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SOBOLEV ORTHOGONAL POLYNOMIALS AND SECOND-ORDER DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor H.L. Krall (1907–1994)

ABSTRACT. There has been a considerable amount of recent research on the subject of Sobolev orthogonal polynomials. In this paper, we consider the problem of when a sequence of polynomials that are orthogonal with respect to the (Sobolev) symmetric bilinear form

$$(p,q)_1 = \int_{\mathbf{R}} pq\, d\mu_0 + \int_{\mathbf{R}} p'q'\, d\mu_1$$

satisfies a second-order differential equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) = \lambda_n y(x).$$

We shall obtain necessary and sufficient conditions for this to occur. Moreover, we will characterize all sequences of polynomials satisfying these conditions. Included in this classification are some, in a sense, *new* orthogonal polynomials. As a consequence of this work, we obtain a new characterization of the classical orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel.

1. Introduction. The study of Sobolev orthogonal polynomials has been the subject of a considerable amount of recent interest. This area deals with the study of sequences of polynomials which are orthogonal with respect to quasi-definite symmetric bilinear forms of the type

$$(p,q)_N = \sum_{k=0}^{N} \int_{\mathbf{R}} p^{(k)}(x)q^{(k)}(x)\, d\mu_k, \tag{1.1}$$

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where each (signed) Borel measure μ_k, $0 \le k \le N$, has finite moments on the real line $\mathbf{R}$. When $N > 0$, the algebraic and analytic theory of these polynomials is quite different from that of the classical theory ($N = 0$). For example, Sobolev orthogonal polynomials, in general, will not satisfy a three-term recurrence relation, unlike their classical counterparts; see, for example, the contributions [**6**] and [**7**]. More work needs to be done in order to unify the subject of Sobolev orthogonality and to understand the many subtle differences and similarities it has with the classical theory.

In this paper, among other results, we shall characterize all sequences of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ which are orthogonal with respect to the bilinear form (1.1) when $N = 1$ and which are solutions to a second-order differential equation of the form

$$a_2(x)y''(x) + a_1(x)y'(x) = \lambda y(x), \quad n = 0, 1, \dots ; \tag{1.2}$$

here $a_1(x)$ and $a_2(x)$ are real-valued functions and λ is a real (eigenvalue) parameter (see Equation (2.9) below for more details). This will generalize the well-known characterization result of the classical orthogonal polynomials due to Bochner [**2**], see Theorem 1.3 below.

We note that the classical orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel are Sobolev orthogonal, and they all satisfy differential equations of the form (1.2), see Example 4.1 in Section 4. For example, the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $\alpha, \beta > -1$, are orthogonal with respect to the inner product

$$\begin{aligned}(p, q) = &\int_{\mathbf{R}} p(x)q(x)(1-x)^{\alpha}(1+x)^{\beta}H(1-x^2)\,dx \\ &+ \int_{\mathbf{R}} p'(x)q'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}H(1-x^2)\,dx;\end{aligned}$$

here $H(x)$ denotes the Heaviside function. This orthogonality follows immediately from the well-known results that the Jacobi polynomials are orthogonal on $\mathbf{R}$ with respect to the weight function

$$w_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}H(1-x^2), \quad \alpha, \beta > -1,$$

and the fact that the first derivative $\{dP_n^{(\alpha,\beta)}(x)/dx\}_{n=0}^{\infty}$ of the Jacobi polynomials are orthogonal on $\mathbf{R}$ with respect to the weight function

$w_{\alpha+1,\beta+1}(x)$. We remark here that the Sobolev orthogonality of the classical orthogonal polynomials has been discussed in detail in [**28**] and, more recently, in [**10**] and [**11**]. In these latter two references, the author discusses the Sobolev orthogonality of the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ and the Laguerre polynomials $\{L_n^\alpha(x)\}$ for all real values, and, in particular, all negative integer values, of the parameters α and β. For example, in [**11**], the author discusses the orthogonality of the Laguerre polynomials $\{L_n^{-1}(x)\}$ of degree ≥ 1 with respect to the inner product

$$H(p,q) = \int_0^\infty \left\{ e^{-x}p'(x)q'(x) + \frac{1}{x}e^{-x}p(x)q(x) \right\} dx.$$

As part of our main results, we show in this paper that the Laguerre polynomials $\{L_n^{-1}(x)\}$ of degree ≥ 0 are orthogonal with respect to the inner product

$$\phi(p,q) = Ap(0)q(0) + \int_0^\infty e^{-x}p'(x)q'(x)\,dx,$$

where A is any positive real number; see (4.7) and Proposition 4.1 below.

Along this line, and for later comparison, we mention the well-known Hahn-Sonine characterization theorem (see [**9**] and [**29**]) for the classical orthogonal polynomials, which we restate as follows:

Theorem 1.1. *The only polynomial sequences $\{\phi_n(x)\}_{n=0}^\infty$ (up to a complex linear change of variable) that are simultaneously orthogonal with respect to bilinear forms of the type*

(i) $(p,q)_0 = \int_{\mathbf{R}} p(x)q(x)\,d\mu_0$ *and*

(ii) $(p,q)_1 = (p,q)_0 + \int_{\mathbf{R}} p'(x)q'(x)\,d\mu_1$, *with μ_0 and μ_1 being real, (possibly signed) Borel measures, are the classical polynomials of*

(a) *Jacobi* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $-\alpha, -\beta, -(\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$,

(b) *Laguerre* $\{L_n^\alpha(x)\}_{n=0}^\infty$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$,

(c) *Hermite* $\{H_n(x)\}_{n=0}^\infty$, *and*

(d) *Bessel* $\{y_n^a(x)\}_{n=0}^\infty$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$.

Remark 1. Actually, in 1887, Na.J. Sonine [**29**] showed that, for positive Borel measures μ_0 and μ_1, the only polynomials satisfying the conditions of Theorem 1.1 are those of Jacobi, $\alpha, \beta > -1$, Laguerre, $\alpha > -1$, and Hermite. W. Hahn re-discovered this result in 1935 and F.S. Beale [**3**] and H.L. Krall [**16**] independently extended this result to the general quasi-definite case in 1941, see Section 2 below for the definition of quasi-definite.

Among other results in this paper, we offer a new characterization of the classical orthogonal polynomials. More specifically, we shall prove:

Theorem 1.2. *Consider the symmetric bilinear form*

$$(p,q)_1 = \int_{\mathbf{R}} p(x)q(x)\,d\mu_0 + \int_{\mathbf{R}} p'(x)q'(x)\,d\mu_1, \tag{1.3}$$

where μ_0 and μ_1 are real, (possibly signed) Borel measures on the real line $\mathbf{R}$, each having finite moments and where μ_0 is quasi-definite. Then the only polynomial sequences, up to a real *linear change of variable, that are orthogonal with respect to this form and satisfy a second-order differential equation of the type* (1.2) *are the orthogonal polynomials of*

(a) *Jacobi* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $-\alpha, -\beta, -(\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$,

(b) *Laguerre* $\{L_n^\alpha(x)\}_{n=0}^\infty$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$,

(c) *Hermite* $\{H_n(x)\}_{n=0}^\infty$,

(d) *Bessel* $\{y_n^a(x)\}_{n=0}^\infty$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$,

(e) *twisted Jacobi* $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $-(\alpha+\beta+1) \in \mathbf{C}\backslash\mathbf{N}$ *and* $\beta = \bar{\alpha}$, *and*

(f) *twisted Hermite* $\{\check{H}_n(x)\}_{n=0}^\infty$.

Furthermore, up to a complex *linear change of variable, the only polynomial sets that are orthogonal with respect to a bilinear form of the type* (1.3) *and satisfy a second-order differential equation of the form* (1.2) *are the classical orthogonal polynomial sets listed in* (a), (b), (c), *and* (d) *above.*

The orthogonal polynomials listed in (a), (b), (c), and (d) above are called the *classical* orthogonal polynomials. It is natural, therefore, to refer to all of the polynomial sets listed in Theorem 1.2 as the *real classical* orthogonal polynomials. In Example 4.1 in Section 4, we shall review some relevant properties of the twisted Jacobi and twisted Hermite polynomials; these polynomials are discussed at length in [**20**] where they are first introduced.

Besides being close in appearance to Theorem 1.1, the above theorem generalizes the following classification theorem which is usually attributed to Bochner [**2**] in the positive-definite case and which was extended to the quasi-definite case by H.L. Krall [**16**].

Theorem 1.3. *The only polynomial sequences, up to a complex change of variable, which are orthogonal with respect to a bilinear form of the type*

$$(p,q)_0 = \int_{\mathbf{R}} p(x)q(x)\,d\mu_0, \tag{1.4}$$

where μ_0 is a real, (*possibly signed*) *Borel measure on* $\mathbf{R}$*, and satisfy a second-order differential equation of the type* (1.2) *are the classical orthogonal polynomials of*

(a) *Jacobi* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $-\alpha, -\beta, -(\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$,

(b) *Laguerre* $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$,

(c) *Hermite* $\{H_n(x)\}_{n=0}^{\infty}$, *and*

(d) *Bessel* $\{y_n^{a}(x)\}_{n=0}^{\infty}$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$.

We remark that, in [**20**], we generalize Theorem 1.3 in the following sense:

Theorem 1.4. *The only polynomial sets, up to a* real *linear change of variable, which are orthogonal to the bilinear form* (1.4) *and satisfy a second-order differential equation of the type* (1.2) *are the*

(a) *Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $-\alpha, -\beta, -(\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$,

(b) *Laguerre polynomials* $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$,

(c) *Hermite polynomials* $\{H_n(x)\}_{n=0}^{\infty}$,

(d) *Bessel polynomials* $\{y_n^{a}(x)\}_{n=0}^{\infty}$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$,

(e) *twisted Jacobi polynomials* $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $-(\alpha+\beta+1) \in \mathbf{C}\backslash\mathbf{N}$ *and* $\beta = \overline{\alpha}$, *and*

(f) *twisted Hermite polynomials* $\{\check{H}_n(x)\}_{n=0}^{\infty}$.

In this paper, we further generalize Theorem 1.4. Indeed, in Section 5 below, we prove the following classification theorem.

Theorem 1.5. *The only polynomial sets, up to a* real *linear change of variable, which are orthogonal to a bilinear form of the type* (1.3) *and satisfy a second-order differential equation of the form* (1.2) *are the*

(a) *Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $(-\alpha, -\beta, -\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$,

(b) *Laguerre polynomials* $\{L_n^{\alpha}(x)\}_{n=0}^{\infty}$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$,

(c) *Hermite polynomials* $\{H_n(x)\}_{n=0}^{\infty}$,

(d) *Bessel polynomials* $\{y_n^{a}(x)\}_{n=0}^{\infty}$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$,

(e) *twisted Jacobi polynomials* $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $-(\alpha+\beta+1) \in \mathbf{C}\backslash\mathbf{N}$ *and* $\beta = \bar{\alpha}$,

(f) *twisted Hermite polynomials* $\{\check{H}_n(x)\}_{n=0}^{\infty}$.

(g) *Laguerre polynomials* $\{L_n^{-1}(x)\}_{n=0}^{\infty}$,

(h) *Jacobi polynomials* $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$, $-(\beta+1) \in \mathbf{R}\backslash\mathbf{N}$ (*or* $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$, $-(\alpha+1) \in \mathbf{R}\backslash\mathbf{N}$), *and*

(i) *twisted Jacobi polynomials* $\{\check{P}_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$.

Furthermore, up to a complex *linear change of variable, the only polynomial sets that are orthogonal with respect to a bilinear form of the type* (1.3) *and satisfy a second-order differential equation of the form* (1.2) *are the polynomial sets listed in* (a), (b), (c), (d), (g) *and* (h) *above.*

The work in this paper is, in part, motivated by a recent and important example of R. Koekoek [**13**]. Koekoek produced the first known example of a differential equation having a full sequence of polynomial solutions which are orthogonal with respect to a Sobolev inner product of the form (1.3) and which are not orthogonal with respect to a bilinear form of the type given in (1.4). This differential equation, which is of order eight, is interesting from the viewpoint of spectral theory since it gives the first known example of a right-definite formally symmetric differential operator defined in some Sobolev space which has a sequence of orthogonal polynomial eigenfunctions; see [**8**]. It is natural to ask if there are differential equations of this type but of smaller order. Indeed, we shall produce such examples below in Section 4.

