ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 1, Spring 1998

HIGHER DIMENSIONAL AHLFORS REGULAR SETS AND CHORDARC CURVES IN $\mathbf{R}^n$

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1. Introduction. Recall that a Jordan curve C is *chordarc*, abbreviated CA, if there is a constant c such that for each pair of points $x, y \in C$ the arclength of one of the components A of $C \setminus \{x, y\}$ satisfies

$$l(A) \leq c|x-y|.$$

Since $l(A) \geq \operatorname{diam}(A)$ for any arc A it is immediate that CA plane curves are quasicircles. (See Section 2 for many definitions and terminology.) Obviously CA curves are locally rectifiable. In fact, every CA curve is *Ahlfors regular*, abbreviated AR, which means that there is a constant b such that for all z and all $r > 0$ we have

$$l(C \cap B(z; r)) \leq br.$$

It is folklore that Ahlfors regular quasicircles are chordarc. Another important property of CA curves, established by Tukia [**9**], and independently by Jerison and Kenig [**8**, 1.13], is that each one is bilipschitz equivalent to the circle $\mathbf{S}^1$ via a global homeomorphism of the plane. We summarize these comments as follows.

Theorem. *For a Jordan curve $C \subset \mathbf{R}^2$, the following are equivalent.*

(a) *C is chordarc.*

(b) *C is an Ahlfors regular quasicircle.*

(c) *$C = f(S^1)$ where $f : \mathbf{R}^2 \to \mathbf{R}^2$ is bilipschitz.*

Moreover, all constants depend only on each other and $\operatorname{diam}(C)$*.*

This paper is a product of our efforts to generalize the above theorem. We consider Jordan curves in $\mathbf{R}^n$ with Hausdorff dimension $\alpha \in [1, n)$.

Received by the editors on June 19, 1995.
The second author was partially supported by the University of Cincinnati's Taft Memorial Fund.

In this setting *quasicircle* is replaced with *bounded turning*, abbreviated BT. In Sections 3 and 4 we generalize the Ahlfors regular and chordarc notions to deal with the situation where C has Hausdorff dimension α. Thus C is α-dimensional Ahlfors regular, or AR^α, if the $\mathcal{H}^\alpha$-measure of that part of C inside a ball of radius r is comparable to r^α and C is α-dimensional chordarc, or CA^α, provided a chordarc-like condition holds where now we require that the $\mathcal{H}^\alpha$-measure of A be comparable to $|x-y|^\alpha$.

This document is organized as follows: Section 2 contains preliminary information including definitions and terminology descriptions. In Sections 3 and 4, respectively, we examine α-dimensional Ahlfors regular sets and α-dimensional chordarc curves in $\mathbf{R}^n$. We conclude Section 4 with a long list of equivalent conditions for certain Jordan curves in $\mathbf{R}^n$. In Section 5 we exhibit numerous examples which illustrate our results.

Now we advertise some of our conclusions. Naturally there arises the question of the existence of CA^α curves. A generalization of the well-known von Koch snowflake provides examples for dimensions $1 \le \alpha < 2$. For general $\alpha \in [1, n)$ we construct self-similar Jordan curves with the desired properties, which gives our initial result; see Examples 5.1, 5.2.

Theorem A. *Given $1 \le \alpha < n$, there exist α-dimensional chordarc Jordan curves in $\mathbf{R}^n$.*

Below we write $\mathbf{S}^\alpha$ for the CA^α curve given by Theorem A. The following is a consequence of Theorem 4.6 and Corollary 4.7.

Theorem B. *For a Jordan curve $C \subset \mathbf{R}^n$, the following are equivalent.*

(a) *C is α-dimensional chordarc.*

(b) *C is α-dimensional Ahlfors regular and bounded turning.*

(c) *C is bounded turning and $\mathcal{H}^\alpha(A) \approx (\operatorname{diam}(A))^\alpha$ for all arcs $A \subset C$.*

(d) *There is a BL^α homeomorphism from C to $\mathbf{S}^1$.*

(e) *There is a BL homeomorphism from C to $\mathbf{S}^\alpha$.*

All constants depend only on each other, α and $\operatorname{diam}(C)$. *When $n = 2$ the map in* (e) *extends to a BL self-homeomorphism of* $\mathbf{R}^2$.

In Section 3 we study AR^α sets and allow $0 < \alpha < n$. We first verify the Möbius invariance of AR^α sets; see Theorem 3.1.

Theorem C. *A set $E \subset \mathbf{R}^n$ is α-dimensional Ahlfors regular if and only if $\varphi(E)$ is α-dimensional Ahlfors regular for some (all) Möbius transformation(s) φ.*

Next we establish an analog of a result of Meyer (see Zinsmeister's thesis [**11**, Proposition 2]) who characterized regular curves in terms of their reflections across certain circles. We summarize our results as follows. See the discussion in Section 3 for precise terminology; here E_x denotes the image of E under the reflection in the sphere centered at x with radius the distance from x to E. See Theorem 3.4 and Corollary 3.5.

Theorem D. *For $E \subset \mathbf{R}^n$, the following are equivalent*:

(a) *E is α-dimensional Ahlfors regular.*

(b) *E is porous and $\mathcal{H}^\alpha(E_x) \approx (\operatorname{diam}(E_x))^\alpha$ for all $x \notin \overline{E}$.*

(c) *E is porous and $\mathcal{H}^\alpha(\varphi(E)) \approx 1$ for all Möbius φ with* $\operatorname{diam}(\varphi(E)) = 1$.

We also investigate the relationship between the α-dimensional Ahlfors regular property of a set and uniform bounds on its upper and lower $\mathcal{H}^\alpha$-densities. We believe the following helps clarify the situation; see Example 5.4.

Theorem E. *Given $0 < \alpha < n$ there exists a compact set $E \subset \mathbf{R}^n$ (in fact a bounded turning Jordan curve when $\alpha \geq 1$) which is a porous α-set with lower/upper $\mathcal{H}^\alpha$-densities which are everywhere in E bounded away from $0/\infty$, yet E fails to be α-dimensional Ahlfors regular. When $\alpha = 1$ we can take E to be a Jordan curve which has linear density one at each of its points.*

In the positive direction the succeeding holds. Here we declare E to be *bilipschitz homogeneous* provided there is a constant K such that for each pair of points $x, y \in E$ there exists a K-bilipschitz homeomorphism $f : E \to E$ satisfying $f(x) = y$. See Theorem 3.7 and Example 5.3.

Theorem F. *A bilipschitz homogeneous α-set with positive lower $\mathcal{H}^\alpha$-density at some point is α-dimensional Ahlfors regular.*

We point out that there are bilipschitz homogeneous Jordan curves which are α-sets yet fail to be AR^α.

2. Preliminaries. Our notation is relatively standard. We write $c = c(a, \dots)$ to indicate a constant c which depends only on the parameters $a, \dots$. We write $a \approx b$ to mean there exists a positive finite constant c with $a/c \leq b \leq ac$; typically c will depend on various parameters, and we try to make this as clear as possible often giving explicit values. We let $B(x; r) = \{y : |x-y| < r\}$ and $S(x; r) = \partial B(x; r)$ denote the open ball and sphere of radius r centered at the point x.

Let C be a Jordan curve in $\mathbf{R}^n$; thus C is a simple, closed, bounded curve in $\mathbf{R}^n$. Given two points x, y in C we let $C(x, y)$ and $C^\alpha(x, y)$ denote the components of $C \backslash \{x, y\}$ with minimal diameter and with minimal $\mathcal{H}^\alpha$-measure respectively. (Abusing notation we often assume that $C(x, y)$ or $C^\alpha(x, y)$ include the endpoints x, y!) Here $\mathcal{H}^\alpha$ denotes Hausdorff α-dimensional measure, which is defined below.

A *quasicircle* is the image of $\mathbf{S}^1$ via a global quasiconformal homeomorphism of the plane. Ahlfors characterized quasicircles as the Jordan plane curves which satisfy a *bounded turning* condition; that is, a Jordan curve $C \subset \mathbf{R}^2$ is a quasicircle if and only if there is a constant a such that for all $x, y \in C$ we have

$$\operatorname{diam}(C(x, y)) \leq a|x - y|.$$

In the sequel we say that a Jordan curve C in $\mathbf{R}^n$ is a-BT if the above condition holds for all points $x, y \in C$.

Recall that f is *K-bilipschitz*, abbreviated K-BL, if

$$|x - y|/K \leq |f(x) - f(y)| \leq K|x - y| \quad \text{for all } x, y.$$

Next, we say that a map f is K-BL$^\alpha$ if

$$|x-y|^\alpha/K \le |f(x)-f(y)| \le K|x-y|^\alpha \quad \text{for all } x, y.$$

We denote *Hausdorff measure* by $\mathcal{H}^\alpha$, but often write $l = \mathcal{H}^1$ for Euclidean arclength. Thus

$$\mathcal{H}^\alpha(A) = \lim_{r\to 0}\left[\inf\left\{\sum_1^\infty \operatorname{diam}(U_i)^\alpha : A \subset \bigcup_1^\infty U_i, \operatorname{diam}(U_i) \le r\right\}\right].$$

We call A an *α-set* provided $0 < \mathcal{H}^\alpha(A) < \infty$. The *Hausdorff dimension* of A is

$$\dim_{\mathcal{H}}(A) = \inf\{\alpha > 0 : \mathcal{H}^\alpha(A) < \infty\}.$$

The *upper* and *lower $\mathcal{H}^\alpha$-densities* of A at x are

$$\overline{D}^\alpha(A,x) = \limsup_{r\to 0} \frac{\mathcal{H}^\alpha(A\cap B(x;r))}{(2r)^\alpha},$$

$$\underline{D}^\alpha(A,x) = \liminf_{r\to 0} \frac{\mathcal{H}^\alpha(A\cap B(x;r)}{(2r)^\alpha}.$$

For further details we refer to [**4,5**].

We make extensive use of Hausdorff measure preserving maps. Let A, B be α-, β-sets respectively. Put $a = \mathcal{H}^\alpha(A), b = \mathcal{H}^\beta(B)$. We call $f : A \to B$ an *$\mathcal{H}^\alpha/\mathcal{H}^\beta$-measure preserving* map provided $a\mathcal{H}^\beta(fE) = b\mathcal{H}^\alpha(E)$ for all α-sets $E \subset A$.

