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APPROXIMATIONS OF UPPER SEMICONTINUOUS MAPS ON PARACOMPACT SPACES

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ABSTRACT. We prove theorems on graphic approximations of upper semi-continuous mappings which are natural analogues of Michael's selection theorems for lower semi-continuous mappings. Our convex-valued approximation theorem gives a generalization of Cellina's theorem in the sense that we omit the metrizability hypothesis. We also introduce a weakening of upper semi-continuity, the so-called quasi upper semi-continuity, and we show that approximation theorems are also valid for the class of quasi upper semi-continuous mappings. We obtain a finite-dimensional version of Kakutani's fixed-point theorem as a corollary of our finite-dimensional approximation theorem.

1. Introduction. In the theory of continuous selections of multi-valued lower semi-continuous maps, the key results are the following four theorems of E. Michael: the *convex-valued*, the 0-*dimensional*, the *compact-valued* and the *finite-dimensional* selection theorem. Recall that a *selection* of a multi-valued map $F : X \to Y$ is a (multi-valued) map $G : X \to Y$ such that, for every $x \in X$, $G(x) \subset F(x)$. The four theorems are summarized in Table 1.

In general, continuous selections do *not* exist for *upper* semi-continuous maps. Nevertheless, it makes sense to ask in this case about the existence of *approximations* of the given upper semi-continuous map F by a map whose graph is "close" to the graph of the map F. The following is known to be true [**1**–**4**], [**12**]:

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Theorem 1.1. *Let $F : X \to Y$ be an upper semi-continuous map of a metric space (X, ρ) into a normed space $(Y, \| \ \|)$ with convex values. Then, for every $\varepsilon > 0$, there exists a continuous single-valued map $f : X \to Y$ such that, for every point $p = (x, y)$ of the graph Γ_f of the map f, there exists a point $q = (x', y')$ of the graph Γ_F of the map F such that $\rho(x, x') < \varepsilon$ and $\|y - y'\| < \varepsilon$.*

TABLE 1.

<table>
<tr><th>Type of the selection theorem</th><th>Hypotheses on X</th><th>Hypotheses on Y</th><th>Hypotheses on $F(x)$</th><th>Conclusions concerning the existence of selections</th></tr>
<tr><td>convex-valued theorem</td><td>paracompact space</td><td>Banach space</td><td>closed convex</td><td rowspan="2">single-valued continuous selection</td></tr>
<tr><td>0-dimensional theorem</td><td>0-dimensional paracompact space</td><td rowspan="3">completely metrizable space</td><td rowspan="2">closed</td></tr>
<tr><td>compact-valued theorem</td><td>paracompact space</td><td>compact-valued semi-continuous selections</td></tr>
<tr><td>finite-dimensional theorem</td><td>$(n+1)$-dimensional paracompact space</td><td>closed n-connected, $\{F(x)\}$ equi-n-connected</td><td>single-valued continuous selection</td></tr>
</table>

This theorem is an analogue of the Michael *convex-valued* selection theorem, although not completely since the hypothesis on X is *stronger*. It turns out that Theorem 1.1 can be generalized to the case when X is *paracompact* and Y is a (nonmetrizable) *topological vector space*. Moreover, the same kind of analogous theorems also exist for other selection theorems mentioned above. In order to state them, we must first introduce a generalized concept of an ε-approximation of a multivalued map.

Definition 1.2. Let $F : X \to Y$ be a multi-valued map between topological spaces X and Y, the values $F(x)$ are nonempty, $\Gamma_F \subset X \times Y$

the graph of the map F and α some open cover of Γ_F in $X \times Y$. A multi-valued map $G : X \to Y$ is said to be an *α-approximation* of F if for every point $p \in \Gamma_G$, there exists a point $q \in \Gamma_F$ such that p and q lie in some common element of the cover α.

If X and Y are metric spaces and the cover α consists of the Cartesian products of $\varepsilon/2$-balls in X and Y, then the α-approximation in the sense of Definition 1.2 above is the usual ε-approximation. Hereafter, open coverings of the space $X \times Y$ which consist of the Cartesian products of the elements of the cover Ω of X and the cover Λ of Y, will be denoted by $\Omega \times \Lambda$. In the case when Y is a topological vector space and V is an open neighborhood of the origin $O \in Y$, we shall denote the cover $\Omega \times \{y + V\}_{y \in Y}$, by $\Omega \times V$.

