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HIGHER ORDER FLATNESS OF IMMERSED MANIFOLDS

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ABSTRACT. We prove that if a manifold M can be immersed into Euclidean space of codimension one, then the rth-order jet bundle $\mathrm{Jet}^r(TM)$ is flat for some $r \geq 0$. This is false if the codimension is greater than one: we give an example of a 4-manifold M^4 that immerses in $\mathbf{R}^6$ but for which none of the bundles $\mathrm{Jet}^r(M^4)$ is flat.

1. Introduction. The purpose of this note is to point out an interesting difference between manifolds that can be immersed in Euclidean space of codimension one and those that cannot. We show that manifolds that admit codimension one Euclidean immersions must satisfy a "higher-order flatness" condition not necessarily satisfied by other manifolds. The best way to describe this condition is in terms of the Andreotti invariant, which is defined as follows. Let $M = M^m$ be a C^∞ real manifold of dimension m. The *Andreotti invariant* $\mathcal{A}(M)$ is the smallest nonnegative integer r (if one exists) such that the rth order jet bundle

$$\mathrm{Jet}^r(M) := \mathrm{Jet}^r(TM) \cong TM \otimes S^r(TM \oplus 1),$$

where S^r is the rth symmetric power operator, admits a flat structure (i.e., admits an affine connection having curvature identically zero). If no such r exists, then we put $\mathcal{A}(M) = \infty$. The manifold M is said to be rth-order flat if $\mathrm{Jet}^r(M)$ is flat.

The Andreotti invariant and a similar invariant, the *alpha invariant*, of manifolds of constant positive curvature have been studied extensively using K-theory (see [**1, 4, 7, 8**]). These studies were motivated by earlier work, [**2, 3**] of Fredricks on the relationship between partial differential equations and higher-order differential geometry.

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Let us establish some notation and terminology. Let $\mathrm{Vect}(M^m)$ be the set of isomorphism classes of real vector bundles over M^m. The rank, or fiber dimension, of $E \in \mathrm{Vect}(M^m)$ is denoted by $\mathrm{rk}(E)$, and $[E]$ is the class of E in $KO(M)$ (see [**6**] for the definitions of the various K-rings). The trivial line bundle is denoted by 1, and for $n > 1$ the trivial vector bundle of rank n is denoted by 1_n. An n-fold direct sum $E \oplus \cdots \oplus E$ will be written as $n \cdot E$. We say that E is in the *stable range* if $\mathrm{rk}(E) > m$. We can cancel in the stable range: if E_1 and E_2 are in the stable range and $E_1 \oplus E = E_2 \oplus E$ for some E, then $E_1 = E_2$. It will be convenient to let $p(r,m) = \mathrm{rk}\, S^r(TM^m \oplus 1) = \binom{m+r}{r}$; then, for example, $\mathrm{rk}\,\mathrm{Jet}^r(M) = m\,p(r,m)$. Of fundamental importance for us is the fact that $\mathrm{Jet}^r(M)$ is in the stable range if $r \geq 1$.

A vector bundle E is *stably trivial* if $E \oplus 1_n$ is trivial for some n. Similarly, if $E \oplus 1_n$ is flat for some n, then E is *stably flat*.

Flatness is preserved by the standard operations on vector bundles: direct sums, tensor products, symmetric powers and pullbacks of flat vector bundles are flat.

2. Codimension one. Much information about vector bundles on a manifold M^m immersed in Euclidean space $\mathbf{R}^{m+1}$ can be obtained by studying vector bundles over real projective spaces; for our purposes, the following result on the Andreotti invariant of RP^m will be sufficient:

Lemma [**8**, Theorem 3]. *Let $m \geq 1$. Then $\mathcal{A}(RP^m) = s$, where s is the smallest integer such that*

$$(m+1)p(s,m) \geq |\tilde{K}O(RP^m)|.$$

Moreover, $\mathrm{Jet}^r(RP^m)$ *is flat for all* $r \geq s$.