Lemma 2.1. *Let $C \subset \mathbf{R}^n$ be a Jordan curve. There exists an $\mathcal{H}^1/\mathcal{H}^\alpha$-measure preserving homeomorphism from $\mathbf{S}^1$ onto C if and only if each nondegenerate subarc of C is an α-set.*

Proof. The necessity of this condition is apparent; we confirm its sufficiency. Since C is a Jordan curve, there is a homeomorphism $\varphi : \mathbf{S}^1 \to C$. Define $h(t) = \mathcal{H}^\alpha(\varphi\{e^{i\theta} : 0 \le \theta \le t\})$ for $0 \le t \le 2\pi$. Then $h : [0, 2\pi] \to [0, c]$ is a homeomorphism, where $c = \mathcal{H}^\alpha(C)$. The injectivity and continuity of h follow from

$$h(s) = h(t) + \mathcal{H}^\alpha(\varphi\{e^{i\theta} : t \le \theta \le s\})$$

which holds for all $0 \leq t < s \leq 2\pi$. (Note that $\mathcal{H}^\alpha$ is additive on subarcs of C and continuous with respect to monotone sequences of subarcs.) Then

$$\Phi(e^{i\tau}) = \varphi(\exp(ih^{-1}(c\tau/2\pi)))$$

is the desired homeomorphism from $\mathbf{S}^1$ onto C. $\square$

Of course even bilipschitz homeomorphisms generally fail to preserve measure. Thus we require information about how BL, BL$^\alpha$, and other maps distort Hausdorff measure. Minor modifications in the proof of [**4**, 1.8] yields our next result which describes the change in Hausdorff measure due to a Hölder map.

Lemma 2.2. *Suppose $f : E \to F$ is surjective and $|f(x) - f(y)| \leq K|x-y|^\alpha$ for all $x, y \in E$. Then*

$$\mathcal{H}^\beta(F) \leq K^\beta \mathcal{H}^{\alpha\beta}(E) \quad \textit{for all } \beta > 0,$$

and thus $\dim_{\mathcal{H}}(E) \geq \alpha \dim_{\mathcal{H}}(F)$.

Corollary 2.3. *Suppose there is a K-BL^α homeomorphism from E to F. Then* $\dim_{\mathcal{H}}(E) = \alpha\dim_{\mathcal{H}}(F)$ *and*

$$\mathcal{H}^{\alpha\beta}(E)/K^\beta \leq \mathcal{H}^\beta(F) \leq K^\beta \mathcal{H}^{\alpha\beta}(E) \quad \textit{for all } \beta > 0.$$

Now we examine the distortion in Hausdorff measure caused by Möbius transformations. It suffices to consider inversion in the unit sphere and we write $x^* = x/|x|^2$ to denote the point symmetric to x with respect to $\mathbf{S}^{n-1}$.

Lemma 2.4. *Let $\sigma(x) = x^* = x/|x|^2$. Suppose $A = \{r/2 \leq |x| \leq 2r\}$. Then*

$$\sigma(x) = \frac{\psi(x)}{r^2} \qquad \textit{where} \quad \psi : A \to A \quad \textit{is } 4\text{-}BL.$$

Thus $\sigma : A \to \sigma(A)$ is a composition of a 4-BL map followed by a dilation.

Proof. Define $\psi(x) = r^2\sigma(x)$. Using the basic identity [**1**, (3.1.5)]

$$|x^* - y^*| = \frac{|x-y|}{|x||y|}$$

we obtain

$$|\psi(x) - \psi(y)| = \frac{|x-y|}{|x/r||y/r|},$$

which yields $(1/4)|x-y| \leq |\psi(x)-\psi(y)| \leq 4|x-y|$ when $x, y \in A$. $\square$

3. Ahlfors regular sets in space. We call $E \subset \mathbf{R}^n$ an α-*dimensional Ahlfors regular* set provided there exists a constant b such that for each $x \in E$ and each $0 < r < \operatorname{diam}(E)$ we have

$$\text{(AR)} \qquad r^\alpha/b \leq \mathcal{H}^\alpha(E \cap B(x;r)) \leq br^\alpha.$$

We abbreviate this by saying that E is b-AR$^\alpha$. Note that here $0 < \alpha < n$.

The one-dimensional Ahlfors regular Jordan curves were introduced by Ahlfors in his 1935 *Acta* paper on the theory of covering surfaces. These regular curves are of significant interest in harmonic analysis and singular integral theory because of David's theorem [**3**] that they are precisely the curves for which the Cauchy integral defines a bounded operator from L^2 into L^2. Regular curves also have an important connection with the so-called level set problem for conformal mappings; see [**2**, Theorem 3] and the discussion and references mentioned therein.

We begin this section with the observation that the class of AR$^\alpha$ sets is invariant with respect to Möbius transformations of $\overline{\mathbf{R}}^n$. Then we give a characterization in terms of reflections across certain spheres.

Theorem 3.1. *The Möbius image of a b-AR$^\alpha$ subset of $\mathbf{R}^n$ is c-AR$^\alpha$ where $c = c(b, \alpha)$.*

Proof. Suppose $E \subset R^n$ is b-AR$^\alpha$; let φ be a Möbius transformation of $\overline{\mathbf{R}}^n$, and put $F = \varphi(E)$. According to [**1**, p. 41] we can write

$$\varphi(x) = \lambda A\sigma(x) + z$$

where $\lambda > 0$, $z \in \mathbf{R}^n$, $A \in O(n)$ is an orthogonal linear transformation, and either σ is the identity (if $\varphi(\infty) = \infty$) or σ is the reflection across the sphere $S(\varphi^{-1}(\infty); 1)$ (if $\varphi(\infty) \neq \infty$). In the first case it is easy to verify that F is b-AR$^\alpha$, because translations do not change Hausdorff measure, nor does the map $x \mapsto Ax$ (since this is an isometry and so preserves distances), and the inequalities in (AR) are homogeneous with respect to dilations. In the latter case we reason similarly and realize that it suffices to consider the situation where φ is reflection across the unit sphere:

$$\varphi(x) = x^* = \frac{x}{|x|^2}.$$

Fix $y_0 = \varphi(x_0) \in F$, $0 < R < \operatorname{diam}(F)$ and let $B = B(y_0; R)$. We consider two cases; first suppose that $|y_0| \geq 2R$. In this situation we have $B \subset A = \{y : |y_0|/2 \leq |y| \leq 2|y_0|\}$. Now by Lemma 2.4 we can write $\varphi|_A = \psi/|y_0|^2$ where $\psi : A \to A$ is 4-BL. It follows that

$$B(x_0; R/4|y_0|^2) \subset \varphi(B) \subset B(x_0; 4R/|y_0|^2).$$

Appealing to Lemma 2.2 we obtain

$$\mathcal{H}^\alpha(F \cap B) \leq (4|y_0|^2)^\alpha \mathcal{H}^\alpha(E \cap \varphi(B)) \leq 16^\alpha b R^\alpha$$

and

$$\mathcal{H}^\alpha(F \cap B) \geq (|y_0|^2/4)^\alpha \mathcal{H}^\alpha(E \cap \varphi(B)) \geq R^\alpha/16^\alpha b.$$

Thus when $|y_0| \geq 2R$ we find that (AR) holds with E, b replaced by $F, 16^\alpha b$.

Next suppose that $|y_0| < 2R$. First we establish a lower bound for $\mathcal{H}^\alpha(F \cap B)$. Now there is a constant $\lambda = \lambda(c, \alpha) \in (0, 1/2]$ with the property that $F \cap B(y_0; R/2)$ contains a point y_1 with $|y_1| \geq \lambda R$. (Here is one way to see this: It is immediate that if $A = B(x; r)\backslash B(x; s)$ is a spherical ring with $A \cap E = \varnothing$, $x \in E$, and if A separates the points of E, then $r/s \leq c^{2/\alpha}$. Using this it is not difficult to see that a similar fact holds for 'Möbius rings'; i.e., if $A = B(x; r)\backslash B(y; s) \subset \mathbf{R}^n\backslash E$ and A separates the points of E, then the 'conformal modulus' of A cannot be arbitrarily large.) Setting $R_1 = \lambda R/2$ we have $B(y_1; R_1) \subset B$, and since $|y_1| \geq 2R_1$, the first part of our proof yields

$$\begin{aligned}\mathcal{H}^\alpha(F \cap B) &\geq \mathcal{H}^\alpha(F \cap B(y_1; R_1)) \\ &\geq \frac{R_1^\alpha}{16^\alpha b} = \frac{\lambda^\alpha}{32^\alpha b} R^\alpha.\end{aligned}$$

It remains to produce an upper bound for $\mathcal{H}^\alpha(F\cap B)$. Note that $B\subset B(t)$, $t=3R$. Thus $\mathcal{H}^\alpha(F\cap B)\le\sum\mathcal{H}^\alpha(F_i)$ where

$$\begin{aligned}F_i&=F\cap\{t/4^{i+1}\le|y|\le t/4^i\}=\varphi(E_i),\\ E_i&=E\cap\{4^i/t\le|x|\le 4^{i+1}/t\}.\end{aligned}$$

Since E is b-AR$^\alpha$,

$$\mathcal{H}^\alpha(E_i)\le b(2(4^{i+1}/t))^\alpha.$$

Next, by Lemma 2.4 we can write $\varphi(x)=\psi(x)/r_i^2$ where ψ is 4-BL in E_i and $r_i=2(4^i/t)$. It follows from Lemma 2.2 that

$$\mathcal{H}^\alpha(F_i)\le\frac{4^\alpha}{r_i^{2\alpha}}\mathcal{H}^\alpha(E_i)$$

and hence

$$\mathcal{H}^\alpha(F)\le 8^\alpha bt^\alpha\sum_0^\infty\frac{1}{4^{\alpha i}}=\frac{96^\alpha}{4^\alpha-1}bR^\alpha.$$

Having considered all cases we conclude that F is c-AR$^\alpha$ with $c=c(b,\alpha)=b\max\{16^\alpha,(32/\lambda)^\alpha,96^\alpha/4^\alpha-1\}$. □

Below we characterize the α-dimensional Ahlfors regular sets in terms of reflections across spheres. This is an analog of work by Meyer and Zinsmeister [**11**, Proposition 2]. Our proof requires some preliminary information which we now establish.