In Section 2 below, we discuss various background results and notations that we shall need for the rest of the paper. One of the main results of this paper, see Theorem 3.3 in Section 3, is to give necessary and sufficient conditions for the existence of real Borel measures μ_0 and μ_1 for which the polynomials orthogonal with respect to the bilinear form (1.3) will satisfy a differential equation of the type (1.2). This work generalizes previous work of Littlejohn [**22**], Krall and Littlejohn [**14**] and Kwon, Kim and Han [**18**], which all deal with symmetry equations associated with differential equations. This work on symmetry equations has proven to be a valuable technique in constructing weight functions for certain systems of orthogonal polynomials. Indeed, the important contribution [**18**] uses this method to construct a real-valued signed measure of bounded variation for the Bessel polynomials. At the end of Section 3, we give a proof of the above-mentioned Theorem 1.2. In Section 4 we construct, using Theorem 3.3, some nonclassical polynomials which satisfy second-order differential equations of the form (1.2) and which are orthogonal to a bilinear form of the type (1.3). Lastly, in Section 5, we prove Theorem 1.5.

2. Preliminaries. The real and complex number fields will be denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively, with $\bar{z}$ denoting the complex conjugate of $z \in \mathbf{C}$. The set of positive integers $\{1, 2, \dots\}$ will be denoted by $\mathbf{N}$ while the set of nonnegative integers $\{0, 1, 2, \dots\}$ will be represented by $\mathbf{N}_0$.

All polynomials throughout this paper are assumed to be real-valued polynomials of the real variable x; the collection of all such polynomials will be denoted by $\mathcal{P}$. We shall denote the degree of a polynomial $\pi \in \mathcal{P}$ by $\deg(\pi)$, with the convention that $\deg(0) = -1$; hence, if $\deg(\pi) = 0$, then π is necessarily a *nonzero* constant. We call any linear functional $\sigma : \mathcal{P} \to \mathbf{R}$ a *moment functional* and denote its action on a polynomial

π by $\langle \sigma, \pi \rangle$. In particular, the *nth moment* of σ is given by $\langle \sigma, x^n \rangle$, $n = 0, 1, \ldots$. With this action, any moment functional σ will define a bilinear quadratic form through the formula $\langle \sigma, pq \rangle$, $p, q \in \mathcal{P}$, which may or may not be a (positive-definite) inner product.

By various representation results like Boas' moment theorem [**1**] or Duran's generalization [**5**] of Boas' theorem, any moment functional σ will have a representation of the form

$$\langle \sigma, \pi \rangle = \int_{\mathbf{R}} \pi(x)\, d\sigma_0, \quad \pi \in \mathcal{P},$$

or

$$\langle \sigma, \pi \rangle = \int_{\mathbf{R}} \pi(x) w_\sigma(x)\, dx, \quad \pi \in \mathcal{P},$$

where σ_0 is, in general, a signed Borel measure having bounded variation on $\mathbf{R}$, and where w_σ is a C^∞ weight function of the Schwartz class. With this in mind, we note that the bilinear form given in (1.3) can be rewritten as

$$(2.1) \qquad (p, q)_1 = \langle \sigma, pq \rangle + \langle \tau, p'q' \rangle.$$

As we shall see, it is somewhat advantageous for us to use this abstract notation involving moment functionals instead of using one of the above integral representations for σ and τ.

For a moment functional σ and $\pi \in \mathcal{P}$, we let σ', the derivative of σ, and $\pi\sigma$, multiplication of σ by a polynomial, be those moment functionals defined by

$$(2.2) \qquad \langle \sigma', p \rangle = -\langle \sigma, p' \rangle, \quad p \in \mathcal{P},$$

and

$$(2.3) \qquad \langle \pi\sigma, p \rangle = \langle \sigma, \pi p \rangle, \quad p \in \mathcal{P}.$$

It is easy then to obtain the following Leibniz rule for any moment functional σ and polynomial π:

$$(2.4) \qquad (\pi\sigma)' = \pi'\sigma + \pi\sigma'.$$

We say that a moment functional σ is *quasi-definite*, respectively *positive-definite*, if the *moments*

$$\sigma_n := \langle \sigma, x^n \rangle, \quad n \in \mathbf{N_0},$$

of σ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det \begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_n \\ \sigma_1 & \sigma_2 & \dots & \sigma_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n & \sigma_{n+1} & \dots & \sigma_{2n} \end{bmatrix} \neq 0, \quad n \in \mathbf{N}_0, \tag{2.5}$$

respectively $\Delta_n(\sigma) > 0, n \in \mathbf{N}_0$.

It is well known, for example, see [**4**, Chapter 1], that a moment functional σ is quasi-definite, respectively positive-definite, if and only if there is a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, with $\deg(P_n) = n$, such that

$$\langle \sigma, P_n P_m \rangle = K_n \delta_{nm}, \quad m, n \in \mathbf{N}_0, \tag{2.6}$$

where $K_n \neq 0$, respectively $K_n > 0$. In this case, it is customary, see [**15**], to call $\{P_n(x)\}_{n=0}^{\infty}$ a *Tchebycheff polynomial system*, TPS for short, relative to σ. When each $K_n > 0$, a TPS is usually referred to as an *orthogonal polynomial system*, OPS for short. In either case, we say that σ is an *orthogonalizing moment functional* for the polynomial system $\{P_n(x)\}_{n=0}^{\infty}$. For more information on this and other terminology, we refer the reader to the paper of Everitt and Littlejohn [**8**].

By a *polynomial system* (PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $\deg(\phi_n) = n$, $n \in \mathbf{N}_0$. Notice, then, that a PS is a basis for $\mathcal{P}$. Any PS $\{\phi_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a nonzero constant multiple), called a *canonical moment functional*, see [**26**], for $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle \sigma, \phi_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, \phi_n \rangle = 0, \quad n \in \mathbf{N}. \tag{2.7}$$

Note that if a PS $\{\phi_n(x)\}_{n=0}^{\infty}$ is a TPS relative to σ, then σ must be a canonical moment functional for $\{\phi_n(x)\}_{n=0}^{\infty}$.

If $\{\phi_n(x)\}_{n=0}^\infty$ is a PS and, for each integer $n \geq 0$, $\phi_n(x)$ satisfies the second-order differential equation

$$L[y](x) := a_2(x)y''(x) + a_1(x)y'(x) = \lambda_n y(x), \tag{2.8}$$

where $a_2(x) \neq 0$, see Remark 3 below, and λ_n is a real parameter depending only on n, we shall refer to this PS as a differential polynomial system of order two, DPS(2) for short. Of course, Theorem 1.3 precisely characterizes the systems of polynomials that are in the intersection class TPS $\cap$ DPS(2). For the remainder of this section, we shall concentrate on stating several algebraic results concerning the class PS $\cap$ DPS(2); these results will be necessary in subsequent sections.

Firstly, if $\{\phi_n(x)\}_{n=0}^\infty$ is a PS and, for each $n \in \mathbf{N}_0$, $\phi_n(x)$ satisfies (2.8), then it is necessary that the coefficients $a_1(x), a_2(x)$ and λ_n be given by

$$\begin{gathered} a_j(x) = \sum_{i=0}^{j} l_{j,i}x^i, \quad j = 1, 2, \\ \lambda_n = nl_{1,1} + n(n-1)l_{2,2}, \quad n \in \mathbf{N}_0;\ l_{1,1}^2 + l_{2,2}^2 \neq 0. \end{gathered} \tag{2.9}$$

Indeed, this was first observed by Bochner in [**2**].

By direct calculation, it is easy to see that (2.8) has a unique monic polynomial solution of degree n for each nonnegative integer n except possibly for a finite number of values of n. For these exceptional cases of n, there may be no polynomial solution of (2.8) of degree n or there will be infinitely many monic polynomial solutions of degree n.

Lemma 2.1. *If the differential equation* (2.8) *has a* PS $\{P_n(x)\}_{n=0}^\infty$ *of solutions, then any canonical moment functional* σ *of* $\{P_n(x)\}_{n=0}^\infty$ *must satisfy*

$$(a_2(x)\sigma)' - a_1(x)\sigma = 0, \tag{2.10}$$

which is equivalent to the recurrence relation

$$\begin{gathered} (nl_{2,2} + l_{1,1})\sigma_{n+1} + (nl_{2,1} + l_{1,0})\sigma_n + nl_{2,0}\sigma_{n-1} = 0, \\ n \in \mathbf{N}_0;\ \sigma_{-1} = 0, \end{gathered} \tag{2.11}$$

where $\{\sigma_n\}_{n=0}^{\infty}$ *are the moments of* σ.

Proof. Let σ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then we have for each integer $n \geq 1$,

$$\begin{aligned} 0 &= \lambda_n\langle \sigma, P_n\rangle = \langle \sigma, \lambda_n P_n\rangle \\ &= \langle \sigma, a_2 P_n'' + a_1 P_n'\rangle \\ &= -\langle (a_2\sigma)' - a_1\sigma, P_n'\rangle, \end{aligned}$$

which implies (2.10) since $\{P_n'(x)\}_{n=0}^{\infty}$ is also a PS. Furthermore, equation (2.10) means that

$$\langle (a_2\sigma)' - a_1\sigma, x^n\rangle = 0, \quad n \in \mathbf{N}_0,$$

which, when written out, is the recurrence relation (2.11) in terms of the moments $\{\sigma_n\}_{n=0}^{\infty}$ of σ. □

Remark 2. Notice that no assumptions are made in the above lemma concerning orthogonality of the PS $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ; see Theorem 2.5 below, which covers the case when a PS is a TPS. Indeed, a PS is a TPS if and only if its canonical moment functional is quasi-definite.

We call (2.10) the *weight equation* for the differential expression $L[\cdot]$ defined in (2.8), while (2.11) is called the *moment equation* for $L[\cdot]$. It must be understood that the lefthand side of (2.10) is interpreted as being the zero moment functional; that is to say, all of the moments of the functional $(a_2(x)\sigma)' - a_1(x)\sigma$ are zero. There are several examples of zero moment functionals available; for example, the functional ω defined on $\mathcal{P}$ by

$$\langle \omega, p\rangle := \int_{\mathbf{R}} p(x)g(x)\,dx,$$

where $g(x)$ is Stieltjes *ghost* function, given by

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x^{1/4}}\sin(x^{1/4}) & \text{if } x \geq 0, \end{cases} \tag{2.12}$$

is the zero moment functional; see [**31**, p. 126] and [**4**, p. 73].

If, in Lemma 2.1, the sequence $\{P_n(x)\}_{n=0}^\infty$ is actually a TPS and σ is an orthogonalizing moment functional for $\{P_n(x)\}_{n=0}^\infty$, then the weight equation (2.10) may be used to find such a σ. The idea is to interpret (2.10) as a distributional differential equation. To do this, however, the righthand side of this equation is no longer necessarily the *function* that is identically zero. Indeed, the zero functional should be replaced, in general, by a function that has zero moments, for example, the one given in (2.12). We then seek a nontrivial solution $w \in \mathcal{D}'$, the space of Schwartz distributions, to this nonhomogeneous equation. This solution w, of course, must have finite moments which satisfy the moment equation (2.11). This method has been effectively used by Littlejohn [**22**] and Kwon, Kim and Han [**18**] to construct orthogonalizing weight functions for certain systems of Tchebycheff polynomials. In [**18**], the authors use this technique to construct an orthogonalizing Borel measure of bounded variation for the simple Bessel polynomials $\{y_n^0(x)\}_{n=0}^\infty$; see Example 4.1 in Section 4 below.

Definition 2.1 (Krall and Scheffer [**17**]). The differential expression $L[\cdot]$, defined in (2.8), is called *admissible* if

$$\lambda_n \neq \lambda_m \qquad \text{for } m \neq n. \tag{2.13}$$

Lemma 2.2. *For the differential expression* $L[\cdot]$, *given in* (2.8) *with coefficients satisfying the conditions in* (2.9), *the following four statements are equivalent*:

(i) $L[\cdot]$ *is admissible*;

(ii) $\lambda_n = nl_{1,1} + n(n-1)l_{2,2} \neq 0$, $n \in \mathbf{N}$;

(iii) $l_{1,1} \notin \{-nl_{2,2} \mid -n \in \mathbf{N}_0\}$;

(iv) *For each* $n \in \mathbf{N}_0$, *the differential equation* (2.8) *has a unique monic polynomial solution of degree* n.

Moreover, if the differential expression $L[\cdot]$ *is admissible, then the weight equation* (2.10), *or equivalently the moment equation* (2.11), *is uniquely solvable in the sense that it has only one linearly independent solution* σ.

Proof. The proofs of (i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) are trivial.

