of X, has B_3 being codense since X has no isolated points. However, $I_{\omega_1} = \Sigma(I_\omega)$, the σ-ideal of countable subsets, fails to have B_3 since X is an uncountable locally countable space.

Further, Jakub Jasinski (University of Scranton) and Irek Recław (Gdansk University), read a preprint of this article and supplied the following example showing that B_3 is not generally preserved by the Σ_κ operator for an arbitrary infinite cardinal κ.

Example 25. Let κ be any infinite cardinal with the discrete topology, and let $F = \{f \in \kappa^\omega | f(\alpha) = 0$ for all but finitely many $\alpha \in \omega\}$ have the subspace topology induced by the product topology on κ^ω. Let $X = \kappa^+ \times F$ have the product topology where κ^+ has the discrete topology. Since the dispersion character (least cardinal of any nonempty open set) of X is κ, the ideal I_κ of subsets of X of cardinality less than κ is codense and hence has B_3. However, $\Sigma_\kappa(I_\kappa) = I_{\kappa^+}$ contains every open set of the form $\{\alpha\} \times F$ but fails to contain $X = \cup_{\alpha \in \kappa^+}(\{\alpha\} \times F)$ and hence cannot have property B_3. It may also be observed that I_κ is a σ-ideal and is κ-complete (closed under union of fewer than κ members) if κ is uncountable and regular.

The following example continues the contrast by indicating the unlikelihood of obtaining a generalized Banach category theorem from B_3.

Example 26. Property B_3 does not imply B_2 even in case $N(\tau) \subset I \subset M(\tau)$. Let (X, τ) be the product $\omega_1 \times K$ of the first uncountable ordinal ω_1 with the usual real unit interval $K = [0,1]$. Let $I = I_{\omega_1} \vee N(\tau)$ be the join of the ideal I_{ω_1} of countable subsets of X with the ideal $N(\tau)$ of nowhere dense subsets of X. Since (X, τ) has uncountable dispersion character, I is τ-codense and thus satisfies B_3. To see that I is codense, suppose on the contrary that $G \in I \cap \tau$ and $G \neq \varnothing$. Then $G = C \cup E$ with $C \in I_{\omega_1}$ and $E \in N(\tau)$. Since $\operatorname{int}(\operatorname{cl}(E)) = \varnothing$, $G - \operatorname{cl}(E)$ is a nonempty countable open set contradicting the uncountable dispersion character. To show that I does not satisfy B_2, it is equivalent to show that $I \neq I^2$. Let $Q_0 = Q \cap K$ where Q is the set of rationals, and consider $A = \omega_1 \times Q_0$. For each point $x = (\alpha, r) \in A$, $U = (\alpha + 1) \times K$ is an open neighborhood of x and $U \cap A = (\alpha + 1) \times Q_0 \in I_{\omega_1} \subseteq I$. So A locally belongs to I and hence $A \in I^2$. But $A \notin I$ for, otherwise, $A = D \cup E$ with $D \in I_{\omega_1}$ and $E \in N(\tau)$, and if $\pi_1 : X \to \omega_1$ is the first projection mapping, $\pi_1(D)$ is a countable and thus bounded subset of ω_1. If $\beta < \omega_1$ is an upper bound for $\pi_1(D)$, $W = \{\gamma < \omega_1 \mid \beta < \gamma\}$ is an open subset of ω_1 and $(W \times Q_0) \cap D = \varnothing$. So $W \times Q_0 \subseteq A - D \subseteq E \Rightarrow W \times Q_0$ is nowhere dense. But this is impossible since $\varnothing \neq W \times K \subseteq (\operatorname{int}(\operatorname{cl}(W))) \times (\operatorname{int}(\operatorname{cl}(Q_0))) = \operatorname{int}(\operatorname{cl}(W \times Q_0))$.

We conclude with the promised construction of I^3 from I very similar to the one given above for I^1. Let $E^0(I) = I$, and for any ideal J, let $E(J) = J \vee P(\cup(J \cap \tau))$. For each ordinal α, let $E^{\alpha+1}(I) = E(E^\alpha(I))$, and if β is a limit ordinal, let $E^\beta(I) = \cup\{E^\alpha(I) \mid \alpha < \beta\}$. From here forward, the construction is so similar to that for I^1 that we state the theorem without further ado.

Theorem 27. *For any topological space (X, τ) and any ideal $I \subseteq P(X)$, there exists an ordinal γ such that $E^\gamma(I) = I^3$.*

Remark 28. Recently K. Ciesielski and J. Jasinski published *Topologies making a given ideal nowhere dense or meager*, [**2**], where a somewhat similar problem of finding a topology τ on X such that τ-nowhere dense sets are exactly the sets of an ideal I on X.

Remark 29. A follow-up paper, *Ideal Banach category theorems and functions*, by the second-named author appeared recently in Mathematica Bohemica [**9**].

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