Recall that a set $E\subset\mathbf{R}^n$ is *a-porous* provided for each ball $B(x;r)$ there exists a ball $B(y;ar)\subset B(x;r)\backslash E$.

Proposition 3.2. *If $E\subset\mathbf{R}^n$ is b-AR^α, then E is a-porous, $a=a(b,\alpha,n)$.*

Proof. Suppose that E is b-AR$^\alpha$. Fix a ball $B(x;r)$; assume $x\in E$ and $0<r<\operatorname{diam}(E)/\sqrt{n}$. Consider the cube $Q=Q(x;s)\subset B(x;r)$ with center x and edge length $s=2r/\sqrt{n}$. Let i be a large integer. Divide each face of Q into 2^i pieces to get 2^{in} subcubes $\tilde{Q}$ each having edge length $t=s/2^i$. Let $\tilde{Q}=Q(y;t)$ be one of these subcubes.

If $E \cap Q(y; t/3) = \varnothing$, then $B(y; ar) \subset B(x; r) \backslash E$ with $a = 1/(3\sqrt{n}2^i)$, so assume that there exists a point $z \in E \cap Q(y; t/3)$. Since $B(z; t/3) \subset \tilde{Q}$ we find that

$$\mathcal{H}^\alpha(E \cap \tilde{Q}) \geq (t/3)^\alpha / b = \frac{1}{b}\left(\frac{2}{3}\right)^\alpha \frac{r^\alpha}{n^{\alpha/2} 2^{\alpha i}}.$$

Summing over all 2^{in} such subcubes $\tilde{Q}$ we obtain

$$\frac{2^{(n-\alpha)i} r^\alpha}{2^\alpha b n^{\alpha/2}} \leq \mathcal{H}^\alpha(E \cap B(x; r)) \leq b r^\alpha.$$

Thus $i \leq \log(2^\alpha b^2 n^{\alpha/2})/(n-\alpha)\log 2$, whence our claim. $\square$

Another property of b-AR$^\alpha$ sets $A \subset \mathbf{R}^n$, which is trivial to validate, is the following $\mathcal{H}^\alpha/\mathrm{diam}$ inequality:

$$\text{(HD)} \qquad (\mathrm{diam}\,(A))^\alpha / b \leq \mathcal{H}^\alpha(A) \leq b(\mathrm{diam}\,(A))^\alpha.$$

What is somewhat surprising is that we can actually use (HD) to characterize AR$^\alpha$ sets in $\mathbf{R}^n$. Our proof employs the following.

Lemma 3.3. *Suppose $E \subset \mathbf{R}^n$ enjoys the property that*

$$\mathcal{H}^\alpha(E \cap B(r)) \leq c r^\alpha$$

for all $r > d$. Let φ be the reflection in the sphere $S(d)$ of radius d. Then for all $t > 1$,

$$\mathcal{H}^\alpha(\varphi[E \backslash B(dt)]) \leq 2^{\alpha+1} c \left(\frac{d}{t}\right)^\alpha.$$

Proof. As in the proof of Lemma 2.4, we observe that $|\varphi(x) - \varphi(y)| = d^2|x-y|/|x||y| \leq |x-y|/4^i t^2$ for points $|x|, |y| \geq 2^i\, dt$. Utilizing

Lemma 2.2, we thus obtain

$$\begin{aligned}\mathcal{H}^\alpha(\varphi[E\backslash B(dt)]) &= \sum_{i=0}^{\infty} \mathcal{H}^\alpha(\varphi[E\cap(B(2^{i+1}\,dt)\backslash B(2^i\,dt))]) \\ &\le \sum_{i=0}^{\infty}\left(\frac{1}{4^i t^2}\right)^\alpha \mathcal{H}^\alpha(E\cap B(2^{i+1}\,dt)) \\ &\le c\sum_{i=0}^{\infty}\left(\frac{1}{4^i t^2}\right)^\alpha (2^{i+1}\,dt)^\alpha \\ &= 2^\alpha c\left(\frac{d}{t}\right)^\alpha \sum_{i=0}^{\infty} 2^{-\alpha i},\end{aligned}$$

as claimed. □

The following notation will be in force: For $x \notin \overline{E}$ set $d_x = \operatorname{dist}(x, E)$, let φ_x be the reflection in the sphere $S(x; d_x)$, and put $E_x = \varphi_x(E)$.

Theorem 3.4. *A set $E \subset \mathbf{R}^n$ is c-AR^α if and only if E is a-porous and for each $x \notin \overline{E}$, the $\mathcal{H}^\alpha/\mathrm{diam}$ inequality* (HD) *holds with $A = E_x$. Here the constants a, b, c depend only on each other and α.*

Proof. The necessity of these conditions is apparent from Proposition 3.2, Theorem 3.1 and the preceding remarks. We substantiate their sufficiency. Thus we assume that E is a-porous and that (HD) is true with $A = E_x$ for all $x \notin \overline{E}$. Fix $z \in E$, $0 < r < \operatorname{diam}(E)$, and let $B = B(z; r)$.

First we demonstrate that

$$\mathcal{H}^\alpha(E\cap B) \le c_1 r^\alpha \quad \text{where} \quad c_1 = b(8/a)^\alpha.$$

Since E is porous, we can choose a point $x \in B\backslash\overline{E}$ with

$$d_x = ar \quad \text{and} \quad B(x; d_x) \subset B.$$

Now $E \cap B$ is contained in the spherical ring $R = B(x; 2r)\backslash B(x; d_x)$, so as in the proof of Lemma 2.4 we see that φ_x is $4/a^2$–BL in R. According to Corollary 2.3, in conjunction with (HD), we can now assert that

$$\begin{aligned}\mathcal{H}^\alpha(E\cap B) &\le \left(\frac{4}{a^2}\right)^\alpha \mathcal{H}^\alpha(\varphi_x[E\cap B]) \le \left(\frac{4}{a^2}\right)^\alpha \mathcal{H}^\alpha(E_x)\\ &\le b\left(\frac{4}{a^2}\right)^\alpha (\operatorname{diam}(E_x))^\alpha \le b\left(\frac{4}{a^2}\right)^\alpha (2d_x)^\alpha\\ &= c_1 r^\alpha.\end{aligned}$$

It remains to confirm that

$$\mathcal{H}^\alpha(E\cap B)\ge r^\alpha/c_3 \quad \text{where} \quad c_3 = 8b^3 128^\alpha(1+1/a)^\alpha.$$

We begin by determining t so that

$$t^\alpha = 4b^2 64^\alpha (1+1/a)^\alpha.$$

Again we use the porosity hypothesis, this time to select a point $x\in B(z;r/t)\backslash\overline{E}$ with

$$d_x = ar/t \quad \text{and} \quad B(x;d_x)\subset B(z;r/t).$$

Now since $B(x;\rho)\subset B(z;r+\rho)$ for all $\rho>0$, the first part of our proof yields $\mathcal{H}^\alpha(E\cap B(x;\rho))\le c_2\rho^\alpha$, where $c_2 = c_1(1+t/a)^\alpha$; thus, we can appeal to Lemma 3.3 and deduce that

$$\mathcal{H}^\alpha(\varphi_x[E\backslash B(x;d_xt/2a)])\le 2^{\alpha+1}c_2(2ad_x/t)^\alpha.$$

Next note that, as $t>4$, $E\not\subset B(x;2d_x)$, so $\operatorname{diam}(E)>4d_x$ and therefore $\operatorname{diam}(E_x)\ge d_x/2$. Hence by (HD)

$$\mathcal{H}^\alpha(E_x)\ge \frac{(d_x)^\alpha}{2^\alpha b}.$$

Now $|y-x|<r/2$ gives $|y-z|<|y-x|+|x-z|<r/2+r/t<r$, so $B(x;d_xt/2a)=B(x;r/2)\subset B(z;r)=B$. Thus

$$\begin{aligned}\mathcal{H}^\alpha(\varphi_x[E\cap B]) &\ge \mathcal{H}^\alpha(\varphi_x[E\cap B(x;d_xt/2a)])\\ &= \mathcal{H}^\alpha(E_x) - \mathcal{H}^\alpha(\varphi_x[E\backslash B(x;d_xt/2a)])\\ &\ge \frac{(d_x)^\alpha}{2^\alpha b} - 2^{\alpha+1}c_2\left(\frac{2ad_x}{t}\right)^\alpha\\ &= \frac{d_x^\alpha}{2^{\alpha+1}b} = \frac{r^\alpha}{c_3}\end{aligned}$$

by our choice of t.

As above, $E\cap B$ is contained in the spherical ring $R=B(x;2r)\backslash B(x;d_x)$, so using the ideas in the proof of Lemma 2.4 we see that φ_x is Lipschitz in R with constant 1. Appealing to Corollary 2.3 we can now assert that

$$\mathcal{H}^\alpha(E\cap B)\geq\mathcal{H}^\alpha(\varphi_x[E\cap B])\geq r^\alpha/c_3$$

which completes the proof. □

Here is a list of equivalent notions which combines Theorems 3.1 and 3.4.

Corollary 3.5. *For $E\subset\mathbf{R}^n$, the following are equivalent*:

(a) *E is AR^α.*

(b) *$\varphi(E)$ is AR^α for some (all) Möbius transformation(s) φ.*

(c) *E is porous and $\mathcal{H}^\alpha(E_x)\approx(\operatorname{diam}(E_x))^\alpha$ for all $x\notin E$.*

(d) *E is porous and $\mathcal{H}^\alpha(\varphi(E))\approx 1$ for all Möbius φ with $\operatorname{diam}(\varphi(E))=1$.*

We conclude this section by considering several questions related to the definition of AR$^\alpha$ sets. First we point out that sets which satisfy a uniform local AR$^\alpha$ condition are AR$^\alpha$.

Lemma 3.6. *Suppose $0<\varepsilon<1$ and (AR) holds for $x\in E$, $0<r<\varepsilon\operatorname{diam}(E)$. Then E is c-AR^α with $c=ab, a=a(\varepsilon,\alpha,n)\approx\varepsilon^{-n\alpha}$.*

Next we turn our attention to any possible link between regularity and bounds on the $\mathcal{H}^\alpha$-densities. Obviously a b-AR$^\alpha$ set has lower, upper $\mathcal{H}^\alpha$-densities which are bounded away from 0, ∞ by $2^{-\alpha}/b$, $2^{-\alpha}b$ respectively at each of its points. We are interested in determining when this necessary condition is sufficient.