(ii) $\Rightarrow$ (i). This follows from the identity

$$(2.14)\quad \begin{aligned}(n+m)(\lambda_n-\lambda_m) &= (n-m)(n+m)(l_{2,2}(n+m-1)+l_{1,1}) \\ &= (n-m)\lambda_{n+m}.\end{aligned}$$

(i) $\Rightarrow$ (iv). For any fixed integer $n \geq 1$, let

$$P_n(x) = \sum_{k=0}^{n} C_k^n x^k, \quad C_n^n = 1$$

be a monic polynomial of degree n. Then $P_n(x)$ satisfies (2.8) if and only if

$$(2.15)\quad \begin{aligned}l_{2,0}(k+2)(k+1)C_{k+2}^n &+ (k+1)(l_{2,1}k+l_{1,0})C_{k+1}^n \\ &+ (\lambda_k-\lambda_n)C_k^n = 0, \quad k=0,1,\dots n-1,\end{aligned}$$

where $C_{n+1}^n = 0$. If the operator $L[\cdot]$ is admissible, then all C_k^n, $k = 0,1,\dots n-1$, are uniquely and successively determined by the equation (2.15) and $C_n^n = 1$.

(iv) $\Rightarrow$ (i). Assume that the differential equation (2.8) has a unique monic polynomial system $\{P_n(x)\}_{n=0}^{\infty}$ of solutions but $L[\cdot]$ is not admissible. Hence, from (ii), we have $\lambda_p = \lambda_0 = 0$ for some integer $p \geq 1$. But then

$$L[P_p + kP_0] = \lambda_p P_p + k\lambda_0 P_0 = 0 = \lambda_p(P_p + kP_0)$$

for any constant k. Hence $L[y] = \lambda_p y$ has infinitely many monic polynomial solutions of degree p, which contradicts our assumption.

Finally, the last statement in the Lemma follows immediately from Equation (2.11). □

Proposition 2.3. *If the differential equation* (2.8) *has a* TPS $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, then* $L[\cdot]$ *is admissible.*

In order to prove this Proposition, we need the following fact:

Lemma 2.4. *Let* σ *be a moment functional.*

(i) *Then* $\sigma = 0$ *if and only if* $\sigma' = 0$.

(ii) *If* σ *is quasi-definite and* $p(x)\sigma = 0$ *for some* $p \in \mathcal{P}$, *then* $p(x) \equiv 0$.

Proof. (i) Suppose $\sigma = 0$. Then, for any $n \in \mathbf{N}_0$, $\langle \sigma', x^n \rangle = -n\langle \sigma, x^{n-1}\rangle = 0$. Conversely, if $\sigma' = 0$, the required result follows from

$$\langle \sigma, x^n \rangle = \langle \sigma, \frac{(x^{n+1})'}{n+1} \rangle = \frac{-1}{n+1}\langle \sigma', x^{n+1}\rangle = 0.$$

(ii) Suppose $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS relative to σ satisfying the orthogonality condition $\langle \sigma, P_n P_m \rangle = K_n \delta_{nm}$ for all $m, n \in \mathbf{N}_0$, where $K_n \neq 0$, $n \in \mathbf{N}_0$. Write $p(x) = \sum_{k=0}^{N} c_k P_k(x)$, where we suppose that $\deg(p) = N$. Then, for $n = 0, 1, \dots N$, we have

$$0 = \langle p\sigma, P_n \rangle = \sum_{k=0}^{N} c_k \langle \sigma, P_k P_n \rangle = c_n K_n.$$

It now follows that $c_n = 0$, $n = 0, 1, \dots N$, and hence $p(x) \equiv 0$. □

Proof of Proposition 2.3. Suppose σ is a quasi-definite moment functional with TPS $\{P_n(x)\}_{n=0}^{\infty}$. Then σ is a canonical moment functional for $\{P_n(x)\}_{n=0}^{\infty}$ and, by Lemma 2.1, σ satisfies the weight equation

$$(a_2\sigma)' - a_1\sigma = 0.$$

If $L[\cdot]$ is not admissible then, from Lemma 2.2(ii), there exists an integer $N \geq 1$ such that $\lambda_N = 0$. Consequently,

$$\begin{aligned} 0 = \lambda_N P_N(x)\sigma &= (a_2(x)P_N''(x) + a_1(x)P_N'(x))\sigma \\ &= (a_2(x)P_N'(x)\sigma)' - P_N'(x)(a_2(x)\sigma)' + P_N'(x)(a_1(x)\sigma) \\ &= (a_2(x)P_N'(x)\sigma)'. \end{aligned}$$

Hence, by Lemma 2.4(i), $a_2 P_N'\sigma = 0$ and, by Lemma 2.4(ii), we see that $a_2 P_N' \equiv 0$. However, $a_2 \not\equiv 0$ so we must have $P_N'(x) \equiv 0$. Of course, this forces $N = 0$, contradicting the fact that $N \geq 1$. □

We are now in a position to state the following special case of a general characterization theorem of H.L. Krall [**14**] for second-order differential equations.

Theorem 2.5. *A* PS $\{P_n(x)\}_{n=0}^{\infty}$ *is a* TPS *satisfying the differential equation* (2.8) *if and only if its canonical moment functional* σ *is quasi-definite and satisfies the moment equation* (2.11).

In view of Lemma 2.2, we can restate the above theorem as:

Corollary 2.6. *A* PS $\{P_n(x)\}_{n=0}^{\infty}$ *is a* TPS *satisfying the differential equation* (2.8) *if and only if the weight equation* (2.10) *has only one linearly independent solution* σ *which is quasi-definite.*

For a new and simple proof of Theorem 2.5 above, the reader is encouraged to consult [**19**]; for another proof of the general Krall result, see [**20**].

The rest of this section is concerned with results necessary to establish the proof of Theorem 1.5 given in Section 5.

For any monic PS $\{P_n(x)\}_{n=0}^{\infty}$, there are constants $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ such that

$$P_{n+1}(x) - (x-\alpha_n)P_n(x) + \beta_n P_{n-1}(x), \quad n \in \mathbf{N},$$

is a polynomial of degree $\leq n-2$. In fact, if $P_n(x) = \sum_{k=0}^{n} C_k^n x^k$ ($C_n^n = 1$), then

$$\alpha_n = C_{n-1}^n - C_n^{n+1},$$

and

$$\beta_n = C_{n-2}^n - (C_{n-1}^n - C_n^{n+1})C_{n-1}^n - C_{n-1}^{n+1}, \qquad C_{-1}^1 = 0. \tag{2.16}$$

At this point, we recall Favard's theorem, see [**4**, Chapter 1.4], which asserts that a monic PS $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS if and only if $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation of the form

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x), \quad n \in \mathbf{N},$$

where $\beta_n \neq 0$.

As defined in [**17**], we shall call a PS $\{P_n(x)\}_{n=0}^{\infty}$ a *weak Tchebycheff polynomial system* (or WTPS) if there is a nontrivial moment functional σ such that

$$\langle \sigma, P_n P_m \rangle = 0, \quad m \neq n.$$

In this case, we say that $\{P_n(x)\}_{n=0}^\infty$ is a WTPS relative to σ.

If $\{P_n(x)\}_{n=0}^\infty$ is a WTPS relative to σ, then σ must be a canonical moment functional of $\{P_n(x)\}_{n=0}^\infty$; however, $<\sigma, P_n^2>$ may or may not be zero for $n \in \mathbf{N}$. The following improvement of Favard's theorem for WTPS's is due to Krall and Scheffer. The proof can be found in [**17**, Lemma 1.1].

Lemma 2.7. *A monic* WTPS $\{P_n(x)\}_{n=0}^\infty$ *is a* TPS *if and only if*

$$\beta_n \neq 0, \quad n \in \mathbf{N},$$

where β_n *is defined in* (2.16).

The next lemma is important in establishing a key result (Theorem 2.12), which is necessary for our proof of Theorem 1.5.

Lemma 2.8. *Assume that* $L[\cdot]$ *is not admissible, where* $L[\cdot]$ *is defined in* (2.8)*, but there exists a* PS $\{P_n(x)\}_{n=0}^\infty$ *such that*

$$L[P_n(x)] = \lambda_n P_n(x), \quad n \in \mathbf{N}_0. \tag{2.17}$$

Then $\beta_n = 0$ *for some integer* $n \geq 2$*, where* β_n *is defined in* (2.16).

Proof. For each $n \in \mathbf{N}_0$, let $P_n(x) = \sum_{k=0}^n C_k^n x^k$, $C_n^n = 1$, be a monic polynomial of degree n. Then $P_n(x)$ satisfies the differential equation (2.17) if and only if

$$l_{2,0}(k+2)(k+1)C_{k+2}^n + (k+1)(l_{2,1}k + l_{1,0})C_{k+1}^n + (\lambda_k - \lambda_n)C_k^n = 0, \tag{2.18}$$

$k = 0, 1, \dots, n-1$, where $C_{n+1}^n = 0$. Since $L[\cdot]$ is not admissible, it follows from Lemma 2.2 that there exists an integer $N \geq 1$ such that $\lambda_N = 0$. This implies that $\lambda_k \neq \lambda_n$ for $k = 0, 1, \dots n-1$, if $n > N$. It follows that equation (2.18) is uniquely solvable for $\{C_k^n\}_{k=0}^n$ when $n > N$, beginning with $C_n^n = 1$. It follows that $\beta_{N+1} = 0$ by solving (2.18) for C_{N+1}^{N+2}, C_N^{N+1}, and C_{N-1}^{N+1} and substituting these back into (2.16). $\square$

The following theorem, established in [**20**, Theorem 2.9], is used to prove Theorem 1.4. It plays an important role, as well, in our proof of Theorem 1.5.

Theorem 2.9. *The differential equation* (2.8) *has a* TPS *of solutions if and only if*

(i) $l_{1,1} \notin \{-nl_{2,2} \mid n \in \mathbf{N}_0\}$, *and*

(ii) $\beta_n \neq 0$ *for* $n \in \mathbf{N}$, *where* β_n *is given in* (2.16), *with*

$$C_{n-1}^n = \frac{n[l_{1,0} + l_{2,1}(n-1)]}{l_{1,1} + 2l_{2,2}(n-1)},$$

and

$$\begin{aligned} C_{n-2}^n = {} & \frac{n(n-1)[l_{2,0}(l_{1,1} + 2l_{2,2}(n-1))}{2[l_{1,1} + 2l_{2,2}(n-1)][l_{1,1} + l_{2,2}(2n-3)]} \\ & + \frac{(l_{1,0} + l_{2,1}(n-2))(l_{1,0} + l_{2,1}(n-1))]}{2[l_{1,1} + 2l_{2,2}(n-1)][l_{1,1} + l_{2,2}(2n-3)]}. \end{aligned}$$

The following result, interesting in its own right, is important in subsequent discussions in this section.

Lemma 2.10. *Let* $L[\cdot]$ *denote the differential expression in* (2.8). *Suppose* p *and* q *are polynomials with* $L[p] = \lambda p$ *and* $L[q] = \mu q$, *where* $\lambda \neq \mu$. *Then*

$$\langle \sigma, pq \rangle = 0$$

for any solution σ *to the weight equation* (2.10). *In particular, if* $L[\cdot]$ *is admissible and* $\{P_n(x)\}_{n=0}^\infty$ *is a* PS *of solutions, then* $\{P_n(x)\}_{n=0}^\infty$ *is a* WTPS.

Proof. The first part of this Lemma follows from the calculation

$$\begin{aligned} (\lambda - \mu)\langle \sigma, pq \rangle &= \langle \sigma, L[p]q - pL[q] \rangle \\ &= \langle \sigma, a_2 p'' q + a_1 p' q - a_2 q'' p - a_1 q' p \rangle \\ &= \langle [(a_2\sigma)' - a_1\sigma]' q + 2[(a_2\sigma)' - a_1\sigma] q', p \rangle \\ &= 0, \end{aligned}$$

and from Lemma 2.4. The last statement of the lemma follows immediately since the weight equation (2.10) can have only one linearly independent solution when $L[\cdot]$ is admissible. $\square$

For any PS $\{P_n(x)\}_{n=0}^{\infty}$, write

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x) - \sum_{k=0}^{n-2} d_n^k P_k(x), \quad n \in \mathbf{N}, \tag{2.19}$$

where $d_1^0 = d_1^{-1} = 0$ and $P_{-1}(x) = 0$. The following result is necessary for Theorem 2.12 which follows.

Lemma 2.11. *Assume that the differential equation* (2.8) *has a monic* PS $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions. Let* $N \geq 0$ *be the largest integer such that* $\lambda_N = 0$. *Then we have*

(i) $d_n^0 = 0$ *if* $n \geq 2$ *and* $n+1 \neq N$, *and*

(ii) *for any moment functional* σ *of the weight equation* (2.10),

$$\langle \sigma, P_n^2 \rangle = \beta_n \cdot \beta_{n-1} \cdots \beta_{N+1} \langle \sigma, P_N^2 \rangle, \qquad n \geq N+1, \tag{2.20}$$

where d_n^0 *and* β_n *are the constants defined in* (2.19).

Proof. Observe that $\lambda_n = \lambda_m$ for $m \neq n$ if and only if $m+n = N$. Hence, by Lemma 2.10, we have

$$\langle \sigma, P_m P_n \rangle = 0, \quad m \neq n \text{ and } m+n \neq N, \tag{2.21}$$

for any solution σ of the weight equation (2.10).