Theorem 1.3. *Let $F : X \to Y$ be an upper semi-continuous convex-valued map of a paracompact space X into a topological vector space Y. Then, for every open cover Ω of X and every convex open neighborhood $V \subset Y$ of $O \in Y$, there exists a continuous single-valued $(\Omega \times V)$-approximation of the map F.*

Analogs of zero-dimensional and compact-valued selection theorems are true for multi-valued mappings without continuity-type restrictions.

Theorem 1.4. *Let $F : X \to Y$ be a mapping of a 0-dimensional paracompact space X into a topological space Y. Then, for every cover Ω of X, there exists a continuous single-valued mapping $f : X \to Y$ such that the graph of f is a subset of the union $\cup\{U \times F(U) \mid U \in \Omega\}$.*

Theorem 1.5. *Let $f : X \to Y$ be a mapping of a paracompact space X into a topological space Y. Then, for every cover Ω of X there exists a compact-valued upper semi-continuous mapping $G : X \to Y$ and a compact-valued lower semi-continuous selection H of G such that the graphs of G and H are subsets of the union $\cup\{U \times F(U) \mid U \in \Omega\}$.*

We shall prove the analogue of the finite-dimensional selection theorem with some additional assumptions on the mapping F, namely, the $*$-paracompactness of F.

Theorem 1.6. *Let $F : X \to Y$ be an upper semi-continuous map of an $(n+1)$-dimensional paracompact space X into a topological space Y, and suppose that all values $F(x)$, $x \in X$, are UV^n subsets of Y. Let F be a $*$-paracompact mapping. Then, for every cover Ω of X and for every cover Λ of Y, there exists a continuous single-valued $(\Omega \times \Lambda)$-approximation of the map F.*

We recall, see [**11**], that A is said to be a UV^n-subset of Y if, for every open $U \supset A$, there exists an open V such that $U \supset V \supset A$ and every continuous mapping $g : S^k \to V$ can be extended to a continuous mapping $\hat{g} : B^{k+1} \to U$. Here B^{k+1} denotes the $(k+1)$-dimensional closed ball in $\mathbf{R}^{n+1}$ and S^k denotes its boundary, $k \leq n$. (A is a PC^n-subset in Y in terminology of [**2**].)

As usual, we denote $F_{-1}(Z) = \{x \in X \mid F(x) \subset Z\}$, and the upper semi-continuity of F means that $F_{-1}(U)$ is open for every open U.

Definition 1.7. Let $F : X \to Y$ be an upper semi-continuous multivalued mapping. Then

(a) A family $\Lambda = \{L_\gamma\}_{\gamma \in \Gamma}$ of open subsets of Y is said to be an *F-covering* if the sets $F_{-1}(L_\gamma)$ are nonempty for all $\gamma \in \Gamma$ and $F_{-1}(\Lambda) = \{F_{-1}(L_\gamma)\}_{\gamma \in \Gamma}$ is a covering of X;

(b) F is said to be *$*$-paracompact* if, for every F-covering Λ and for every star-refinement Ω of the covering $F_{-1}(\Lambda)$ there exists an F-covering $\hat{\Lambda}$ such that Ω is a refinement of $F_{-1}(\hat{\Lambda})$ and $F_{-1}(\hat{\Lambda})$ is a star-refinement of $F_{-1}(\Lambda)$.

A simple example of a $*$-paracompact mapping is an open upper semi-continuous mapping F, i.e., a mapping with the property that the image of every open subset of X is an open subset of Y. In fact, one can then put $\hat{\Lambda} = F(\Omega)$ in the definition 1.7 (b). As a special case, one can consider the quotient mapping of a continuous decomposition into UV^n subsets. Another example is provided by any upper semi-continuous mapping theorem between compact metric spaces X and Y compact values.

Some remarks concerning the proofs of these theorems. The proof of Theorem 1.3 is similar to that of Theorem 1.1; the only difference is due to the fact that one must substitute the triangle inequality by the star-refinement of the necessary locally finite covers. Similar substitution was made, for example, in [**17**]. But here we in fact prove Theorem 1.3 for the class of quasi upper semi-continuous mappings, see Definition 1.9 and Theorem 1.10 below. Theorem 1.5 follows from Theorem 1.4 by an application of our earlier theorem [**18**].