An immediate upshot of the definition of lower $\mathcal{H}^\alpha$-density is that, whenever $\delta=\underline{D}^\alpha(E,x)>0$, $\mathcal{H}^\alpha(E\cap B(x;r))\geq cr^\alpha$ for all $0<r<R$ (where $c=2^{\alpha-1}\delta$ and $R=R(x,E)$) and an analogous statement holds concerning finiteness of the upper $\mathcal{H}^\alpha$-density. Based on these considerations we initially expected that somehow regularity could be

deduced from having uniform positive/finite bounds on the lower/upper $\mathcal{H}^\alpha$-densities. However, as Theorem E shows, there is absolutely no hope for such a result.

With the above in mind we believe that Theorem F is of importance; below we present a more quantitative version of this. Recall that E is said to be K-*bilipschitz homogeneous*, abbreviated K-BLH, if for each pair of points $x, y \in E$ there exists a K-bilipschitz homeomorphism $f : E \to E$ satisfying $f(x) = y$. (We examine this condition in greater detail in a forthcoming paper.) We emphasize that each of the hypotheses stated below is essential: Example 5.3 gives a BLH set with densities bounded away from 0 and ∞ which fails to be AR^α, and (in a forthcoming paper we verify that) there are BLH Jordan curves which are α-sets yet are not AR^α.

Theorem 3.7. *Let $E \subset \mathbf{R}^n$ be a K-BLH α-set. Suppose $\underline{D}^\alpha(E, x) \geq \delta > 0$ for some point $x \in E$. Then E is b-AR^α where $b = aK^{2\alpha}$ and $a = a(\alpha, \delta, x, E)$.*

Proof. By definition there is a constant $R_1 = R_1(x, E) > 0$ such that

$$\mathcal{H}^\alpha(E \cap B(x; r)) \geq \frac{\delta}{2}(2r)^\alpha \quad \text{for all} \quad 0 < r < R_1.$$

We assume that $d = \operatorname{diam}(E) > R_1$. The above then gives

$$\mathcal{H}^\alpha(E \cap B(x; r)) \geq b_1 r^\alpha \quad \text{for all} \quad 0 < r < d,$$

where $b_1 = 2^{\alpha-1}\delta(R_1/d)^\alpha$.

Now fix any point $y \in E$ and $0 < r < d$. Choose a K-BL $f : E \to E$ with $f(x) = y$. Then $f(E \cap B(x; r/K)) \subset E \cap B(y; r)$, so

$$\mathcal{H}^\alpha(E \cap B(y; r)) \geq K^{-\alpha}\mathcal{H}^\alpha(E \cap B(x; r/K)) \geq b_1 K^{-2\alpha} r^\alpha.$$

Mimicking the preceding argument, and using the fact that $\overline{D}^\alpha(E, z) \leq 1$ for $\mathcal{H}^\alpha$almost everywhere $z \in E$, we deduce that

$$\mathcal{H}^\alpha(E \cap B(y; r)) \leq b_2 K^{2\alpha} r^\alpha,$$

where $b_2 = \max\{2^{\alpha+1}, \mathcal{H}^\alpha(E)/R_2^\alpha\}$ and $R_2 = R_2(z, E)$. Thus E is b-AR^α with $b = aK^{2\alpha}$, $a = \max\{2^{\alpha+1}, \mathcal{H}^\alpha(E)/R_2^\alpha, 2^{1-\alpha}\delta^{-1}(d/R_1)^\alpha\}$. $\square$

4. Chordarc curves in space. The Ahlfors regular quasicircles are precisely the *chordarc* curves, also called Lavrentiev or quasismooth curves. These have been studied by David, Jerison and Kenig, Lavrentiev, Pommerenke, Semmes, Tukia, Warchawski, and Zinsmeister. Tukia [**9**], and independently Jerison and Kenig [**8**, 1.13, p. 227], established that a Jordan curve in $\mathbf{R}^2$ is chordarc if and only if it is the image of $\mathbf{S}^1$ or $\mathbf{R}$ under a global bilipschitz self-homeomorphism of $\mathbf{R}^2$.

In this section we generalize certain properties of chordarc curves, replacing the original chordarc condition by our $\mathcal{H}^\alpha$-chordarc condition (see the paragraph preceding Proposition 4.4) and regularity by Ahlfors α-dimensional regularity. In particular we confirm that CA^α Jordan curves are precisely the BT AR^α curves.

Let $C \subset \mathbf{R}^n$ be a Jordan curve. The natural α-dimensional generalization of the ordinary chordarc definition would be to require that there is a constant c such that for each pair of points $x, y \in C$ one of the components $A = A(x, y)$ of $C \backslash \{x, y\}$ satisfies

$$\text{(CA)} \qquad |x-y|^\alpha / c \leq \mathcal{H}^\alpha(A) \leq c|x-y|^\alpha.$$

We are immediately confronted with the question: Can we take $A = C(x, y)$ or $A = C^\alpha(x, y)$? (Recall that these are the components of $C\backslash\{x, y\}$ with minimal diameter and minimal $\mathcal{H}^\alpha$-measure respectively.) Proposition 4.4 answers this in the affirmative, and we base our formal definition on this result. Our proofs require the ensuing technical results.

The bisection method can be used to corroborate the following.

Lemma 4.1. *Let $C \subset \mathbf{R}^n$ be a Jordan curve. Suppose that* (CA) *holds for one of the components A of $C\backslash\{x, y\}$ for all $x, y \in C$. Then C is an α-set.*

Lemma 4.2. *Let C be as in Lemma* 4.1. *Put $d = \operatorname{diam}(C), m = \mathcal{H}^\alpha(C)$ and $\delta = (m/2c)^{1/\alpha}$. Let I be any subarc of C, let x, y be the endpoints of I, and let $A = A(x, y)$ be the component of $C\backslash\{x, y\}$ which satisfies* (CA). *Then*:

(a) *If* $\operatorname{diam}(I) < d/2$, *then* $I = C(x, y)$.

(b) *If* $\mathcal{H}^\alpha(I) < m/2$*, then* $I = C^\alpha(x,y)$.

(c) *If* $|x-y| \le \delta$*, then* $\mathcal{H}^\alpha(A) = \mathcal{H}^\alpha(C^\alpha(x,y))$.

(d) *If* $|x-y| \ge \delta$*, then* $\mathcal{H}^\alpha(I) \ge \delta^\alpha/c$.

Proof. Parts (a) and (b) are obvious; (c) follows from $\mathcal{H}^\alpha(A) + \mathcal{H}^\alpha(C^\alpha(x,y)) < 2\mathcal{H}^\alpha(A) \le 2c|x-y|^\alpha$. We verify (d); assume $|x-y| \ge \delta$. If either $I = A$ or $A = C^\alpha(x,y)$, then (d) is an easy consequence of (CA). Suppose $I \ne A$ and $A \ne C^\alpha(x,y)$; so $I = C^\alpha(x,y)$. Select $z \in I \cap S(x;\delta)$. Then by (c), $A(x,z) = C^\alpha(x,z)$. Also, since $z \in I$, $C^\alpha(x,z) \subset C^\alpha(x,y)$. Thus

$$\mathcal{H}^\alpha(I) \ge \mathcal{H}^\alpha(C^\alpha(x,z)) = \mathcal{H}^\alpha(A(x,z)) \ge |x-z|^\alpha/c = \delta^\alpha/c$$

as desired. $\square$

Corollary 4.3. *Let* C *be as in Lemma* 4.1*. Then*:

(a) $\mathcal{H}^\alpha(I) > 0$ *for all non-degenerate arcs* $I \subset C$.

(b) $\operatorname{diam}(C)^\alpha/c \le \mathcal{H}^\alpha(C) \le 2c \operatorname{diam}(C)^\alpha$.

Proof. Put $d = \operatorname{diam}(C)$, $m = \mathcal{H}^\alpha(C)$ and $\delta = (m/2c)^{1/\alpha}$.

(a) Let x, y be the endpoints of a nonempty subarc I of C, and let A be the component of $C\backslash\{x,y\}$ which satisfies (CA). If $|x-y| \ge \delta$, then Lemma 4.2(d) asserts that $\mathcal{H}^\alpha(I) \ge \delta^\alpha/c > 0$. On the other hand, if $|x-y| \le \delta$, then Lemma 4.2(c), in conjunction with (CA), implies $\mathcal{H}^\alpha(I) \ge \mathcal{H}^\alpha(A(x,y)) \ge |x-y|^\alpha/c > 0$.

(b) First choose $x, y \in C$ with $|x-y| = d$. Then

$$\mathcal{H}^\alpha(C) \ge \mathcal{H}^\alpha(A(x,y)) \ge |x-y|^\alpha/c = d^\alpha/c.$$

Next choose $x, y \in C$ with $\mathcal{H}^\alpha(C_1) = \mathcal{H}^\alpha(C_2)$ where C_1, C_2 are the components of $C\backslash\{x,y\}$. Since $\mathcal{H}^\alpha(C) < \infty$, we obtain

$$\mathcal{H}^\alpha(C) = 2\mathcal{H}^\alpha(C_i) \le 2c|x-y|^\alpha \le 2cd^\alpha,$$

where $i = 1$ or $i = 2$ is chosen so that $C_i = A(x,y)$. $\square$

We are now in a position to establish the ensuing result upon which we base our formal definition; to wit, we declare a Jordan curve $C \subset \mathbf{R}^n$

to be an *α-dimensional chordarc* curve provided there is a constant c such that one of (a) or (b) or (c), and hence all of these, holds for each pair of points $x, y \in C$. We abbreviate this by saying that C is c-CA^α.

Proposition 4.4. *For a Jordan curve $C \subset \mathbf{R}^n$, the following are equivalent*:

(a) (CA) *holds for some component $A = A(x, y)$ of $C \backslash \{x, y\}$.*

(b) (CA) *holds for $A = C^\alpha(x, y)$.*

(c) (CA) *holds for $A = C(x, y)$.*

Of course, here we require that (CA) *holds for all $x, y \in C$; the constant c will vary from* (a) *to* (c)*, but α is fixed.*

Proof. Clearly (c) implies (a). We demonstrate (a) $\implies$ (b) and (b) $\implies$ (c).