(i) Let σ be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then σ satisfies the weight equation (2.10) by Lemma 2.1. If we apply σ to the equation (2.19) we obtain, for $n \geq 2$,

$$\begin{aligned} 0 &= \langle \sigma, P_{n+1} \rangle \\ &= \langle \sigma, xP_n \rangle - \alpha_n \langle \sigma, P_n \rangle - \beta_n \langle \sigma, P_{n-1} \rangle - \sum_{k=0}^{n-2} d_n^k \langle \sigma, P_k \rangle \\ &= \langle \sigma, xP_n \rangle - d_n^0 \langle \sigma, P_0 \rangle, \end{aligned}$$

since $\langle\sigma, P_n\rangle = 0$ for $n \geq 1$. Hence, we have proven (i) since $\langle\sigma, P_0\rangle \neq 0$ and $\langle\sigma, xP_n\rangle = \langle\sigma, P_1P_n\rangle = 0$ for $n \geq 2$ and $n+1 \neq N$ by (2.21).

(ii) Let σ be any solution of (2.10). If we multiply equation (2.19) by $P_{n-1}(x)$ and apply σ we obtain, by (2.21),

$$\begin{aligned} 0 &= \langle\sigma, P_{n+1}P_{n-1}\rangle \\ &= \langle\sigma, P_nxP_{n-1}\rangle - \alpha_n\langle\sigma, P_nP_{n-1}\rangle \\ &\quad - \beta_n\langle\sigma, P_{n-1}^2\rangle - \sum_{k=0}^{n-2} d_n^k\langle\sigma, P_kP_{n-1}\rangle \\ &= \langle\sigma, P_n^2\rangle - \beta_n\langle\sigma, P_{n-1}^2\rangle - d_n^0\langle\sigma, P_0P_{n-1}\rangle \end{aligned}$$

for $n \geq N+1$. If $n > N+1$, then $\langle\sigma, P_0P_{n-1}\rangle = 0$ by (2.21). If $n = N+1$ and $N \geq 1$, then $d_{N+1}^0 = 0$ by (i). Finally, if $N = 0$, then $d_1^0 = 0$. Therefore, we have

$$\langle\sigma, P_n^2\rangle = \beta_n\langle\sigma, P_{n-1}^2\rangle, \qquad n \geq N+1,$$

from which (2.20) follows. □

Theorem 2.12. *Assume that the differential equation* (2.8) *has a* PS $\{P_n(x)\}_{n=0}^\infty$ *of solutions. If* $\{P_n(x)\}_{n=0}^\infty$ *is not a* TPS, *then for any solution* σ *of the weight equation* (2.10), *there is an integer* $m \geq 0$ *such that*

$$\langle\sigma, P_n^2\rangle = 0, \quad \text{for all } n \geq m+1.$$

Proof. Let $N \geq 0$ be the largest integer such that $\lambda_N = 0$. If $N \geq 1$, then $L[\cdot]$ is not admissible and $\beta_{N+1} = 0$, see the proof of Lemma 2.8. Hence, from equation (2.20), it follows that $\langle\sigma, P_n^2\rangle = 0$ for $n \geq N+1$. If $N = 0$, then $L[\cdot]$ is admissible and hence, from Lemma 2.10, $\{P_n(x)\}_{n=0}^\infty$ is a WTPS. Consequently, $\beta_k = 0$ for some integer $k \geq 1$ by Lemma 2.7. Thus, $\langle\sigma, P_n^2\rangle = 0$ for all $n \geq k$ by (2.20). □

3. Sobolev orthogonality. For any symmetric bilinear form $\phi(\cdot,\cdot)$ defined on $\mathcal{P} \times \mathcal{P}$, we call the double sequence $\{\phi_{m,n} :=$

$\phi(x^m, x^n)\}_{m,n=0}^{\infty}$ the *moments* of ϕ and say that ϕ is quasi-definite, respectively positive-definite, if

$$\Delta_n(\phi) := \det \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \dots & \phi_{0,n} \\ \phi_{1,0} & \phi_{1,1} & \dots & \phi_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n,0} & \phi_{n,1} & \dots & \phi_{n,n} \end{bmatrix} \neq 0, \quad n \in \mathbf{N}_0, \tag{3.1}$$

respectively $\Delta_n(\phi) > 0$, $n \in \mathbf{N}_0$.

Lemma 3.1. *A symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathcal{P} \times \mathcal{P}$ is quasi-definite, respectively positive-definite, if and only if there is a* PS $\{Q_n(x)\}_{n=0}^{\infty}$ *and real constants $K_n \neq 0$, respectively $K_n > 0$, $n \in \mathbf{N}_0$, such that*

$$\phi(Q_m, Q_n) = K_n \delta_{nm}, \quad m, n \in \mathbf{N}_0. \tag{3.2}$$

Proof. Assume that $\phi(\cdot,\cdot)$ is quasi-definite. Define a sequence of polynomials by

$$Q_0(x) = 1$$

$$Q_n(x) = (\Delta_{n-1}(\phi))^{-1} \det \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \dots & \phi_{0,n} \\ \phi_{1,0} & \phi_{1,1} & \dots & \phi_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1,0} & \phi_{n-1,1} & \dots & \phi_{n-1,n} \\ 1 & x & \dots & x^n \end{bmatrix}, \tag{3.3}$$

$$n \in \mathbf{N}.$$

Then $\{Q_n(x)\}_{n=0}^{\infty}$ is a monic PS and we have (3.2) with $K_n = \Delta_n(\phi)/\Delta_{n-1}(\phi)$, $\Delta_{-1} = 1$. Conversely, assume that there is a polynomial set $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying (3.2), which is unique if we assume each $Q_n(x)$ is monic. Writing

$$Q_n(x) = \sum_{k=0}^{n} C_k^n x^k,$$

we see that the orthogonality condition (3.2) gives for $j = 0, 1, \dots n$,

$$\begin{aligned}\sum_{k=0}^{n} \phi_{j,k} C_k^n = \sum_{k=0}^{n} C_k^n \phi(x^j, x^k) &= \phi(x^j, Q_n(x)) \\ &= \phi(Q_j(x), Q_n(x)) = K_n \delta_{jk}.\end{aligned} \tag{3.4}$$

Since the simultaneous equations (3.4) have a unique nontrivial solution, we must have $\Delta_n(\phi) \neq 0$ for any $n \in \mathbf{N}_0$. Finally, we have that $K_n > 0$, $n \in \mathbf{N}_0$, if and only if $\Delta_n(\phi) > 0$, $n \in \mathbf{N}_0$. □

We call a PS $\{Q_n(x)\}_{n=0}^{\infty}$ associated with the bilinear form $\phi(\cdot,\cdot)$ in Lemma 3.1 a *Sobolev-Tchebycheff polynomial system* relative to ϕ, STPS for short. If the bilinear form $\phi(\cdot,\cdot)$ is positive-definite, then we shall refer to $\{Q_n(x)\}_{n=0}^{\infty}$ as a *Sobolev orthogonal polynomial system* relative to ϕ, SOPS for short.

From here on, we shall consider only a symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathcal{P} \times \mathcal{P}$ of the form

$$\phi(p, q) := \langle \sigma, pq \rangle + \langle \tau, p'q' \rangle, \quad p, q \in \mathcal{P}, \tag{3.5}$$

where σ and τ are moment functionals. We call such a form a *symmetric Sobolev bilinear form.*

We note that it is possible that ϕ, given in (3.5), is quasi-definite and yet neither σ nor τ is quasi-definite. Indeed, Duran [**6**] produced the following example.

Example 3.1. Define a moment functional σ by its moments

$$\sigma_0 = 3, \qquad \sigma_1 = 1, \quad \text{and} \quad \sigma_n = \frac{1}{n+1} = \int_0^1 x^n \, dx \qquad n \geq 2.$$

Since $\Delta_2(\sigma) = 0$, σ is not quasi-definite. Similarly $\tau = \delta$, where $\delta(x)$ is the Dirac moment functional defined by

$$\langle \delta(x), x^n \rangle = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \in \mathbf{N}, \end{cases}$$

is not quasi-definite. However, the form ϕ on $\mathcal{P} \times \mathcal{P}$ defined by

$$\phi(p,q) = \langle \sigma, pq \rangle + \frac{1}{8}\langle \delta, p'q' \rangle$$

is positive-definite. Indeed, Duran shows that

$$\phi(p,p) = \int_0^1 p^2(x)\,dx + \left(\sqrt{2}p(0) + \frac{1}{2\sqrt{2}}p'(0)\right)^2 > 0, \quad p \in \mathcal{P}.$$

Lemma 3.2. *Suppose $\phi(\cdot,\cdot)$, as given in* (3.5)*, is quasi-definite and $\{Q_n(x)\}_{n=0}^\infty$ is an* STPS *relative to ϕ. Then σ is a canonical moment functional for $\{Q_n(x)\}_{n=0}^\infty$. In particular, $\langle \sigma, 1 \rangle \neq 0$.*

Proof. In Equation (3.5), let $p = Q_n$ and $q = Q_0$. If $n \geq 1$, then $0 = \phi(Q_n, Q_0) = \langle \sigma, Q_n Q_0 \rangle$ and if $n = 0$, then $0 \neq \phi(Q_0, Q_0) = Q_0^2 \langle \sigma, 1 \rangle$. □

The main question that we wish to address in this section is the following: when does an STPS $\{Q_n(x)\}_{n=0}^\infty$ relative to the form $\phi(\cdot,\cdot)$ defined in (3.5) satisfy a second-order differential equation of the form given in (2.8)?

Theorem 3.3. *Let $\{Q_n(x)\}_{n=0}^\infty$ be an* STPS *relative to the* (*quasi-definite*) *symmetric Sobolev bilinear form $\phi(\cdot,\cdot)$, defined in* (3.5)*. Then the following statements are equivalent*:

(i) *$\{Q_n(x)\}_{n=0}^\infty$ satisfies the differential equation $L[y] = \lambda_n y$, defined in* (2.8)*, i.e.,*

$$(l_{2,2}x^2 + l_{2,1}x + l_{2,0})Q_n''(x) + (l_{1,1}x + l_{1,0})Q_n'(x) = \lambda_n Q_n(x), \quad n \in \mathbf{N}_0.$$

(ii) *The differential operator $L[\cdot]$, defined in* (2.8)*, is symmetric on polynomials relative to $\phi(\cdot,\cdot)$, i.e.,*

$$\phi(L[p], q) = \phi(p, L[q]), \quad p, q \in \mathcal{P}. \tag{3.6}$$

(iii) *The moment functionals* σ *and* τ *satisfy the weight equations*

$$(a_2(x)\sigma)' - a_1(x)\sigma = 0, \tag{3.7}$$

and

$$a_2(x)\tau' - a_1(x)\tau = 0. \tag{3.8}$$

(iv) *The moments of* $\phi(\cdot,\cdot), \sigma$, *and* τ, *given respectively by*

$$\phi_{mn} := \phi(x^m, x^n), \qquad \sigma_n := \langle \sigma, x^n \rangle,$$

and

$$\tau_n := \langle \tau, x^n \rangle, \quad m, n \in \mathbf{N}_0,$$

satisfy the equations

$$\phi_{mn} = \sigma_{n+m} + mn\tau_{m+n-2}, \qquad \tau_{-2} = \tau_{-1} = 0, \tag{3.9}$$

$$\begin{gathered}(nl_{2,2} + l_{1,1})\sigma_{n+1} + (nl_{2,1} + l_{1,0})\sigma_n + nl_{2,0}\sigma_{n-1} = 0, \\ \sigma_{-1} = 0,\end{gathered} \tag{3.10}$$

$$\begin{gathered}((n+2)l_{2,2} + l_{1,1})\tau_{n+1} + ((n+1)l_{2,1} + l_{1,0})\tau_n + nl_{2,0}\tau_{n-1} = 0 \\ \tau_{-1} = 0.\end{gathered} \tag{3.11}$$

Proof. (i) $\Rightarrow$ (ii). If $L[Q_n] = \lambda_n Q_n$ for each integer $n \in \mathbf{N}_0$, then for all integers $m, n \in \mathbf{N}_0$, we have

$$\phi(L[Q_m], Q_n) - \phi(Q_m, L[Q_n]) = (\lambda_m - \lambda_n)\phi(Q_m, Q_n) = 0.$$

Equation (3.6) now follows by linearity since $\{Q_n(x)\}_{n=0}^{\infty}$ is a PS.