Theorem 1.8. *For every paracompact space X there exists a 0-dimensional paracompact space Z and a perfect inductively open map $m : Z \to X$ of Z onto X.*

Here the perfectness of the map m implies the compactness of the values and the upper semi-continuity of its inverse $m^{-1} : X \to Z$, whereas the inductive openness of m is equivalent to the existence of a lower semi-continuous compact-valued selection of the inverse map $m^{-1} : X \to Z$. In fact, the surjection $m : Z \to X$ is also a Milyutin map, and this fact was used in [**18**] for a proof that the convex-valued selection theorem follows from the 0-dimensional selection theorem. Nevertheless, in the case of the upper semi-continuous maps which we have, one cannot derive Theorem 1.3 from Theorem 1.4 in such a way, because the Milyutin property uses essentially the integration of vector-valued functions.

We prove Theorem 1.6 by induction on skeletons of the nerve $\mathcal{N}$ of some suitable covering of the domain X. So, a desired approximation f is constructed as the composition of a canonical mapping from X into the nerve $\mathcal{N}$ and a mapping from $\mathcal{N}$ into Y. In [**5, 2, 6**], such an approximation f was obtained in the case $n = \infty$ via a technique of domination of X by finite polyhedra.

Finally, we introduce the notion of *quasi upper semi-continuous* mappings which extends the notion of upper semi-continuity.

Definition 1.9. Let $F : X \to Y$ be a multi-valued mapping from a topological space X into a metric space (Y, ρ), respectively into a topological vector space Y. We say that F is *quasi upper semi-continuous*, q.u.s.c., at a point $x \in X$ if, for each of its neighborhoods $W(x)$ and for each $\varepsilon > 0$, respectively for each convex neighborhood V of the origin $O \in Y$, there exists a point $q(x) \in W(x)$ such that $x \in \operatorname{Int} F_{-1}(B(F(q(x)), \varepsilon))$, respectively $x \in \operatorname{Int} F_{-1}(F(q(x)) + V)$. F

is said to be a *quasi upper semi-continuous mapping* if it is quasi upper semi-continuous at each point of its domain.

As usual, we denote the open ε-neighborhood of the set $F(q(x))$ in the metric space (Y, ρ) in this definition by $B(F(q(x)), \varepsilon)$. Clearly each upper semi-continuous mapping F is a quasi upper semi-continuous mapping. It suffices to put $q(x) = x$. The converse is false. Indeed, let A be a dense subset of X with $X \backslash A \neq \varnothing$, and let, for a fixed $y_0 \in Y$,

$$F(x) = \begin{cases} \{y_0\} & x \in X \backslash A, \\ Y & x \in A. \end{cases}$$

Then F is upper semi-continuous at points of A and F is quasi upper semi-continuous (and non upper semi-continuous) at points of $X \backslash A$.

Theorem 1.10. *Theorems* 1.3 *and* 1.6 *also hold if, instead of upper semicontinuity of the mapping F, one assumes the quasi upper semicontinuity of F.*

Theorem 1.11. *If $n = 0$, i.e., if X is one-dimensional paracompactum, then Theorem* 1.6 *is true without the assumption of $*$-paracompactness of F.*

2. Proofs. For the proofs of main results, we shall need the following properties of regular spaces [**10**, pp. 156, 171].

Proposition 2.1. *Let X be a regular space. Then the following statements are equivalent.*

a) *X is paracompact*;

b) *Every open cover Ω of X is unique, i.e., there exists in the diagonal of $X \times X$ an open neighborhood Δ such that the covering of X by the sets $\Delta(x) = \{y \mid (x, y) \in \Delta\}$ is finer than Ω, and*

c) *For every open cover Ω of X there exists an open star-refinement $\mathcal{V}$.*

Proof of Theorem 1.10. We first prove a generalization of Theorem 1.3. So let $F : X \to Y$ be a quasi upper semi-continuous convex-valued mapping from a paracompact space X into a topological vector space Y. Let Ω be an open covering of X, and let V be a convex neighborhood of the origin $O \in Y$. For each $x \in X$, fix an arbitrary element $W(x) \in \Omega$ such that $x \in W(x)$.

(1) Using the q.u.s.c. of F, find for each $x \in X$ a point $q(x) \in W(x)$ and a neighborhood $U(x) \subset W(x)$ such that $F(z) \subset F(q(x)) + V$ for all $z \in U(x)$.

(2) Find a unique covering $\{G(x)\}_{x\in X}$ which is a star refinement of the covering $\{U(x)\}_{x\in X}$ of X.

(3) Using q.u.s.c. of F once more, find for each $x \in X$ a point $q'(x) \in G(x)$ and a neighborhood $U'(x) \subset G(x)$ such that $F(z) \subset F(q'(x)) + V$ for all $z \in U'(x)$.