(a) $\implies$ (b). By definition $\mathcal{H}^\alpha(C^\alpha(x, y)) \leq \mathcal{H}^\alpha(A(x, y)) \leq c|x-y|^\alpha$, so we need only establish a lower bound for $\mathcal{H}^\alpha(C^\alpha(x, y))$. First, by Lemma 4.2(c) we know that $\mathcal{H}^\alpha(A(x, y)) = \mathcal{H}^\alpha(C^\alpha(x, y))$ when $|x-y| \leq \delta = (m/2c)^{1/\alpha}$. Suppose $|x-y| \geq \delta$. Applying Lemma 4.2(d) to $I = C^\alpha(x, y)$, and using Corollary 4.3(b), we obtain

$$\mathcal{H}^\alpha(C^\alpha(x, y)) \geq \frac{\delta^\alpha}{c} \geq \frac{d^\alpha}{2c^3} \geq \frac{|x-y|^\alpha}{2c^3}.$$

(b) $\implies$ (c). Since $\mathcal{H}^\alpha(C(x, y)) \geq \mathcal{H}^\alpha(C^\alpha(x, y)) \geq |x-y|^\alpha/c$, it suffices to determine an upper bound for $\mathcal{H}^\alpha(C(x, y))$. Assume that $\mathcal{H}^\alpha(C(x, y)) > \mathcal{H}^\alpha(C^\alpha(x, y))$. Choose a point $z \in C^\alpha(x, y)$ with $2|x-z| \geq \operatorname{diam}(C^\alpha(x, y)) \geq \operatorname{diam}(C(x, y))$. Note that $C^\alpha(x, z) \subset C^\alpha(x, y)$ and so

$$|x-z|^\alpha \leq c\mathcal{H}^\alpha(C^\alpha(x, z)) \leq c^2|x-y|^\alpha.$$

Now choose a point $w \in C(x, y)$ so that $\mathcal{H}^\alpha(A_x) = \mathcal{H}^\alpha(A_y) = \mathcal{H}^\alpha(C(x, y))/2$, where A_u is the u-component of $C(x, y) \backslash \{w\}$ ($u = x$ or y). Finally,

$$\begin{aligned} \mathcal{H}^\alpha(C(x, y)) &= 2\mathcal{H}^\alpha(A_x) \leq 2\mathcal{H}^\alpha(C^\alpha(x, w)) \\ &\leq 2c|x-w|^\alpha \leq 2c[\operatorname{diam}(C(x, y))]^\alpha \\ &\leq 2c[2|x-z|]^\alpha \leq 2^{\alpha+1}c^3|x-y|^\alpha. \qquad \square \end{aligned}$$

Now we corroborate that CA and CA$^\alpha$ curves share analogous properties.

Theorem 4.5. *Let $C \subset \mathbf{R}^n$ be a c-CA$^\alpha$ Jordan curve. Then*:
(a) *C is a-BT with $a = 2c^{2/\alpha}$.*
(b) *C is b-AR^α with $b = 2^{\alpha+1}(1+2a)^\alpha c$, $a = 2c^{2/\alpha}$.*
(c) *C satisfies the condition* (HD) *for all arcs $A \subset C$ with $b = 2c$.*

Proof. Put $d = \operatorname{diam}(C)$, $m = \mathcal{H}^\alpha(C)$ and $\delta = (m/2c)^{1/\alpha}$.

(a) Fix $x, y \in C$ and choose $z \in C(x,y)$ with $2|x-z| \geq \operatorname{diam}(C(x,y))$. Using (CA) applied to each of the arcs $C(x,z) \subset C(x,y)$,

$$[\operatorname{diam}(C(x,y))]^\alpha \leq 2^\alpha c \mathcal{H}^\alpha(C(x,z)) \leq 2^\alpha c^2 |x-y|^\alpha.$$

(b) Fix a point $x \in C$ and $0 < r < d$. First, let A be any subarc of C joining x to a point $y \in C \cap S(x;r)$ with $A \subset B(x;r)$. Then

$$\mathcal{H}^\alpha(C \cap B(x;r)) \geq \mathcal{H}^\alpha(A) \geq \mathcal{H}^\alpha(C^\alpha(x,y)) \geq r^\alpha/c.$$

It remains to produce an upper bound for $\mathcal{H}^\alpha(C \cap B(x;r))$.

Put $\varepsilon = 1/2(1+2a)$ where a is the BT constant for C. If $r \geq \varepsilon d$, then Lemma 4.3(b) yields

$$\mathcal{H}^\alpha(C \cap B(x;r)) \leq \mathcal{H}^\alpha(C) \leq 2cd^\alpha \leq \frac{2c}{\varepsilon^\alpha} r^\alpha.$$

Now suppose $r \leq \varepsilon d$. Choose a point $w \in C$ with $|x-w| \geq d/2$. Let A be the component of $C \backslash B(x;r)$ containing w and let y, z be the endpoints of A. We claim that $C(y,z) \supset C \cap B(x;r)$ and hence

$$\mathcal{H}^\alpha(C \cap B(x;r)) \leq \mathcal{H}^\alpha(C(y,z)) \leq c|y-z|^\alpha \leq 2^\alpha c r^\alpha.$$

Note that if $C(y,z) \not\supset C \cap B(x;r)$, then since C is a-BT we would have

$$\begin{aligned} 2ar \geq a|y-z| &\geq \operatorname{diam}(C(y,z)) \\ &\geq |y-w| \geq |x-w| - |x-y| \\ &\geq d/2 - r, \end{aligned}$$

giving $(1+2a)r \geq d/2$ and contradicting $r \leq \varepsilon d$.

(c) Let x, y be the endpoints of an arc $A \subset C$. If (CA) does not hold for A, then there is a subarc $A' \subset A$ with endpoints x', y' and with $\mathcal{H}^\alpha(A') = m/2 \geq \mathcal{H}^\alpha(A)/2$; thus, as (CA) does hold for A', we obtain

$$(\operatorname{diam}(A))^\alpha \geq |x'-y'|^\alpha \geq \mathcal{H}^\alpha(A')/c \geq \mathcal{H}^\alpha(A')/2c.$$

The above inequality is obvious when (CA) holds for A. We conclude that in all cases $\mathcal{H}^\alpha(A) \leq 2c(\operatorname{diam}(A))^\alpha$.

It remains to produce the opposite inequality. This is an immediate consequence of Lemma 4.3(b) when $\mathcal{H}^\alpha(A) \geq m/2$. Assume $\mathcal{H}^\alpha(A) < m/2$. Choose a subarc A' of A with endpoints x', y' such that $\operatorname{diam}(A) = |x'-y'|$. Then (CA) holds for $A' = C^\alpha(x', y')$, so

$$(\operatorname{diam}(A))^\alpha = |x'-y'|^\alpha \leq c\mathcal{H}^\alpha(A') \leq c\mathcal{H}^\alpha(A)$$

as desired. □

Theorem 4.6. *For a Jordan curve $C \subset \mathbf{R}^n$, the following are equivalent.*

(a) *C is CA$^\alpha$.*

(b) *C is AR$^\alpha$ and BT.*

(c) *C is BT and* (HD) *holds for all subarcs $A \subset C$.*

(d) *There is a BL$^\alpha$ homeomorphism from C to $\mathbf{S}^1$.*

Moreover, all constants depend only on each other, α and $\operatorname{diam}(C)$.

Proof. We know from Theorem 4.5 that (a) implies both (b) and (c). We corroborate (b) $\Longrightarrow$ (a), (c) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ (a).

(b) $\Rightarrow$ (a). Suppose that C is both a-BT and b-AR$^\alpha$. Fix points $x, y \in C$ and put $r = \operatorname{diam}(C(x,y))$. Since $C(x,y) \subset \overline{B}(x;r)$,

$$\mathcal{H}^\alpha(C(x,y)) \leq \mathcal{H}^\alpha(C \cap B(x;r)) \leq br^\alpha \leq a^\alpha b|x-y|^\alpha.$$

For the opposite inequality, let $\rho = r/4a$ and select a point $z \in C(x,y)$ with $|x-z|$, $|y-z| > r/4$. If there exists a point $w \in$

$[C \cap B(z;\rho)] \backslash C(x,y)$, then as either $x \in C(w,z)$ or $y \in C(w,z)$ we deduce that

$$r/4 < \operatorname{diam}(C(w,z)) \leq a|w-z| < a\rho = r/4;$$

since this is impossible, we must have $C \cap B(z;\rho) = C(x,y) \cap B(z;\rho)$. Employing (AR) we conclude that

$$\begin{aligned}\mathcal{H}^\alpha(C(x,y)) &\geq \mathcal{H}^\alpha(C(x,y) \cap B(z;\rho)) \\ &= \mathcal{H}^\alpha(C \cap B(z;\rho)) \geq \rho^\alpha/b \\ &= r^\alpha/b(4a)^\alpha.\end{aligned}$$

Thus C is c-CA$^\alpha$ with $c = 4^\alpha a^\alpha b$.

(c) $\Rightarrow$ (d). We outline the argument, since Falconer and Marsh prove essentially this result in [**6**]. Suppose C is a-BT and (HD) holds. An appeal to Lemma 2.1 furnishes an $\mathcal{H}^\alpha/\mathcal{H}^1$-measure preserving map $f : C \to \mathbf{S}^1$.

Fix $x, y \in C$, and let $I = f(A)$, $A = C(x,y)$. Using standard inequalities comparing the diameter and length of subarcs of $\mathbf{S}^1$ and the fact that I must be the smaller component of $\mathbf{S}^1 \backslash \{f(x), f(y)\}$, we see that

$$|h(x) - h(y)| \approx \mathcal{H}^1(I) \approx \mathcal{H}^\alpha(A) \approx (\operatorname{diam}(A))^\alpha \approx |x-y|^\alpha;$$

here the last two approximations follow from (HD) and the fact that C is BT, respectively. A careful inspection of this argument reveals that the Hölder constant can be chosen as $K = b^2 \max\{2\pi a/d^\alpha, d^\alpha/2\}$ where b is the constant from (HD) and $d = \operatorname{diam}(C)$.