(ii) $\Rightarrow$ (i). Since $L[Q_n]$ is a polynomial of degree $\leq n$, we may write

$$L[Q_n](x) = \sum_{j=0}^{n} c_{j,n} Q_j(x),$$

for some constants $c_{j,n}$. Then for $k = 0, 1, \dots n-1$,

$$\begin{aligned} c_{k,n}\phi(Q_k, Q_k) &= \sum_{j=0}^{n} c_{j,n}\phi(Q_j, Q_k) = \phi(L[Q_n], Q_k) \\ &= \phi(Q_n, L[Q_k]) = 0, \end{aligned}$$

since $\deg(L[Q_k]) \le k$. Hence, $c_{k,n} = 0$ for each $k = 0, 1, \dots n-1$, and thus $L[Q_n](x) = c_{n,n}Q_n(x)$. By comparing coefficients of x^n on both sides, it is easy to see that $c_{n,n} = nl_{1,1} + n(n-1)l_{2,2} = \lambda_n$.

(ii) $\Leftrightarrow$ (iii). The reader may check the following identities, valid for any polynomials $p, q \in \mathcal{P}$:

$$\phi(L[p], q) = \langle L^+[q\sigma], p\rangle - \langle L^+[(q'\tau)'], p\rangle$$

and

$$\phi(p, L[q]) = \langle L[q]\sigma, p\rangle - \langle ((L[q])'\tau)', p\rangle,$$

where $L^+[\cdot]$ is the formal Lagrangian adjoint of the expression $L[\cdot]$, defined by $L^+[y](x) := (a_2(x)y(x))'' - (a_1(x)y(x))'$. Hence, condition (3.6) is equivalent to:

$$L^+[q\sigma] - L^+[(q'\tau)'] - L[q]\sigma + ((L[q])'\tau)' = 0, \quad q \in \mathcal{P},$$

which, when written out and simplified, yields for all $q \in \mathcal{P}$,

$$\begin{aligned} (2a_1\tau - 2a_2\tau')q^{(3)} + (-3(a_2\tau')' + 3(a_1\tau)')q'' \\ + (-(a_2\tau')'' + (a_1\tau)'' + 2(a_2\sigma)' - 2a_1\sigma)q' \\ + ((a_2\sigma)'' - (a_1\sigma)')q = 0. \end{aligned}$$

From this, we see that statement (ii) is equivalent to the fact that σ and τ satisfy the four functional equations

$$a_2\tau' - a_1\tau = 0, \tag{3.12}$$

$$(a_2\tau')' - (a_1\tau)' = 0, \tag{3.13}$$

$$-(a_2\tau')'' + (a_1\tau)'' + 2(a_2\sigma)' - 2a_1\sigma = 0, \tag{3.14}$$

$$(a_2\sigma)'' - (a_1\sigma)' = 0. \tag{3.15}$$

Lemma 2.4 asserts that (3.12) and (3.13) are equivalent. Moreover, (3.14) simplifies to $(a_2\sigma)' - a_1\sigma = 0$, which is equivalent to (3.15) by Lemma 2.4. Thus the four equations (3.12)–(3.15) are equivalent to (3.7) and (3.8).

(iii) $\Leftrightarrow$ (iv). Note that

$$\begin{aligned}\phi_{mn} &= \phi(x^m, x^n) = \langle \sigma, x^{m+n}\rangle + \langle \tau, (x^m)'(x^n)'\rangle \\ &= \sigma_{n+m} + mn\tau_{n+m-2}.\end{aligned}$$

Furthermore, the reader can check that the moment functionals σ and τ satisfy (3.7) and (3.8) if and only if their moments satisfy (3.10) and (3.11), respectively. $\square$

We call the functional equations (3.7) and (3.8) the *Sobolev weight equations* for the differential expression $L[\cdot]$. As we shall soon see, these Sobolev weight equations are the tools for which we construct second-order differential equations having Sobolev orthogonal polynomial solutions. We call equations (3.10) and (3.11) the *Sobolev moment equations*. Observe that, when $\tau = 0$, the Sobolev weight equations (3.7) and (3.8) reduce to the weight equation (2.10). Consequently, in view of Theorem 2.5 and Corollary 2.6, we see that Theorem 3.3 gives a generalization of H.L. Krall's result, Theorem 2.5, for second-order differential equations. More specifically, we have

Corollary 3.4. *Consider the bilinear form $\phi(\cdot,\cdot)$ defined in* (3.5) *with moments $\phi_{mn} = \phi(x^m, x^n) = \sigma_{n+m} + nm\tau_{n+m-2}$, $m, n \in \mathbf{N}_0$, where $\{\sigma_n\}_{n=0}^\infty$ and $\{\tau_n\}_{n=0}^\infty$ are the moments of σ and τ, respectively. Then there exists an* STPS $\{Q_n(x)\}_{n=0}^\infty$ *relative to $\phi(\cdot,\cdot)$ with $Q_n(x)$ satisfying the differential equation* (2.8) *if and only if*

(i) *$\phi(\cdot,\cdot)$ is quasi-definite, and*

(ii) *$\{\sigma_n\}_{n=0}^\infty$ and $\{\tau_n\}_{n=0}^\infty$ satisfy the Sobolev moment equations* (3.10) *and* (3.11)*, respectively.*

Proof. This follows immediately from Lemma 3.1 and Theorem 3.3. $\square$

Remark 3. When $a_2(x) \equiv 0$, the differential equation (2.8) reduces to the first-order equation

$$(l_{1,1}x + l_{1,0})y'(x) = nl_{1,1}y(x),$$

which can have a PS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions only when $l_{1,1} \neq 0$.

The reader can observe that the proof of Theorem 3.3 remains valid if $a_2(x) \equiv 0$. In this case, the Sobolev weight equations (3.7) and (3.8) reduce to

$$(l_{1,1}x + l_{1,0})\sigma = (l_{1,1}x + l_{1,0})\tau = 0,$$

from which we see that

$$\sigma = c_1\delta\left(x + \frac{l_{1,0}}{l_{1,1}}\right),$$

and

$$\tau = c_2\delta\left(x + \frac{l_{1,0}}{l_{1,1}}\right),$$

where c_1 and c_2 are arbitrary constants. When these functionals are substituted into (3.5), the corresponding bilinear form $\phi(\cdot,\cdot)$ cannot be quasi-definite. Consequently, the above first-order differential equation can never have an STPS of solutions. In particular, it can never have a TPS of solutions.

Remark 4. We note that if we differentiate (2.8) with respect to x and replace dy/dx by $z(x)$, we get the second-order differential equation

$$\begin{aligned} M[z](x) &= a_2(x)z''(x) + (a_2'(x) + a_1(x))z'(x) \\ &= (\lambda_{n+1} - a_1'(x))z(x). \end{aligned} \tag{3.16}$$

It is interesting to note that Equation (3.8) is the weight equation for $M[\cdot]$ in the sense of Lemma 2.1.

Remark 5. Observe that Equations (3.7) and (3.8), considered as classical differential equations, will always have solutions $w_1(x) > 0$ and $w_2(x) > 0$, respectively, that are defined on any interval $I = (a,b)$ where $a_2(x) > 0$. Consequently, the differential expression $L[y] =$

$a_2(x)y'' + a_1(x)y'$ will be a formally symmetric operator with respect to the bilinear form $\phi_{w_1,w_2}(\cdot,\cdot)$ defined by

$$\phi_{w_1,w_2}(f,g) := \int_a^b f(x)g(x)w_1(x)\,dx + \int_a^b f'(x)g'(x)w_2(x)\,dx.$$

This generalizes the well-known fact that any second-order differential expression with smooth coefficients can be made formally symmetric in some weighted L^2-space. In view of this, we shall call the pair (w_1, w_2) a *Sobolev symmetry factor* for $L[\cdot]$. For work on symmetry factors in the L^2 sense, see the contributions [**23**] and [**24**].

We are now in a position to prove Theorem 1.2, given in Section 1, which we restate in terms of moment functionals σ and τ.

Theorem 3.5. *Suppose that*

(i) $\{Q_n(x)\}_{n=0}^\infty$ *is an* STPS *relative to the symmetric bilinear form* $\phi(\cdot,\cdot)$ *defined in* (3.5), *and*

(ii) *for each integer* $n \in \mathbf{N}_0$, $Q_n(x)$ *satisfies the second-order differential equation* (2.8). *If the moment functional* σ *is quasi-definite, then* $\{Q_n(x)\}_{n=0}^\infty$ *is orthogonal with respect to* σ *and is, up to a* real *linear change of variable, necessarily one of the following sets of polynomials*:

(a) *Jacobi* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $-\alpha, -\beta, -(\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$,

(b) *Laguerre* $\{L_n^\alpha(x)\}_{n=0}^\infty$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$,

(c) *Hermite* $\{H_n(x)\}_{n=0}^\infty$,

(d) *Bessel* $\{y_n^a(x)\}_{n=0}^\infty$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$,

(e) *twisted Jacobi polynomials* $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $-(\alpha+\beta+1) \in \mathbf{C}\backslash\mathbf{N}$ *and* $\beta = \bar{\alpha}$, *or*

(f) *twisted Hermite polynomials* $\{\check{H}_n(x)\}_{n=0}^\infty$.

Moreover, if σ *is quasi-definite, the moment functional* τ *is necessarily of the form*

$$\tau = ka_2(x)\sigma, \tag{3.17}$$

for some real constant k. *Furthermore, up to a* complex *linear change of variable, the only polynomial sets that are orthogonal with respect to* *a bilinear form of the type* (3.5) *and satisfy a second-order differential equation of the form* (2.8) *are the classical orthogonal polynomial sets listed in* (a), (b), (c)*, and* (d) *above.*

Proof. We may assume that $Q_n(x)$ is monic, $n \in \mathbf{N}_0$. Let $\{P_n(x)\}_{n=0}^\infty$ be the monic TPS relative to the quasi-definite moment functional σ. From Theorem 3.3, we see that σ satisfies the weight equation (3.7). Hence, by Theorem 2.5, $\{P_n(x)\}_{n=0}^\infty$ also satisfies the differential equation in (2.8). By Theorem 1.3 or Theorem 1.4, $\{P_n(x)\}_{n=0}^\infty$ is one of the above mentioned TPS's. By Proposition 2.3, we then see that $L[\cdot]$ is admissible. Since $Q_n(x)$ is a monic polynomial solution to (2.8) for each integer $n \in \mathbf{N}_0$, we apply Lemma 2.2 (iv) to conclude that $P_n(x) \equiv Q_n(x)$, $n \in \mathbf{N}_0$. Lastly, to prove that τ is given as in (3.17), we note that since the operator $L[\cdot]$ is admissible, so is the operator $M[\cdot]$ defined in (3.16). Consequently, by Lemma 2.2, the associated moment equation (3.11) for $M[\cdot]$ is uniquely solvable. Define $\tilde{\tau} = a_2(x)\sigma$; then $\tilde{\tau} \neq 0$ by Lemma 2.4 (ii) and the fact that σ is quasi-definite. Moreover, $\tilde{\tau}$ satisfies the weight equation (3.8). Indeed, since $(a_2\sigma)' = a_1\sigma$ by (3.7), we see that

$$a_2\tilde{\tau}' - a_1\tilde{\tau} = a_2(a_2\sigma)' - a_1(a_2\sigma) = a_2(a_1\sigma) - a_1(a_2\sigma) = 0.$$

However, since τ is a solution to (3.8), which is uniquely solvable by Lemma 2.2, we must have that $\tau = k\tilde{\tau}$ for some constant k. □

Remark 6. The hypothesis that σ is quasi-definite cannot be relaxed. Indeed, in the next section, we produce *nonclassical* (in a sense) examples of STPS's that satisfy second-order differential equations of the form (2.8).

Corollary 3.6. (a) *If* $\tau \neq 0$ *in Theorem* 3.5*, then* τ *is quasi-definite and* $\{Q_n'(x)\}_{n=1}^\infty$ *is a* TPS *with respect to* τ*.*

(b) *Suppose* $\{Q_n(x)\}_{n=0}^\infty$ *is an* STPS *with respect to the bilinear form* $\phi(\cdot,\cdot)$ *defined in* (3.5)*. If* σ *is quasi-definite and* $\tau \neq 0$ *is not quasi-definite, then* $\{Q_n(x)\}_{n=0}^\infty$ *cannot satisfy a differential equation of the form* (2.8)*.*

Proof. It is well-known, see [**9**] or [**30**], that if $\{Q_n(x)\}_{n=0}^\infty$ is classical and orthogonal with respect to a moment functional σ, then

$\{Q'_n(x)\}_{n=1}^{\infty}$ is orthogonal with respect to $a_2(x)\sigma$. This fact, together with the proof of Theorem 3.5, yield both (a) and (b). □

Remark 7. The STPS $\{Q_n(x)\}_{n=0}^{\infty}$ in Corollary 3.6 (b) may satisfy a second-order differential equation with polynomial coefficients depending on the parameter n. One such example is to be found in the recent contribution of Marcellán, Pérez, and Piñar [**25**] in which they study an STPS relative to ϕ, where σ is a quasi-definite moment functional for the generalized Bessel polynomials and $\tau = \lambda\delta(x)$ for some nonzero constant λ. However, from Corollary 3.6, this STPS cannot satisfy a second-order equation of the form (2.8).