A well known result (see [**6**], for example) tells us that the order of $\tilde{K}O(RP^m)$ is

$$\begin{cases} 2^{[m/2]} & \text{if } m \equiv 6,\ 7,\ 8 \mod 8, \\ 2^{[m/2]+1} & \text{otherwise,} \end{cases}$$

from which it follows that $\mathcal{A}(RP^m)$ is finite for all m and that $\mathcal{A}(RP^m) \to \infty$ as $m \to \infty$.

We wish to study $\mathcal{A}(M^m)$ in the case where M^m can be immersed into $\mathbf{R}^{m+1}$. If M is orientable, then the next theorem shows that M is either flat or first-order flat.

Theorem 1. *If M^m is orientable and can be immersed into $\mathbf{R}^{m+1}$, then $\mathcal{A}(M) \leq 1$.*

Proof. Choose an immersion of M^m into $\mathbf{R}^{m+1}$ and let V be the normal line bundle of this immersion. Then $TM \oplus V$ is trivial, because $TM \oplus V$ is isomorphic to the restriction of $T(\mathbf{R}^{m+1})$ to the image of M. Therefore V is itself orientable, but an orientable line bundle is trivial. Hence,

$$\begin{aligned}\mathrm{Jet}^1(M) \oplus 1_{m+1} &= (TM \otimes (TM \oplus 1)) \oplus 1_{m+1} \\ &= (TM \otimes 1_{m+1}) \oplus 1_{m+1} \\ &= (m+1) \cdot (TM \oplus 1) \\ &= 1_{(m+1)^2},\end{aligned}$$

so $\mathrm{Jet}^1(M)$ is stably trivial. But $\mathrm{Jet}^1(M)$ is in the stable range, so $\mathrm{Jet}^1(M)$ is trivial, hence flat. □

The result for nonorientable M is derived from the fact that the real projective spaces are the classifying spaces for real line bundles:

Theorem 2. *If M^m immerses in $\mathbf{R}^{m+1}$, then*

$$\mathcal{A}(M) \leq \max\{1, \mathcal{A}(RP^m)\}.$$

Proof. As in the preceding proof, $TM \oplus V = 1_{m+1}$, where V is the normal line bundle with respect to a chosen immersion. Choose a map $f : M^m \to RP^m$ such that $V = f^*(L)$, where L is the canonical line bundle over RP^m.

We need to write $[\mathrm{Jet}^r(M)]$ in terms of $[V]$ and $[1]$. If E is any vector bundle and F is a line bundle, then

$$[S^r(E)] = [S^r(E \oplus F)] - [S^{r-1}(E \oplus F)] \cdot [F],$$

so

$$\begin{aligned}[S^r(TM \oplus 1)] &= [S^r(1_{m+2})] - [S^{r-1}(1_{m+2})] \cdot [V] \\ &= p(r, m+1)[1] - p(r-1, m+1)[V].\end{aligned}$$

Since $V \otimes V = 1$, we get

$$\begin{aligned}[\mathrm{Jet}^r(M)] &= [(TM \oplus 1) \otimes S^r(TM \oplus 1)] - [S^r(TM \oplus 1)] \\ &= A[1] - B[V],\end{aligned}$$

where

$$\begin{aligned}A &= (m+1)p(r, m+1) + p(r-1, m+1), \\ B &= (m+1)p(r-1, m+1) + p(r, m+1).\end{aligned}$$

Let c be the order of $\tilde{K}O(RP^m)$. Since f induces a ring homomorphism in K-theory, it follows that $c[V] = c[1]$. Let $s = \max\{1, \mathcal{A}(RP^m)\}$.