(d) $\Rightarrow$ (a). Suppose $f : C \to \mathbf{S}^1$ is a BL$^\alpha$ homeomorphism. Let A be the component of $C \backslash \{x,y\}$ which is mapped by f onto the smaller component of $\mathbf{S}^1 \backslash \{f(x), f(y)\}$. From Corollary 2.3 we obtain $\alpha = \dim_{\mathcal{H}}(C)$ and also

$$\mathcal{H}^\alpha(A) \approx \mathcal{H}^1(f(A)) \approx |f(x) - f(y)| \approx |x-y|^\alpha$$

for all $x, y \in C$ where the constants depend only on the Hölder constant for f; the interested reader can readily verify that when f is K-BL$^\alpha$, C is c-AR$^\alpha$ with $c = (\pi/2)K^2$. $\square$

Corollary 4.7. *Suppose A, B are a-CA^α, b-CA^β Jordan curves. Then there exists a homeomorphism $f : A \to B$ with*

$$|x-y|^\alpha/K \le |f(x)-f(y)|^\beta \le K|x-y|^\alpha \quad \text{for all } x, y$$

where $K = ab\max\{\mathcal{H}^\alpha(A)/\mathcal{H}^\alpha(B), \mathcal{H}^\alpha(B)/\mathcal{H}^\alpha(A)\}$. When $\alpha = \beta$ and $A, B \subset \mathbf{R}^2$, we can extend f to a bilipschitz self-homeomorphism of $\mathbf{R}^2$.

Proof. The first part follows by taking f to be an $\mathcal{H}^\alpha/\mathcal{H}^\beta$-measure preserving map. The latter assertion is a consequence of work by Tukia and Väisälä [**10**]. □

Our results enable us to substantiate the following list of equivalent conditions for certain Jordan curves.

Corollary 4.8. *Let $C \subset \mathbf{R}^n$ be a Jordan curve which is a BT α-set with positive lower $\mathcal{H}^\alpha$-density at some point. Then the following are equivalent.*

(a) *C is CA^α.*

(b) *C is AR^α.*

(c) *C is BLH.*

(d) *$\mathcal{H}^\alpha(A) \approx (\operatorname{diam}(A))^\alpha$ for all arcs $A \subset C$.*

(e) *C is porous and $\mathcal{H}^\alpha(C_x) \approx (\operatorname{diam}(E_x))^\alpha$ for all $x \notin E$.*

(f) *There is a BL^α homeomorphism from C to $\mathbf{S}^1$.*

(g) *There is a BL homeomorphism from $\mathbf{S}^\alpha$ to C.*

(h) *There is a homeomorphism $g : \mathbf{S}^1 \to C$ satisfying*

$$|g(z_1)-g(z_2)| \approx |g(w_1)-g(w_2)| \quad \textit{when} \quad |z_1-z_2| \approx |w_1-w_2|$$

for all $z_1, z_2, w_1, w_2 \in \mathbf{S}^1$.

Moreover, all constants depend only on each other, α and $\operatorname{diam}(C)$.

A thorough discussion of condition (h) above will appear in a forthcoming paper.

5. Examples. In this section we exhibit numerous examples which illustrate the concepts presented throughout our paper. In particular, Examples 5.1 and 5.2 substantiate Theorem A, and Theorem E is a consequence of Example 5.4.

We begin by corroborating the existence of CA^α curves. First we briefly mention the following well known construction.

Example 5.1. The α-dimensional von Koch snowflakes K^α and $\mathbf{K}^\alpha$.

Proof. Fix $\alpha \in (1,2)$ and choose $t \in (1/4, 1/2)$ with $4t^\alpha = 1$. Define a sequence $\{J_n\}$ of Jordan arcs as follows. First, $J_0 = [0,1]$. Next, $J_1 = \cup_{k=1}^4 I_1^k$ where $I_1^k = \sigma_k(J_0)$, σ_k is a similarity from J_0 onto the interval $[a_{k-1}, a_k]$, and (in complex notation) $a_0 = 0, a_1 = t, a_2 = 1/2 + i\sqrt{t - 1/4}, a_3 = 1 - t, a_4 = 1$. Then $J_2 = \cup_{k=1}^4 I_2^k$ where $I_2^k = \sigma_k(J_1)$. Iterating this process yields a sequence of Jordan arcs $\{J_n\}$ with the property that J_n converges (in the Hausdorff metric) to a *snowflake* arc which we denote by K^α[**7**, pp. 728–729].

Employing Hutchison's open set condition [**7**, pp. 735–736] we find that $\dim_{\mathcal{H}}(K^\alpha) = \alpha$. A calculation shows that $\mathcal{H}^\alpha(K^\alpha) = 1$. Using an $\mathcal{H}^\alpha/\mathcal{H}^1$-measure preserving map, or by a direct limit process construction, we obtain a BL^α homeomorphism from K^α to $[0,1]$. We obtain a Jordan curve, the α-dimensional von Koch snowflake $\mathbf{K}^\alpha$, by proceeding as above but starting with an equilateral triangle in place of the interval $[0,1]$. $\square$

We have been unable to generalize the above procedure for the situation $n \geq 3$. We are indebted to K.J. Falconer for discussions leading us to a construction which does generalize. Our goal is to communicate an explicit algorithm which yields a self-similar CA^α Jordan curve for an arbitrary dimension α. To facilitate our exposition and your comprehension, we provide a detailed explanation for the case $1 \leq \alpha < 2$ and then we indicate how to handle the higher dimensional situation.

We begin with an outline of the general construction. Again we obtain a Jordan curve by piecing together certain Jordan arcs, each of which is constructed via a self-similar process. Thus we need only describe how to obtain these arcs. We begin by dividing the cube

$Q = [0,1]^n$ into a large number, say m^n, of identical subcubes $\tilde{Q}$. Then we join the opposite vertices $(0, \dots, 0)$ and $(1, \dots, 1)$ of Q by using the diagonals in a chain of M of the subcubes $\tilde{Q}$. This process is now iterated (by subdividing each $\tilde{Q}$ and joining its opposite vertices ...) and we obtain a fractal arc with (similarity) dimension $\log M/\log m$. By choosing m arbitrarily large and varying M appropriately, we can make $\log M/\log m$ arbitrarily close to any given α, and by carefully choosing the chain of subcubes used we can ensure that we get an "$\mathcal{H}^\alpha$-chordarc arc." We get a Jordan curve by constructing four such arcs which join the opposite vertices $(0, \dots, 0)$ to $(1, \dots, 1)$ to $(0, \dots, 0, 2)$ to $(1, \dots, 1, -1)$ to $(0, \dots, 0)$ of four adjacent cubes.

Example 5.2. Given $1 \le \alpha < n$, there exists a c-CA$^\alpha$ Jordan curve $\mathbf{F}^\alpha$ in $\mathbf{R}^n$, where $c = c(\alpha, n)$.

Proof. For each positive integer $q \in \mathbf{N}$ we construct self-similar Jordan curves having dimensions $\alpha_i^{(q)}$ with the property that $\{\alpha_i^{(q)} : i = 0, \dots, N_q; q \in \mathbf{N}\}$ is dense in $[n-1, n]$. The desired result then follows by using induction and approximating in the Hausdorff metric. Each curve is obtained by taking the union of four similar arcs which join the opposite vertices $(0, \dots, 0), (1, \dots, 1)$; $(1, \dots, 1), (0, \dots, 0, 2)$; $(0, \dots, 0, 2), (1, \dots, 1, -1)$; $(1, \dots, 1, -1), (0, \dots, 0)$ of four adjacent cubes. To construct the arc F which joins the opposite vertices of $Q = [0,1]^n$, we divide Q into m^n (where $m = 4q + 3$) subcubes $\tilde{Q}$. We use the diagonals of the cubes $\tilde{Q}$ from a specific chain of M of these subcubes. We iterate this process. By standard techniques ([**5**, Chapter 9] or [**7**, 5.3]) we obtain a self-similar fractal arc F which is an α-set with $\alpha = \dim_{\mathcal{H}}(F) = \log M/\log m$ and which satisfies an $\mathcal{H}^\alpha$-chordarc condition.

We provide a detailed construction for the arc $F = F_{N_q}^{(q)}$ with 'maximal dimension'; we take $M = (4q+1)(2q+1)^{n-1} + 2$, so

$$\begin{aligned}\alpha_{N_q}^{(q)} = \dim_{\mathcal{H}}(F) &= \frac{\log M}{\log m} \\ &= \frac{\log\big((4q+1)(2q+1)^{n-1} + 2\big)}{\log(4q+3)} \approx n\end{aligned}$$

for q sufficiently large. Then we explain how to modify this arc, step by step, to obtain arcs $F_{N_q-1},\dots,F_1,F_0$ with dimensions $\alpha_i^{(q)}$ which satisfy the requirement that $\{\alpha_i^{(q)} : i=0,\dots,N_q; q\in\mathbf{N}\}$ is dense in $[n-1,n]$.

We begin now with a careful description of the $n=2$ case following the above outline. Fix $q\in\mathbf{N}$ and put $p=4q+1, m=p+2$. Divide the unit square $[0,1]^2$ into m^2 subsquares

$$Q_{ij}=\left[\frac{i}{m},\frac{i+1}{m}\right]\times\left[\frac{j}{m},\frac{j}{m+1}\right],\quad 0\le i,j\le p+1.$$

We refer to Q_{ij} as being in 'column i' and 'row j.' Note that each square Q has two diagonals, Δ^+ and Δ^-, with positive and negative slopes respectively. We join the vertices $(0,0),(1,1)$ of $[0,1]^2$ by using a chain consisting of the diagonals $\Delta_{ij}^{\pm}$ of certain of the subsquares Q_{ij}. (Actually the choice of which diagonal to use will be forced and thus we need only specify which subsquares are in the chain.)

The border subsquares $Q_{i0},Q_{i,p+1},Q_{0j},Q_{p+1,j},\ 0\le i,j\le p+1$, form a 'safety zone'; we only use the two subsquares $Q_{00},Q_{p+1,p+1}$ from this border 'safety zone.' In a similar manner, the 'even rows' $Q_{i2},Q_{i4},\dots,Q_{i,p-1},\ 1\le i\le p$, form 'safety zones' and we use only one subsquare from each 'even row' (to get across the row!). In each of the remaining 'odd rows' we use all possible subsquares alternating the diagonals and moving from 'left to right,' then 'right to left,' and so forth.