If we assume that τ, instead of σ, is quasi-definite we have the following result:

Theorem 3.7. *Let* $\{Q_n(x)\}_{n=0}^{\infty}$ *be an* STPS *relative to the form* $\phi(\cdot,\cdot)$ *defined in* (3.5), *and suppose that, for each* $n \in \mathbf{N}_0$, $Q_n(x)$ *satisfies* (2.8). *If the moment functional* τ *is quasi-definite, then*

(i) $\{Q'_n(x)\}_{n=1}^{\infty}$ *is a real classical polynomial sequence, see Theorem* 1.2, *that is orthogonal with respect to* τ *and for each integer* $n \in \mathbf{N}$, $Q'_n(x)$ *satisfies equation* (3.16);

(ii) *the operator* $M[\cdot]$ *in* (3.16) *is admissible, i.e.,* $l_{1,1} \notin \{-nl_{2,2} \mid n \geq 2\}$;

(iii) $\{Q_n(x)\}_{n=0}^{\infty}$ *is weakly orthogonal with respect to* σ, *i.e.,*

$$\langle \sigma, Q_n Q_m \rangle = 0, \quad \text{if } m \neq n \tag{3.18}$$

and $\langle \sigma, Q_n^2 \rangle$ *may or may not be zero* (*but* $\langle \sigma, Q_0^2 \rangle \neq 0$);

(iv) $a_2(x)\sigma = k\tau$ *for some constant* k; *hence, either* $\sigma = 0$ *or* $a_2(x)\sigma$ *is quasi-definite.*

Proof. We may assume that $Q_n(x)$ is monic for each $n \in \mathbf{N}_0$. Since $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies (2.8), it follows that $\{Q'_{n+1}(x)/n+1\}_{n=0}^{\infty}$ is a monic PS which satisfies the differential equation given in (3.16). Let $\{P_n(x)\}_{n=0}^{\infty}$ be the monic TPS relative to τ. By Theorem 3.3, τ satisfies the weight equation given in (3.8). Hence, from Theorem 2.5 and Remark 4, it follows that for each $n \in \mathbf{N}_0$, $P_n(x)$ satisfies equation (3.16) and $\{P_n(x)\}_{n=0}^\infty$ is one of the real classical TPS's. By Proposition 2.3, equation (3.16) is admissible. Hence, by Lemma 2.2 (iv), we must have

$$\frac{Q'_{n+1}(x)}{n+1} = P_n(x), \quad n \in \mathbf{N}_0.$$

This proves (i) and (ii). Part (iii) follows immediately from (i) and the orthogonality of $\{Q_n(x)\}_{n=0}^\infty$ with respect to the form ϕ. Finally, from Lemma 2.2, since equation (3.16) is admissible, the weight equation (3.8) has only one linearly independent solution. Since τ and $a_2(x)\sigma$ both satisfy this weight equation, we must therefore have $a_2(x)\sigma = k\tau$ for some constant k. □

4. Examples in the class $STPS \cap DPS(2)$. From Theorem 3.5 we immediately obtain the following results concerning the Sobolev orthogonality of the real classical orthogonal polynomials; in each case below, the parameter k is an arbitrary real number.

Example 4.1. *The real classical orthogonal polynomials.* (i) The *Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $(-\alpha, -\beta, -(\alpha+\beta+1) \in \mathbf{R}\backslash\mathbf{N}$, satisfy the second-order differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' = \lambda_n y, \tag{4.1}$$

where $\lambda_n = -n(n+\alpha+\beta+1)$, and are orthogonal with respect to the Sobolev bilinear form

$$\begin{aligned}(p,q)_J = \int_{-1}^1 p(x)q(x)(1-x)^\alpha(1+x)^\beta\,dx \\ + k\int_{-1}^1 p'(x)q'(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\,dx \\ p, q \in \mathcal{P}.\end{aligned}$$

(ii) The *Laguerre polynomials* $\{L_n^\alpha(x)\}_{n=0}^\infty$, $-\alpha \in \mathbf{R}\backslash\mathbf{N}$, satisfy the second-order differential equation

$$xy'' + (1+\alpha-x)y' = \lambda_n y, \tag{4.2}$$

where $\lambda_n = -n$, and are orthogonal with respect to the Sobolev bilinear form

$$(p,q)_L = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + k\int_0^\infty p'(x)q'(x)x^{\alpha+1}e^{-x}\,dx$$
$$p, q \in \mathcal{P}.$$

(iii) The *Hermite polynomials* $\{H_n(x)\}_{n=0}^\infty$ satisfy the second-order equation

$$y'' - 2xy' = \lambda_n y,$$

where $\lambda_n = -2n$, and are orthogonal with respect to the Sobolev bilinear form

$$(p,q)_H = \int_0^\infty p(x)q(x)e^{-x^2}\,dx + k\int_0^\infty p'(x)q'(x)e^{-x^2}\,dx$$
$$p, q \in \mathcal{P}.$$

(iv) The *Bessel polynomials* $\{y_n^a(x)\}_{n=0}^\infty$, $-(a+1) \in \mathbf{R}\backslash\mathbf{N}$, are solutions of the second-order equation

$$x^2 y'' + ((a+2)x+2)y' = \lambda_n y,$$

where $\lambda_n = n(n+a+1)$, and are orthogonal with respect to the Sobolev bilinear form

$$(p,q)_B = \langle \sigma, pq\rangle + k\langle x^2\sigma, p'q'\rangle,$$

where σ is any orthogonalizing moment functional for the Bessel polynomials. For example, when $a = 0$, Kwon, Kim, and Han [**18**] have shown that the moment functional σ, defined by

$$\langle \sigma, p\rangle = -\int_0^\infty e^{-2/x}\left(\int_x^\infty e^{2/t}t^{-2}g(t)\,dt\right)p(x)\,dx, \quad p \in \mathcal{P},$$

where $g(t)$ is the function defined in (2.12) is an orthogonalizing moment functional for the Bessel polynomials $\{y_n^0(x)\}_{n=0}^\infty$.

(v) The *twisted Jacobi polynomials* $\{\check{P}_n^{(\alpha,\beta)}(x)\}_{n=0}^\infty$, $-(\alpha+\beta+1) \in \mathbf{C}\backslash\mathbf{N}$ and $\beta = \bar{\alpha}$, are given by $\check{P}_n^{(\alpha,\beta)}(x) = i^n P_n^{(\alpha,\beta)}(-ix)$, where $P_n^{(\alpha,\beta)}(x)$ is the monic Jacobi polynomial of degree n and $i = \sqrt{-1}$.

For each $n \in \mathbf{N}_0$, $y(x) = \check{P}_n^{(\alpha,\beta)}(x)$ is a real polynomial satisfying the (real) second-order differential equation

$$(4.3)\quad (1+x^2)y'' + ((\alpha+\beta+2)x + (\alpha-\beta)i)y' = n(n+\alpha+\beta+1)y.$$

From Theorems 3.3 and 3.5, we see that the twisted Jacobi polynomials are Sobolev orthogonal with respect to the bilinear form

$$(p,q)_{tJ} = \langle \sigma, pq\rangle + k\langle \sigma, (1+x^2)p'q'\rangle, \quad p,q \in \mathcal{P},$$

where σ is any nonzero moment functional satisfying the weight equation

$$(1+x^2)\sigma' + (-(\alpha+\beta)x + (\beta-\alpha)i)\sigma = 0.$$

At this point, an explicit weight function for the twisted Jacobi polynomials is unavailable; it is known (see [**20**]), however, that σ is quasi-definite but never positive-definite.

(vi) The *twisted Hermite polynomials* $\{\check{H}_n(x)\}_{n=0}^{\infty}$ are defined through the formula $\check{H}_n(x) := i^n H_n(-ix)$, where $H_n(x)$ is the monic Hermite polynomial of degree n. For each $n \in \mathbf{N}_0$, $y(x) = \check{H}_n(x)$ is a real polynomial and satisfies the second-order differential equation

$$y'' + 2xy' = 2ny, \quad n \in \mathbf{N}_0.$$

They are Sobolev orthogonal with respect to the bilinear form

$$(p,q)_{tH} = \langle \sigma, pq\rangle + \langle \sigma, p'q'\rangle, \quad p,q \in \mathcal{P},$$

where σ is any nonzero moment functional satisfying the weight equation

$$\sigma' - 2x\sigma = 0.$$

As in the twisted Jacobi case, no orthogonalizing weight function has been explicitly determined at this point; however, it is known that such a σ is quasi-definite but never positive-definite.

As we shall see, the *nonclassical* examples that we discuss in this section are really not new at all! Indeed, they are *almost* (real) classical OPS's except that the parameters used in their definition exclude them from being called (real) classical. A complete classification of polynomial sets in the intersection class $STPS \cap DPS(2)$ will be presented in Section 5.

If $x \in \mathbf{R}$ and k is a nonnegative integer, we remind the reader of the notation

$$\binom{x}{k} = \begin{cases} x(x-1)\cdots(x-k+1)/k! & \text{if } k \geq 1 \\ 1 & \text{if } k = 0. \end{cases}$$

Example 4.2 (*Laguerre*: $\alpha = -1$). The differential equation

$$xy'' - xy' = -ny \tag{4.4}$$

is a special case of the general Laguerre differential given in (4.2). By Lemma 2.2, equation (4.4) is admissible and has a unique monic set of polynomial solutions which are the Laguerre polynomials $\{L_n^{-1}(x)\}_{n=0}^{\infty}$, defined explicitly by

$$L_0^{-1}(x) = 1$$

and

$$L_n^{-1}(x) := (-1)^n n! \sum_{k=0}^{n} \binom{n-1}{n-k} \frac{(-1)^k}{k!} x^k, \quad n \in \mathbf{N}.$$

By Theorem 2.5, this PS cannot be a TPS with respect to any moment functional. Indeed, the moments $\{\mu_n\}_{n=0}^{\infty}$ associated with this PS are such that $\mu_n = 0$ for $n \in \mathbf{N}$. However, $\{L_n^{-1}(x)\}_{n=0}^{\infty}$ is an STPS. To see this, we apply Theorem 3.3. The Sobolev weight equations (3.7) and (3.8) are, respectively,

$$(x\sigma)' + x\sigma = 0, \tag{4.5}$$

and

$$x\tau' + x\tau = 0. \tag{4.6}$$

The solutions of these functional equations are given by

$$\sigma = c_1 \delta(x) \quad \text{and} \quad \tau = c_2 H(x) e^{-x},$$

respectively, where c_1 and c_2 are arbitrary constants and $H(x)$ is the Heaviside function. When these functionals are substituted into the form (3.5), we obtain

$$\phi(p,q) = c_1 p(0)q(0) + c_2 \int_0^\infty e^{-x} p'(x)q'(x)\,dx, \quad p,q \in \mathcal{P}.$$

In order for $\phi(\cdot,\cdot)$ to be quasi-definite, it is necessary that $c_2 \neq 0$. Hence, by setting $A = c_1/c_2$, we may consider the one-parameter family of bilinear forms

$$\phi_A(p,q) = Ap(0)q(0) + \int_0^\infty p'(x)q'(x)e^{-x}\,dx. \tag{4.7}$$

Proposition 4.1. *The bilinear form* $\phi_A(\cdot,\cdot)$ *in* (4.7) *is*

(i) *quasi-definite if and only if* $A \neq 0$,

(ii) *positive-definite if and only if* $A > 0$.

In either case, the monic STPS *or monic* SOPS *relative to* $\phi_A(\cdot,\cdot)$ *is* $\{L_n^{-1}(x)\}_{n=0}^\infty$. *Furthermore,*

$$\phi_A(L_n^{(-1,-1)}, L_n^{(-1,-1)}) = \begin{cases} A & \text{if } n = 0 \\ (n!)^2 & \text{if } n \geq 1. \end{cases}$$

Proof. We first recall the following well-known facts, see [**30**]:

(a) $L_n^{-1}(0) = 0$ for all $n \in \mathbf{N}$,

(b) $(L_n^{-1}(x))' = nL_{n-1}(x)$, where $\{L_n(x)\}_{n=0}^\infty$ is the simple ($\alpha = 0$), Laguerre PS defined by

$$L_n(x) = (-1)^n n! \sum_{k=0}^n \binom{n}{n-k} \frac{(-1)^k}{k!} x^k, \quad n \in \mathbf{N}_0,$$

(c) $\int_0^\infty L_n(x)L_m(x)e^{-x}\,dx = (n!)^2\delta_{nm}$, $n,m \in \mathbf{N}_0$.