In order to prove that $\mathrm{Jet}^s(M)$ is flat, it suffices to find an integer a such that

$$(*) \qquad \begin{cases} a \equiv A \mod c \\ 0 \leq a \leq \mathrm{rk}\,\mathrm{Jet}^s(M) = mp(s, m). \end{cases}$$

Indeed, suppose a satisfies $(*)$. Then, since $c[V] = c[1]$, we can write

$$[\mathrm{Jet}^s(M)] = a[1] + b[V] = [(a \cdot 1) \oplus (b \cdot V)]$$

with $b = mp(s, m) - a \geq 0$. Since $\mathrm{Jet}^s(M)$ is in the stable range, it follows that $\mathrm{Jet}^s(M)$ is isomorphic to the flat vector bundle $a \cdot 1 \oplus b \cdot V$.

If $m \leq 7$, then $s = 1$ and $mp(s, m) \geq c$, so $(*)$ obviously has a solution and $\mathcal{A}(M) \leq s$. Henceforth, assume $m > 7$. Then $\mathcal{A}(RP^m) \geq 1$, so $s = \mathcal{A}(RP^m)$. Since $(*)$ has a solution if $mp(s, m) \geq c$, it suffices by the Lemma to consider

$$(**) \qquad mp(s, m) < c \leq (m+1)p(s, m).$$

We now claim that $a = A - c$ is a solution of $(*)$. The second inequality in $(**)$ implies $A - c \geq 0$, so it remains to show that $A - c \leq mp(s, m)$. It suffices by the first inequality in $(**)$ to show that $A \leq 2mp(s, m)$.

Straightforward manipulations show that this inequality is valid if and only if $s \leq m-2$.

But

$$\begin{aligned}(m+1)p(m-2,m) &= (m+1)\left(\frac{m+1}{1}\cdot\frac{m+2}{2}\cdots\frac{m+(m-2)}{m-2}\right)\\ &\geq (m+1)\left(\frac{m+(m-2)}{m-2}\right)^{m-2}\\ &\geq 4\cdot 2^{m-2} \geq 2^{(m/2)+1} \geq c,\end{aligned}$$

so $s \leq m-2$ by the Lemma. □

3. Higher codimension. Theorem 2 fails spectacularly if the codimension of the immersion is greater than one:

Theorem 3. *The connected sum of complex projective planes*

$$M^4 = CP^2\#CP^2\#CP^2\#(-CP^2)$$

can be immersed in $\mathbf{R}^6$, *and* $\mathcal{A}(M^4)=\infty$. (*The minus sign indicates opposite orientation.*)

Proof. The signature of CP^2 is 1, so M^4 has signature $\sigma(M^4) = 3-1=2$, so by Thom's signature formula for closed four-manifolds [**10**],

$$p_1(TM^4) = 3\sigma(M^4) = 6.$$

We prove in [**7**] that the rational Pontryagin classes of a manifold with finite Andreotti invariant must vanish in positive degrees; it follows that $\mathcal{A}(M^4)=\infty$.

A fundamental result of Hirsch [**5**] is that a smooth closed orientable four-manifold N can be immersed in $\mathbf{R}^6$ if and only if there exists $x \in H^2(N;\mathbf{Z})$ with

$$x_{(2)} = w_2(TN) \quad \text{and} \quad x^2 = -p_1(TN),$$

where $x_{(2)}$ is the reduction modulo 2 of x and $w_2 \in H^2(N;\mathbf{Z}_2)$ is the second Stiefel-Whitney class.

By Wu's formula [**9**, p. 136] the second Stiefel-Whitney class of a smooth closed connected oriented four-manifold N is characterized by the property that $w_2 x = x^2$ for all $x \in H^2(N; \mathbf{Z}_2)$. From this it follows that

$$w_2(TM^4) = (1,1,1,1) \in H^2(M^4; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

Put $x = (1,1,1,3) \in H^2(M^4; \mathbf{Z})$. Then $x_{(2)} = w_2(TM^4)$ and

$$x^2 = 1^2 + 1^2 + 1^2 - 3^2 = -6 = -p_1(TM^4),$$

so M^4 can be immersed in $\mathbf{R}^6$. $\square$

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