In the first row we move 'left to right' using the diagonals

$$\Delta_{11}^+,\ \Delta_{21}^-,\ \Delta_{31}^+,\ \dots,\ \Delta_{p1}^+.$$

This chain of diagonals, together with Δ_{00}^+, now joins $(0,0)$ to $(1/m,1/m)$ to $((p+1)/m,2/m)$. We cross the 'row two safety zone' by using Δ_{p2}^-, and then in row three we move 'right to left' via the diagonals

$$\Delta_{p-1,3}^-,\ \Delta_{p-2,3}^+,\ \dots,\ \Delta_{33}^+,\ \Delta_{32}^-.$$

Next we use Δ_{14}^- to cross the 'row four safety zone.' Our chain of diagonals now joins $(0,0)$ to $(1/m,5/m)$.

We repeat the above process: rows 5, 6, 7, 8 and 9, 10, 11, 12 and $\dots 4q-3,4q-2,4q-1,4q$ are just like rows 1, 2, 3, 4. Finally, we cross row $p = 4q + 1$ from 'left to right' using the same pattern as in row one. We have now constructed a piecewise linear arc A_1 which joins the vertices $(0, 0), (1, 1)$ of $[0, 1]^2$ and consists of

$$M = 2 + p(2q + 1)$$

diagonals $\Delta_{ij}^{\pm}$ each of 'size' $1/m$.

We iterate the above process in the usual way thereby obtaining piecewise linear arcs A_k which join $(0, 0), (1, 1)$ and consist of M^k line segments each of 'size' $1/m^k$. By standard techniques [**5**, Chapter 9], [**7**, 5.3(1)] there exists a limit arc $F = F^\alpha = \lim_{k\to\infty} A_k$ which is an α-set where

$$\alpha = \alpha_{N_q}^{(q)} = \frac{\log M}{\log m} = \frac{\log(p(2q + 1) + 2)}{\log(4q + 3)}.$$

It remains to verify that

$$\mathcal{H}^\alpha(F(x, y)) \approx |x - y|^\alpha \qquad \text{for all} \quad x, y \in F,$$

where $F(x, y)$ is the subarc of F joining x, y. (This condition guarantees that the Jordan curve $\mathbf{F}^\alpha$, obtained by taking the union of four copies of F^α, is CA$^\alpha$.) To see that F satisfies the above $\mathcal{H}^\alpha$-chordarc condition, it suffices, according to Theorem 4.6(c), to show that F is BT and satisfies the $\mathcal{H}^\alpha/\mathrm{diam}$-condition (HD). That (HD) holds is an easy consequence of the fact that F consists of M^k subarcs of 'size' m^{-k} where $Mm^{-\alpha} = 1$, and thus if $\mathrm{diam}\,(A) \approx m^{-k}$, then

$$\mathcal{H}^\alpha(A) \approx \frac{\mathcal{H}^\alpha(F)}{M^k} \approx (m^{-\alpha})^k \approx (\mathrm{diam}\,(A))^\alpha.$$

Employing induction, it is not difficult to demonstrate that F is $2m$-BT.

We have now made clear how to build a c-CA$^\alpha$ Jordan curve $\mathbf{F}^\alpha$, where $c = c(\alpha)$ and $\alpha = \alpha_{N_q}^{(q)} = \log M/\log m = \log(8q^2 + 6q + 3)/\log(4q + 3)$. Clearly $\alpha \nearrow 2$ as $q\to\infty$. Next we explain how to modify the construction of $F = F_{N_q}^{(q)}$, i.e., we modify the construction of the arcs A_k, to get the arcs $F_l^{(q)}$, $l = 0, 1, \dots, N_q - 1$. Here $F_0^{(q)}$ is simply the diagonal of $[0, 1]^2$. The idea is simple, although tedious to write down. What we do is to use fewer and fewer of the diagonals $\Delta_{ij}^{\pm}$ during the iteration part of our construction of F.

We start by 'trimming' the diagonals used in row one: precisely, we replace the diagonals

$$\Delta_{p-1,1}^{-},\ \Delta_{p1}^{+},\ \Delta_{p2}^{-},\ \Delta_{p-1,3}^{-},\ \Delta_{p-2,3}^{+} \quad \text{with} \quad \Delta_{p-2,2}^{-}.$$

Iterating this scheme and taking a limit gives us an arc F_{N_q-1} which has dimension

$$\alpha_{N_q-1} = \alpha_{N_q-1}^{(q)} = \frac{\log M_{N_q-1}}{\log m};$$

here $M_{N_q-1} = M - 4$ (where $M = M_{N_q} = 2 + p(2q+1)$ is as above) because our new construction uses four fewer diagonals.

We continue this 'trimming' process obtaining self-similar limit arcs $F_l = F_l^{(q)}$, $l = N_q, N_q - 1, \dots, 2, 1, 0$. Our construction of F_l uses M_l total diagonals where $0 < M_{l+1} - M_l \leq 4$. Thus the dimensions $\alpha_l = \alpha_l^{(q)}$ satisfy

$$0 < \alpha_{l+1} - \alpha_l = \frac{\log M_{l+1} - \log M_l}{\log m} \leq \frac{\log 5}{\log(4q+3)}.$$

It is now apparent that the set of dimensions $\{\alpha_l^{(q)} : l = 0, \dots, N_q; q \in \mathbf{N}\}$ is dense in $[1,2]$, so by approximating in the Hausdorff metric space we obtain c-CA$^\alpha$ Jordan curves $\mathbf{F}^\alpha$ for each $1 \leq \alpha < 2$, $c = c(\alpha)$.

It remains to elucidate how our construction generalizes to higher dimensions. For convenience we illustrate the case $n = 3$; the interested reader can provide the details for $n > 3$. We divide the cube $[0,1]^3$ into m^3 subcubes (again, $m = p + 2, p = 4q + 1$)

$$Q_{ijk} = \left[\frac{i}{m}, \frac{i+1}{m}\right] \times \left[\frac{j}{m}, \frac{j}{m+1}\right] \times \left[\frac{k}{m}, \frac{k+1}{m}\right]$$
$$0 \leq i, j, k \leq p + 1.$$

We refer to Q_{ijk} as being in 'column i,' 'row j' and at 'depth k.' Thus the subcubes $Q_{ij0}, Q_{i0k}, Q_{0jk}$ form the front, bottom, left faces, respectively. In $[0,1]^3$ we have four possible diagonals: $\Delta^1, \Delta^2, \Delta^3, \Delta^4$ which join the vertices $(0,0,0)$ to $(1,1,1)$, $(0,1,1)$ to $(1,0,0)$, $(1,0,1)$ to $(0,1,0)$, $(0,1,0)$ to $(1,1,0)$ respectively.

We join $(0,0,0),(1,1,1)$ by using a chain consisting of the diagonals Δ^1_{ijk} and Δ^2_{ijk} of certain of the subcubes Q_{ijk}. In addition to the border 'safety zones' (formed by the border cubes Q_{ij0}, $Q_{ij,p+1}$, Q_{i0k}, $Q_{i,p+1,k}Q_{0jk}$, $Q_{p+1,jk}$, $0 \le i,j,k \le p+1$) we also have 'safety depth zones' formed by the cubes at 'even depths' $k = 2,4,\dots,4q$. We go 'up' and 'down' the 'odd depths'; $k = 1,5,\dots,4q+1$ are the 'up depths' and $k = 3,7,\dots,4q-1$ are the 'down depths.' In each of these 'up/down depths' we mimic the two-dimensional case, using the diagonals Δ^1 and Δ^2 in place of Δ^+ and Δ^-, respectively. In this way we obtain a piecewise linear arc which joins the vertices $(0,0,0)$, $(1,1,1)$ of $[0,1]^3$ and consists of

$$M = 2 + p(2q+1)^2$$

diagonals each of 'size' $1/m$.

We iterate obtaining piecewise linear arcs which join $(0,0,0),(1,1,1)$ and consist of M^l line segments each of 'size' $1/m^l$, and then the limit arc is an α-set where

$$\alpha = \alpha^{(q)}_{N_q} = \frac{\log M}{\log m} = \frac{\log(p(2q+1)^2+2)}{\log(4q+3)}.$$

For large q this dimension will be approximately 3. As before, we 'trim' our construction by using fewer and fewer of the diagonals in the iteration. It is during this 'trimming' process that we need to use the diagonals Δ^3_{ijk} and Δ^4_{ijk}. Again we start with the maximal dimension construction (outlined immediately above) and work our way through smaller dimensions until we reach the situation where each 'up/down' depth consists of just the 'diagonal' diagonals Δ^1_{iii} and Δ^4_{iii} $(i = 1,\dots,p)$. This 'minimal dimension' construction yields the dimension

$$\alpha_0 = \frac{\log M_0}{\log m} = \frac{\log(p(2q+1)+2)}{\log(4q+3)} < 2.$$

The construction of the 'in between' arcs uses M_l total diagonals where $0 < M_{l+1} - M_l \le 4$. Thus the dimensions $\alpha_l = \alpha^{(q)}_l$ satisfy

$$0 < \alpha_{l+1} - \alpha_l = \frac{\log M_{l+1} - \log M_l}{\log m} \le \frac{\log 5}{\log(4q+3)}.$$

Again, the set of dimensions $\{\alpha^{(q)}_l : l = 0,\dots,N_q; q \in \mathbf{N}\}$ is dense in $[2,3]$, so we can approximate in the Hausdorff metric space to get any desired dimension $\alpha \in [2,3]$.

We can verify the $\mathcal{H}^\alpha$-chordarc condition just as in the $n = 2$ situation. Utilizing induction we find that this entire construction works for all n. We have now demonstrated how to build a c-CA$^\alpha$ Jordan curve $\mathbf{F}^\alpha$, where $c = c(\alpha, n)$ and $\alpha \in [1, n)$ is arbitrary. □

Now we exhibit numerous examples which have various 'nice' properties yet nonetheless fail to be AR$^\alpha$ or CA$^\alpha$. When $\alpha \geq 1$ the following gives examples of bilipschitz homogeneous curves which have uniform bounds on their upper and lower $\mathcal{H}^\alpha$-densities. Among other things, this example illustrates the necessity of the α-set hypothesis in Theorem F.