From these facts, we obtain

$$\phi_A(L_n^{-1}, L_m^{-1}) = \begin{cases} A & \text{if } n = m = 0 \\ (n!)^2 & \text{if } n = m > 0 \\ 0 & \text{if } n \neq m, \end{cases}$$

and the result follows. □

We refer the reader to [**11**] for related work on the Laguerre expression (4.4). There is a connection between our inner product $\phi_1(\cdot,\cdot)$ and the left-definite inner product given in [**11**] when $\alpha = 0$.

Example 4.3 (*Jacobi*: $\alpha = \beta = -1$). The differential equation

$$(1-x^2)y'' = -n(n-1)y \tag{4.8}$$

is a special case of the Jacobi differential equation (4.1); indeed, equation (4.8) is obtained from (4.1) by setting $\alpha = \beta = -1$. Using Lemma 2.2, it is easy to see that equation (4.8) is not admissible but there is a unique monic polynomial solution $P_n^{(-1,-1)}(x)$ of degree n to (4.8) for all $n \neq 1$. Indeed, for $n \in \mathbf{N}_0$ but $n \neq 1$, it is given explicitly by

$$\begin{aligned} P_n^{(-1,-1)}(x) = \binom{2n-2}{n}^{-1} \sum_{k=0}^{n} \binom{n-1}{k} \\ \cdot \binom{n-1}{n-k} (x-1)^{n-k}(x+1)^k. \end{aligned} \tag{4.9}$$

When $n = 1$, any polynomial of the form $P_1^{(-1,-1)}(x) = x + \gamma$, γ an arbitrary constant, is a solution to (4.8). As in Example 4.1, the PS $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$ (with any choice for a monic $P_1^{(-1,-1)}(x)$) cannot be a TPS. However, this PS does form an STPS. In this case, the weight equations (3.7) and (3.8) become, respectively,

$$((1-x^2)\sigma)' = 0, \tag{4.10}$$

and

$$(1-x^2)\tau' = 0. \tag{4.11}$$

From these equations, we find that

$$\sigma = c_1\delta(1-x) + c_2\delta(1+x) \quad \text{and} \quad \tau = c_3 H(1-x^2),$$

where c_i, $i = 1, 2, 3$, are arbitrary constants. The corresponding bilinear form (3.5) becomes

$$\phi(p, q) = c_1 p(1)q(1) + c_2 p(-1)q(-1) + c_3 \int_{-1}^{1} p'(x)q'(x)\,dx.$$

As in the Laguerre case above, in order for ϕ to be quasi-definite, it is necessary that $c_3 \neq 0$. Hence, by setting $A = c_1/c_3$ and $B = c_2/c_3$, we may consider the two-parameter family of bilinear forms

$$\phi_{A,B}(p, q) = Ap(1)q(1) + Bp(-1)q(-1) + \int_{-1}^{1} p'(x)q'(x)\,dx. \tag{4.12}$$

Proposition 4.2. *The bilinear form* $\phi_{A,B}(\cdot,\cdot)$ *given in* (4.12) *is*

(i) *quasi-definite if and only if* $A + B \neq 0$ *and* $2AB + A + B \neq 0$,

(ii) *positive-definite if and only if* $A + B > 0$ *and* $2AB + A + B > 0$.

In either case, the monic STPS *or monic* SOPS *relative to* $\phi_{A,B}(\cdot,\cdot)$ *is* $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$, *where* $P_n^{(-1,-1)}(x)$ *is given in* (4.9) *for* $n \neq 1$ *and* $P_1^{(-1,-1)}(x) = x + \gamma$, *where* $\gamma = (B - A)/(A + B)$. *Furthermore,*

$$\phi_{A,B}(P_n^{(-1,-1)}, P_n^{(-1,-1)}) = \begin{cases} A + B & \text{if } n = 0 \\ A(\gamma+1)^2 + B(\gamma-1)^2 + 2 & \text{if } n = 1 \\ n^2 K_{n-1} & \text{if } n \geq 2, \end{cases}$$

where

$$K_n = \int_{-1}^{1} (P_n^{(0,0)}(x))^2\,dx = \frac{2^{2n+1}(n!)^4}{(2n)!(2n+1)!}, \quad n \in \mathbf{N}_0.$$

Conversely, for any real number γ, $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$, *where* $P_1^{(-1,-1)}(x) = x + \gamma$, *is an* STPS *relative to* $\phi_{A,B}(\cdot,\cdot)$ *where* A *and* B *are real numbers satisfying*

$$A + B \neq 0, \quad A(\gamma+1)^2 + B(\gamma-1)^2 + 2 \neq 0$$

and

$$A(\gamma+1) + B(\gamma-1) = 0.$$

Proof. From Proposition 4.1, we see that

$$\phi_{A,B}(P_m^{(-1,-1)}, P_n^{(-1,-1)}) = 0 \quad \text{for } 0 \le m < n$$

and

$$(m,n) \ne (0,1).$$

With the help of the following well-known facts,

(i) $P_n^{(-1,-1)}(\pm 1) = 0$ for all integers $n \ge 2$,

(ii) $d(P_n^{(-1,-1)}(x))/dx = nP_{n-1}(x)$, where $P_n(x)$ is the Legendre polynomial defined by

$$P_n(x) = \binom{2n}{n}^{-1} \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{(n-k)!k!(n-2k)!} x^{n-2k}, \quad n \in \mathbf{N}_0,$$

(iii) $\int_{-1}^{1} P_n(x)P_m(x)\,dx = K_n\delta_{nm}$, $n, m \in \mathbf{N}_0$, where K_n is defined as above, we obtain the identity

$$\phi_{A,B}(P_n^{(-1,-1)}, P_m^{(-1,-1)}) = \begin{cases} 0 & \text{if } 0 \le m < n, (n,m) \ne (1,0) \\ A(\gamma+1) + B(\gamma-1) & \text{if } (n,m) = (1,0) \\ A+B & \text{if } n = m = 0 \\ A(\gamma+1)^2 + B(\gamma-1)^2 + 2 & \text{if } n = m = 1 \\ n^2 K_{n-1} & \text{if } n = m \ge 2, \end{cases}$$

from which the result follows. □

Example 4.4 (*Jacobi*: $\alpha = -1, -\beta \in \mathbf{R}\backslash\mathbf{N}$). Consider the Jacobi differential equation (4.1) when $\alpha = -1$:

$$(1-x^2)y'' + (\beta+1)(1-x)y' = -n(n+\beta)y, \tag{4.13}$$

where $-\beta \in \mathbf{R}\backslash\mathbf{N}$. Unlike the previous example when $\alpha = \beta = -1$, equation (4.13) is admissible and the PS of solutions to (4.13) is the Jacobi polynomials $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$, defined explicitly by

$$P_n^{(-1,\beta)}(x) = \binom{2n+\beta-1}{n}^{-1} \sum_{k=0}^{n} \binom{n-1}{k} \cdot \binom{n+\beta}{n-k} (x-1)^{n-k}(x+1)^k, \quad n \in \mathbf{N}_0.$$

As in the previous two cases, this PS cannot be orthogonal with respect to any quasi-definite moment functional but it does form an STPS. The weight equations (3.7) and (3.8) are, in this case,

$$((1-x^2)\sigma)' - (\beta+1)(1-x)\sigma = 0, \tag{4.14}$$

and

$$(1-x^2)\tau' - (\beta+1)(1-x)\tau = 0. \tag{4.15}$$

These equations have solutions

$$\sigma = c_1\delta(x-1) \quad \text{and} \quad \tau = c_2(1+x)_+^{\beta+1}H(1-x),$$

where c_1 and c_2 are arbitrary constants; see Remark 8 below for information on the distribution $(1+x)_+^{\beta+1}$. The corresponding bilinear form (3.5) is

$$\phi(p,q) = c_1p(1)q(1) + c_2\langle(1+x)_+^{\beta+1}H(1-x), p'(x)q'(x)\rangle, \quad p,q \in \mathcal{P}.$$

In order for $\phi(\cdot,\cdot)$ to be quasi-definite, it is necessary that $c_2 \neq 0$; hence, by setting $A = c_1/c_2$, we may consider the one-parameter family of bilinear forms

$$\phi_B^\beta(p,q) = Ap(1)q(1) + \langle(1+x)_+^{\beta+1}H(1-x), p'(x)q'(x)\rangle \quad p,q \in \mathcal{P}. \tag{4.16}$$

The following result can be proved in a similar way that we established Propositions 4.1 and 4.2.

Proposition 4.3. *The bilinear form* $\phi_A^\beta(\cdot,\cdot)$, *defined in* (4.16), *is*

(i) *quasi-definite if and only if* $A \neq 0$ *and* $-\beta \notin \mathbf{N}$,

(ii) *positive-definite if and only if* $A > 0$ *and* $\beta > -2$, $\beta \neq -1$.

In either case, the monic STPS *or monic* SOPS *relative to* $\phi_A^\beta(\cdot,\cdot)$ *is* $\{P_n^{(-1,\beta)}(x)\}_{n=0}^\infty$. *Furthermore,*

$$\phi_A^\beta(P_n^{(-1,\beta)}, P_n^{(-1,\beta)}) = \begin{cases} A & \text{if } n=0 \\ n^2K_{n-1}(0,\beta+1) & \text{if } n \geq 1, \end{cases}$$

where

$$K_n(0,\beta+1) = \langle (1+x)_+^{\beta+1} H(1-x), (P_n^{(0,\beta+1)}(x))^2 \rangle$$
$$= \frac{2^{2n+\beta+2}\,(n!\Gamma(n+\beta+2))^2}{\Gamma(2n+\beta+3)\Gamma(2n+\beta+2)}, \quad n \in \mathbf{N}_0.$$

Remark 8. For any real number a, consider the function

$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

There always exists a distribution x_+^a on $\mathbf{R}$ which coincides with $f_a(x)$ on $\mathbf{R} \setminus \{0\}$. If $a > -1$, then $f_a(x)$ is locally integrable on $\mathbf{R}$ so that $x_+^a = f_a(x)$ and, if otherwise, x_+^a is the regularization of $f_a(x)$. For more details on the distribution x_+^a, we refer to Hörmander [**12**, 3.3.2].

Example 4.5 (*Jacobi*: $-\alpha \in \mathbf{R}\backslash\mathbf{N}$, $\beta = -1$). By switching the roles of α and β in the previous example, it is easy to see that the Jacobi polynomials $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$ forms an STPS. More specifically, we have the following result:

Proposition 4.4. *The Jacobi polynomials* $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$ *are orthogonal with respect to the bilinear form* $\phi_B^\alpha(\cdot,\cdot)$, *defined by*

$$\phi_B^\alpha(p,q) = Bp(-1)q(-1) + \langle (1-x)_+^{\alpha+1} H(1+x), p'(x)q'(x) \rangle \quad p,q \in \mathcal{P}. \tag{4.17}$$

This bilinear form is

(i) *quasi-definite if and only if* $B \neq 0$ *and* $-\alpha \notin \mathbf{N}$,

(ii) *positive-definite if and only if* $B > 0$ *and* $\alpha > -2$, $\alpha \neq -1$.

Furthermore,

$$\phi_B^\alpha(P_n^{(\alpha,-1)}, P_n^{(\alpha,-1)}) = \begin{cases} B & \text{if } n = 0 \\ n^2 K_{n-1}(\alpha+1,0) & \text{if } n \geq 1, \end{cases}$$

where

$$K_n(\alpha+1,0) = \langle (1-x)_+^{\alpha+1} H(1+x), (P_n^{(\alpha+1,0)}(x))^2\rangle$$
$$= \frac{2^{2n+\alpha+2}(n!\Gamma(n+\alpha+2))^2}{\Gamma(2n+\alpha+3)\Gamma(2n+\alpha+2)}, \quad n \in \mathbf{N}_0.$$

Example 4.6 (*Twisted Jacobi*: $\alpha = -1, \beta = -1$). Consider the twisted Jacobi equation (4.3) when $\alpha = \beta = -1$:

$$(1+x^2)y'' = n(n-1)y. \tag{4.18}$$

From Lemma 2.2, this equation is not admissible; however, there is a unique (real) monic polynomial solution $\check{P}_n^{(-1,-1)}(x)$ for each $n \neq 1$ given by

$$\check{P}_n^{(-1,-1)}(x) = \binom{2n-2}{n}^{-1} \sum_{k=0}^{n} \binom{n-1}{k} \cdot \binom{n-1}{n-k} (x-i)^{n-k}(x+i)^k.$$

For $n = 1$, any polynomial $\check{P}_1^{(-1,-1)}(x) = x + \gamma$, where γ is an arbitrary real number, is a solution to (4.18). From Theorem 1.4, the PS $\{\check{P}_n^{(-1,-1)}\}_{n=0}^{\infty}$ cannot be a TPS but it does form an STPS for any choice of $\gamma \in \mathbf{R}$. Indeed, the Sobolev weight equations (3.7) and (3.8) associated with equation (4.18) are, respectively,

$$((1+x^2)\sigma)' = 0, \tag{4.19}$$

and

$$(1+x^2)\tau' = 0. \tag{4.20}$$

Equation (4.19) has two linearly independent solutions (see Remark 9 below):

$$\sigma_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \delta^{(2n)}(x)$$

and

$$\sigma_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \delta^{(2n+1)}(x).$$

The reader can check that neither of these moment functionals is quasi-definite. Equation (4.20) has only one linearly independent solution τ, given by

$$\tau = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \delta^{(2n)}(x);$$

this moment functional is quasi-definite. Hence, as before, we consider the two-parameter family of bilinear forms

$$\check{\phi}_{A,B}(p,q) = A\langle \sigma_1, pq\rangle + B\langle \sigma_2, pq\rangle + \langle \tau, p'q'\rangle, \quad p, q \in \mathcal{P}. \tag{4.21}$$

The proof of the following proposition is similar to that of the proofs of Propositions 4.1 and 4.2.

Proposition 4.5. *The bilinear form* $\check{\phi}_{A,B}(\cdot,\cdot)$, *defined in* (4.21), *is quasi-definite* (*but never positive-definite*) *if and only if* $A \neq 0$ *and* $A - A^2 - B^2 \neq 0$. *In this case, the monic* STPS *relative to* $\check{\phi}_{A,B}(\cdot,\cdot)$ *is* $\{\check{P}_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$, *where* $\check{P}_1^{(-1,-1)}(x) = x + \gamma$ *and* $\gamma = B/A$. *Furthermore,*

$$\check{\phi}_{A,B}(\check{P}_n^{(-1,-1)}, \check{P}_n^{(-1,-1)}) = \begin{cases} A & \text{if } n = 0 \\ A(\gamma^2 - 1) + 2B\gamma + 1 & \text{if } n = 1 \\ n^2 \check{K}_{n-1}(0,0) & \text{if } n \geq 2, \end{cases}$$

where

$$\check{K}_n(0,0) = \langle \tau, (\check{P}_n^{(0,0)}(x))^2 \rangle = \frac{(-4)^n (n!)^4}{(2n)!(2n+1)!}, \quad n \in \mathbf{N}_0.$$

Conversely, with any choice of γ, $\{\check{P}_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$ *is an* STPS *relative to* $\check{\phi}_{A,B}(\cdot,\cdot)$ *if* A *and* B *are such that*

$$A \neq 0, \quad A(\gamma^2 - 1) + 2B\gamma + 1 \neq 0, \quad \text{and} \quad A\gamma + B = 0.$$

Remark 9. For any moment functional σ with moments $\{\sigma_n\}_{n=0}^{\infty}$, the formal δ-series expansion of σ is

$$\sigma \approx \sum_{n=0}^{\infty} \frac{(-1)^n \sigma_n}{n!} \delta^{(n)}(x).$$

This was first introduced by Morton and Krall in [**27**].

5. A Proof of Theorem 1.5. In this section, we shall show that the only polynomials that satisfy a second-order differential equation of the form (2.8) and are orthogonal with respect to a Sobolev bilinear form of the type (3.5) are the examples listed in Section 4; this will establish a proof of Theorem 1.5. In order to carry out this classification, we shall consider three cases:

(i) Type A: σ is quasi-definite;

(ii) Type B: σ is not quasi-definite and τ is quasi-definite;

(iii) Type C: both σ and τ are not quasi-definite.

Type A. *σ is quasi-definite.* In this case, we apply Theorem 3.5 to conclude that the only such Sobolev orthogonal polynomials are those listed in Example 4.1 (or those listed in Theorem 3.5).

Type B. *σ is not quasi-definite* and τ is *quasi-definite.*

Case B.1. $\deg(a_2) = 0$. In this case, we see from Theorem 3.7 (i) and (iv) that $\sigma = 0$ and the PS orthogonal with respect to

$$\phi(p, q) = \langle \tau, p'q' \rangle, \quad p, q \in \mathcal{P},$$

is necessarily one of the real classical orthogonal polynomial sequences listed in Example 4.1.

Case B.2. $\deg(a_2) \geq 1$. Through a real linear change of variables, we may assume that $a_2(x)$ is one of the following: (a) $1 - x^2$, (b) $1 + x^2$, (c) x^2, or (d) x.

Case B.2(a). $a_2(x) = 1 - x^2$. In this case, equation (1.2) has the form

$$(1 - x^2)y'' + (Ax + B)y' = n(A - n + 1)y,$$

where A and B are arbitrary constants. Without loss of any generality, we let $A = -(\alpha + \beta + 2)$ and $B = \beta - \alpha$, where α and β are arbitrary real constants. In this case, the above equation becomes

$$(5.1) \quad (1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' = -n(n + \alpha + \beta + 1)y;$$

that is to say, the Jacobi equation given in equation (4.1). The Sobolev weight equations (3.7) and (3.8) become, respectively,

$$(5.2) \qquad ((1 - x^2)\sigma)' - (\beta - \alpha - (\alpha + \beta + 2)x)\sigma = 0,$$

and

$$(5.3) \qquad (1 - x^2)\tau' - (\beta - \alpha - (\alpha + \beta + 4)x)\tau = 0.$$

It is easy to check that equation (5.2) has no quasi-definite moment functional solution if and only if $-(\alpha + \beta + 1)$ or $-\alpha$ or $-\beta$ is a positive integer. Similarly, equation (5.3) has a quasi-definite moment functional solution if and only if none of $-(\alpha + \beta + 3)$, $-(\alpha + 1)$, and $-(\beta + 1)$ are positive integers. Hence, the four following distinct cases arise for the real parameters α and β:

(i) $\alpha + \beta + 1 = -1$, $-\alpha \notin \{2, 3, 4, \dots\}$ and $-\beta \notin \{2, 3, 4, \dots\}$,

(ii) $\alpha + \beta + 1 = -2$, $-\alpha \notin \{2, 3, 4, \dots\}$ and $-\beta \notin \{2, 3, 4, \dots\}$,

(iii) $\alpha = -1$ and $-\beta \notin \{2, 3, 4, \dots\}$,

(iv) $\beta = -1$ and $-\alpha \notin \{2, 3, 4, \dots\}$.

Case B.2a(i). $\alpha + \beta + 1 = -1$, $-\alpha \notin \{2, 3, 4, \dots\}$ *and* $-\beta \notin \{2, 3, 4, \dots\}$. In this case, we see that $a_1(x) = \beta - \alpha$. In general, equation (2.8) has no polynomial solution of degree 1 if $\deg(a_1) = 0$. Consequently, we must have $a_1(x) \equiv 0$ so that $\beta = \alpha = -1$. This leads to the Jacobi PS $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$ discussed in Example 4.3.

Case B.2a(ii). $\alpha + \beta + 1 = -2$, $-\alpha \notin \{2, 3, 4, \dots\}$ *and* $-\beta \notin \{2, 3, 4, \dots\}$. In this case, it is easy to check that equation (5.1) has no polynomial solution of degree 2; hence, there is no STPS in this case.

Case B.2a(iii). $\alpha = -1$ *and* $-\beta \notin \{2, 3, 4, \dots\}$. If $-\beta = 1$, then this case reduces to Case B.2a(i). Hence we assume that $-\beta \notin \mathbf{N}$. In this case, we obtain the Jacobi polynomials $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$ discussed in Example 4.4.

Case B.2a(iv). $\beta = -1$ *and* $-\alpha \notin \{2, 3, 4, \dots\}$. Similar to the previous case, this leads to the Jacobi PS discussed in Example 4.5.

Case B.2(b). $a_2(x) = 1 + x^2$. When $a_2(x) = 1 + x^2$, equation (1.2) has the form

$$(1 + x^2)y'' + (bx + c)y' = n(n + b - 1)y; \tag{5.4}$$

moreover, the Sobolev weight equations (3.7) and (3.8) become, respectively,

$$((1 + x^2)\sigma)' - (bx + c)\sigma = 0, \tag{5.5}$$

and

$$(1 + x^2)\tau' - (bx + c)\tau = 0. \tag{5.6}$$

Equation (5.5) has no quasi-definite moment functional if and only if $-b \in \mathbf{N}_0$, whereas equation (5.6) has a quasi-definite moment functional solution if and only if $-b \notin \{2, 3, 4, \dots\}$. Hence, there arise two cases: (i) $b = 0$ and (ii) $b = -1$. However, if $b = -1$, then equation (5.4) has no polynomial solution of degree two. When $b = 0$, it must be the case that $c = 0$. Indeed, as remarked earlier, equation (2.8) cannot have a polynomial solution of degree 1 when $\deg(a_1(x)) = 0$. Consequently, it follows that we obtain the twisted Jacobi case discussed in Example 4.6.

Case B.2(c). $a_2(x) = x^2$. In this case, equation (2.8) becomes

$$x^2y'' + (ax + b)y' = n(n + a - 1)y, \quad a, b \in \mathbf{R}. \tag{5.7}$$

The two Sobolev weight equations associated with (5.7) are

$$(x^2\sigma)' - (ax + b)\sigma = 0 \tag{5.8}$$

and

$$x^2\tau' - (ax+b)\tau = 0. \tag{5.9}$$

Equation (5.8) has no quasi-definite moment functional if and only if $-a \in \mathbf{N}_0$ or $b = 0$, whereas equation (5.9) has a quasi-definite moment functional solution if and only if $-(a+2) \notin \mathbf{N}_0$ and $b \neq 0$. It follows that there are two cases to consider: (i) $a = 0$, $b \neq 0$, and (ii) $a = -1$, $b \neq 0$. However, in both cases, equation (5.7) has either no polynomial solution of degree one or of degree two. Consequently, this case does not yield any STPS's.

Case B.2(d). $a_2(x) = x$. In this case, we may assume that equation (2.1) has the form

$$xy'' + (1+\alpha - x)y' = -ny, \quad \alpha \in \mathbf{R}. \tag{5.10}$$

The two Sobolev weight equations are

$$(x\sigma)' - (1+\alpha-x)\sigma = 0, \tag{5.11}$$

and

$$x\tau' - (1+\alpha-x)\tau = 0. \tag{5.12}$$

Equation (5.11) has no quasi-definite moment functional solution if and only if $-\alpha \in \mathbf{N}$, whereas equation (5.12) has a quasi-definite moment functional solution if and only if $-\alpha \notin \{2,3,4\dots\}$. Hence, in this case, $\alpha = -1$ and we obtain the Laguerre polynomials discussed in Example 4.2.

Type C. *Both* σ *and* τ *are not quasi-definite.* We remind the reader of Example 3.1. It is possible that the bilinear form $\phi(\cdot,\cdot)$ in (3.5) is quasi-definite and yet neither moment functional σ nor τ is quasi-definite. However, in this case, we prove:

Theorem 5.1. *Suppose* $\{P_n(x)\}_{n=0}^{\infty}$ *is an* STPS *relative to the quasi-definite bilinear form* $\phi(\cdot,\cdot)$, *defined in* (3.5). *If both* σ *and* τ *are not* *quasi-definite, then* $\{P_n(x)\}_{n=0}^{\infty}$ *cannot satisfy a differential equation of the form* (2.8).

Proof. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ does satisfy a differential equation of the type (2.8). By Theorem 3.3, the moment functionals σ and τ satisfy the Sobolev moment equations (3.7) and (3.8). From Lemma 3.2, σ is a canonical moment functional for $\{P_n(x)\}_{n=0}^{\infty}$. However, since σ is not quasi-definite, $\{P_n(x)\}_{n=0}^{\infty}$ cannot be a TPS (see Corollary 2.6). On the other hand, $\{P_n'(x)\}_{n=0}^{\infty}$ is a PS satisfying equation (3.16) and has τ as a canonical moment functional. Hence, by Corollary 2.6, the PS $\{P_n'(x)\}_{n=0}^{\infty}$ cannot be a TPS with respect to τ. Therefore, by Theorem 2.12, we have

$$\langle \sigma, P_n^2 \rangle = \langle \tau, (P_n')^2 \rangle = 0$$

for all sufficiently large n. Hence, there exists $N \in \mathbf{N}$ such that for $n \geq N$,

$$\phi(P_n, P_n) = \langle \sigma, P_n^2 \rangle + \langle \tau, (P_n')^2 \rangle = 0,$$

which contradicts our assumption that $\{P_n(x)\}_{n=0}^{\infty}$ is an STPS relative to $\phi(\cdot,\cdot)$. □

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