Example 5.3. A union of AR$^\alpha$ sets need not be AR$^\alpha$.

Proof. We consider $1 < \alpha < 2$ leaving the remaining cases as exercises for the reader. Let C be the union of infinitely many copies of von Koch snowflake arcs K^α placed at each integer. Then

$$N^{-\alpha}\mathcal{H}^\alpha(C \cap B(0; N)) = N^{-\alpha} 2N\mathcal{H}^\alpha(K^\alpha) = 2N^{1-\alpha} \to 0,$$

so C is not AR$^\alpha$. □

At the end of Section 3 we investigated the relationship between the AR$^\alpha$ property of a set and uniform bounds on its upper and lower $\mathcal{H}^\alpha$-densities. Now we furnish an example which illustrates that even for 'nice' sets it is not enough to simply have bounds on the densities.

Example 5.4. Given $0 < \alpha < n$ there exist a compact set $E \subset R^n$ (a BT Jordan curve when $\alpha \geq 1$) which is a porous α-set with lower/upper $\mathcal{H}^\alpha$-densities which are everywhere in E bounded away from $0/\infty$, yet E fails to be AR$^\alpha$. When $\alpha = 1$ we can take E to be a Jordan curve which has linear density one at each of its points.

Proof. For simplicity we assume that $1 \leq \alpha < 2 = n$. The case $2 \leq \alpha < n$ can be dealt with by using the arcs F^α in place of K^α below, and the Cantor dusts described in Example 5.6 can be used for the case $0 < \alpha < 1$.

Fix $\alpha < \beta < 2$. We construct α-dimensional arcs A_i by placing appropriately shrunken copies of K^α on the 'frames' of the polygonal arcs which approximate K^β. We obtain our desired Jordan curve C by taking two copies of K^α together with an infinite union of arcs $B_i = \rho_i K^\alpha$ and $C_i = r_i A_i$. Here C_i will have an endpoint x_i and since

$$r_i^{-\alpha}\mathcal{H}^\alpha(C \cap B(x_i; r_i)) \geq r_i^{-\alpha}\mathcal{H}^\alpha(C_i) = \mathcal{H}^\alpha(A_i) \to \infty \quad \text{as} \quad i \to \infty,$$

we see why C fails to be AR^α.

Choose t, s so that $4t^\alpha = 1 = 4s^\beta$. Let J_i be the polygonal arc obtained at the ith stage in the construction of K^β; thus J_i consists of 4^i segments each of size s^i; see Example 5.1. On each of these segments of J_i we place a copy of $s^i K^\alpha$ and let A_i be the resulting arc. Thus A_i is a BT, porous Jordan arc joining 0,1 with $\mathcal{H}^\alpha$-densities which satisfy

$$0 < \varepsilon \leq \underline{D}^\alpha(A_i, x) \qquad \overline{D}^\alpha(A_i, x) \leq M < \infty \quad \text{for all} \quad x \in A_i,$$

where $\varepsilon = \varepsilon(\alpha)$, $M = M(\alpha)$, and with measure

$$\mathcal{H}^\alpha(A_i) = 4^i \mathcal{H}^\alpha(s^i K^\alpha) = 4^i s^{\alpha i} = 4^{(1-\alpha/\beta)i}.$$

Now take $x_i = 1/2^i, r_i = 1/\left(2^i 4^{(1/\alpha - 1/\beta)i}\right) < 1/4^i$ (because $0 < 1/\alpha - 1/\beta < 1/2$) and put $\rho_i = x_i - x_{i+1} - r_{i+1}$. Next let C_i be the arc $r_i A_i$ placed at x_i, so C_i joins x_i to $x_i + r_i$, and let B_i be the arc $\rho_i K^\alpha$ placed from $x_{i+1} + r_{i+1}$ to x_i. Finally, let C be the Jordan curve obtained by taking $\mathbf{K}^\alpha$ and replacing the 'top' arc–the one joining 0,1–with $\cup_{i=1}^\infty (B_i \cup C_i)$. Then C is a BT porous Jordan curve which fails to be AR^α because

$$r_i^{-\alpha}\mathcal{H}^\alpha(C \cap B(x_i; r_i)) \geq \mathcal{H}^\alpha(A_i) = 4^{(1-\alpha/\beta)i} \to \infty \quad \text{as} \quad i \to \infty.$$

It remains to verify that C is an α-set with lower/upper $\mathcal{H}^\alpha$-densities which are everywhere in C bounded away from $0/\infty$. Note that

$$\mathcal{H}^\alpha(C_i) = r_i^\alpha \mathcal{H}^\alpha(A_i) = 1/2^{\alpha i} = x_i^\alpha \geq \rho_i^\alpha = \mathcal{H}^\alpha(B_i)$$

and thus

$$\sum_{i=k}^\infty \mathcal{H}^\alpha(C_i) = \frac{1}{2^{\alpha k}} \frac{2^\alpha}{2^\alpha - 1} \geq \sum_{i=k}^\infty \mathcal{H}^\alpha(B_i).$$

In particular,

$$2 < \mathcal{H}^\alpha(C) = 2 + \sum_{i=1}^{\infty} (\mathcal{H}^\alpha(B_i) + \mathcal{H}^\alpha(C_i)) \le \frac{2^\alpha}{2^\alpha - 1}$$

and we see that C is indeed an α-set. Now we establish the bounds on the $\mathcal{H}^\alpha$-densities. Obviously we need only examine these at the origin. It is immediate that

$$\underline{D}^\alpha(C, 0) \ge \underline{D}^\alpha(K^\alpha, 0) \ge \varepsilon > 0;$$

in fact we also have $\underline{D}^\alpha(\bigcup(B_i \cup C_i), 0) > 0$. To show that the upper $\mathcal{H}^\alpha$-density is finite it suffices to demonstrate that the $\mathcal{H}^\alpha$-measure of $\cup(B_i \cup C_i) \cap [0, r]$ is comparable to r^α and we need only do this for $r = x_k + r_k$ for arbitrary k. This follows from our calculations above since

$$\begin{aligned} \mathcal{H}^\alpha(\cup(B_i \cup C_i) \cap [0, r]) &= \sum_{i=k}^{\infty} (\mathcal{H}^\alpha(B_i) + \mathcal{H}^\alpha(C_i)) \\ &\le \frac{2}{2^{\alpha k}} \frac{2^\alpha}{2^\alpha - 1} \le \frac{2r^\alpha}{2^\alpha - 1}. \end{aligned}$$

The truly interested reader can readily check that we do get linear density one everywhere when $\alpha = 1$, provided we change our choice of r_i to $r_i = 1/(4^i 4^{(1-1/\beta)i})$. $\square$

Our next example reveals the necessity of the BT hypothesis in part (c) of Theorems B and 4.6.

Example 5.5. There are Jordan curves C for which the $\mathcal{H}^\alpha$/diam condition (HD) holds for all arcs $A \subset C$, yet C fails to be BT.

Proof. We assume $\alpha = 1$; for $\alpha > 1$ just replace each line segment by an appropriately scaled snowflake arc K^α. We construct C as a limit in the Hausdorff metric of a sequence of Jordan curves C_k. We begin with an equilateral triangle C_1 with vertices $0, z, w$ where $z = e^{\pi i/6}$ and $w = \bar{z}$. The curve C_{k+1} will be obtained from C_k by constructing certain isosceles triangles inside C_1 with bases on the edges $[0, z], [0, w]$.

Let $a_1 = (z+w)/3 = 1/\sqrt{3}$ be the centroid of C_1 and let $z_1 = z/2, w_1 = w/2$ be the midpoints of $[0,z],[0,w]$. Choose points $v_1 \in [z_1,a_1], u_1 \in [w_1,a_1]$ with $|v_1 - a_1| = |u_1 - a_1| = 1/10$. Let T_1 and S_1 be the isosceles triangles with vertices $v_1, (1 \pm 1/5)z_1$ and $u_1, (1 \pm 1/5)w_1$, respectively. Put $C_2 = C_1 \triangle (T_1 \cup S_1)$, where $\triangle$ stands for symmetric difference. Assume that C_k as been defined. Let $z_k = z/2^k, w_k = w/2^k, a_k = a_1/2^{k-1}$ and choose points $v_k \in [z_k, a_k]$ and $u_k \in [w_k, a_k]$ with $|v_k - a_k| = |u_k - a_k| = 1/10^k$. Let T_k and S_k be the isosceles triangles with vertices $v_k, (1 \pm 1/5^k)z_k$ and $u_k, (1 \pm 1/5^k)w_k$, respectively. Put $C_{k+1} = C_k \triangle (T_k \cup S_k)$.

Let C be the limit, in the Hausdorff metric, of the curves C_k. It is easy to see that C is not BT because $|v_k - u_k| \le |v_k - a_k| + |u_k - a_k| \le 2/10^k$ while $\operatorname{diam}(C(v_k,u_k)) \ge |z_k| = 1/2^k$. Thus it remains to verify that C satisfies the $\mathcal{H}^\alpha/\mathrm{diam}$ condition (HD), and, since $\alpha = 1$, it suffices to establish that

$$l(A) \le c \operatorname{diam}(A) \quad \text{for all arcs} \quad A \subset C.$$

This is clear if A contains any two of the points $0, z, w$ so we assume that A is a subarc of C which lies in $C(0,z)$ with endpoints x, y chosen so that $0, x, y, z$ lie in order on C. When x and y are close enough, e.g., if they both belong to the same triangle T_k, our desired conclusion is an easy consequence of the triangle inequality. Thus we may assume, e.g., that x and y belong to triangles T_i and T_j respectively with $i > j$. Easy estimates now show that

$$l(A) \le l(C(0,y)) \le c|z_j| \le c \operatorname{diam}(A),$$

thus verifying the $\mathcal{H}^\alpha/\mathrm{diam}$ condition (HD). □

Finally, we turn our attention to Cantor dusts. Falconer gives a detailed description for constructing generalized Cantor dusts; see [**5**, pp. 56–59]. We restrict our attention to his Example 4.4 on page 57 where each kth stage basic interval is replaced by m equally spaced equal length subintervals; here m is allowed to vary considerably. Verification of the following is a fun exercise which we leave to the reader.

Example 5.6. The above mentioned Cantor dust is AR^α if and only if m is bounded.

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