(4) Let $\{e_\alpha\}_{\alpha\in A}$ be a locally finite continuous partition of unity inscribed into the covering $\{U'(x)\}_{x\in X}$ of X. For each $\alpha \in A$ we can choose $x_\alpha \in X$ such that $\operatorname{supp} e_\alpha \subset U'(x_\alpha)$, and we fix $y_\alpha \in F(q'(x_\alpha))$, and

(5) Finally, put $f(x) = \sum e_\alpha(x) y_\alpha$ where the sum is taken over all $\alpha \in A$ with $e_\alpha(x) > 0$.

Let us check that f is the desired $(\Omega \times V)$-approximation of F. For a fixed $x_0 \in X$ we have that

$$\begin{aligned} x_0 \in \operatorname{St}\{x_0, \{\operatorname{supp} e_\alpha\}_{\alpha\in A}\} &\subset \operatorname{St}\{x_0, \{U'(x)\}_{x\in X}\} \\ &\subset \operatorname{St}\{x_0, \{G(x)\}_{x\in X}\} \subset U(x') \subset W(x') \end{aligned}$$

for some $x' \in X$. According to the Definition 1.9 of quasi upper semicontinuity, we have that $q(x') \in W(x')$. Hence the points x_0 and $q(x')$ are Ω-close.

Analogously, if $e_\alpha(x_0) > 0$, then $x_0 \in G(x_\alpha)$ and $q'(x_\alpha) \in G(x_\alpha)$, see (3) above. Hence, $q'(x_\alpha) \in \operatorname{St}\{x_0, \{G(x)\}_{x\in X}\} \subset U(x')$. Therefore,

$$y_\alpha \in F(q'(x_\alpha)) \subset F(q(x')) + V$$

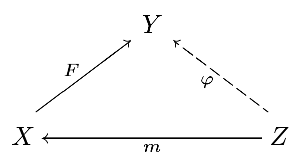
i.e., $y_\alpha - v_\alpha \in V$ for some $v_\alpha \in F(q(x'))$. But then, for $v = \sum e_\alpha(x_0) v_\alpha \in F(q(x'))$, we have that

$$f(x_0) - v = \sum e_\alpha(x_0)(y_\alpha - v_\alpha) \in V.$$

Hence, the point $(x_0, f(x_0)) \in \Gamma_f$ is $(\Omega \times V)$-close to the point $(q(x'), v) \in \Gamma_F$. $\square$

Proof of Theorem 1.4. Let $\{V_\beta\}_{\beta\in B}$ be a disjoint open cover of X which refines the cover Ω. Choose $y_\beta \in F(V_\beta)$ for each $\beta \in B$ and define $f : X \to Y$ by $f|_{V_\beta} = y_\beta$. $\square$

Proof of Theorem 1.5. Given a paracompact space X, let $m : Z \to X$ be a perfect, inductively open map of a 0-dimensional paracompact space Z onto X. The existence of m and Z is guaranteed by Theorem 1.8. Apply Theorem 1.4 for the map $F \circ m : Z \to Y$, the open cover $m^{-1}(\Omega)$ of Z.



Let $\varphi : Z \to Y$ be a single-valued continuous mapping with the graph Γ_φ being a subset of the union $\cup\{V \times (F \circ m)(V) \mid V = m^{-1}(U)$ for some $U \in \Omega\}$. Set $G = \varphi \circ m^{-1}$ and $H = \varphi \circ S$, where S is a compact-valued lower semi-continuous selection of the upper semi-continuous map $m^{-1} : X \to Z$.

Let us verify that the graph Γ_G of G is a subset of the union $\cup\{U \times F(U) \mid U \in \Omega\}$. In fact, we have $(x, y) \in \Gamma_G \Rightarrow y \in G(x) \Rightarrow y \in \varphi(m^{-1}(\alpha)) \Rightarrow (z, y) \in \Gamma_\varphi$ for some $z \in m^{-1} \subset Z \Rightarrow z \in m^{-1}(U)$ and $y \in (F \circ m)(m^{-1}(U))$ for some $U \in \Omega \Rightarrow x = m(z) \in m(m^{-1}(U)) = U$ and $y \in F(U) \Rightarrow (x, y) \in U \times F(U)$. The compactness of values of G and H and their semi-continuity is obvious. $\square$

Second proof of Theorem 1.5 (E. Michael). Let $\{V_\beta\}_{\beta\in B}$ be a locally finite, open star-refinement of Ω. Choose $y_\beta \in F(V_\beta)$ for each $\beta \in B$ and define H and G by $H(x) = \{y_\beta \mid x \in V_\beta\}$ and $G(x) = \{y_\beta \mid x \in \text{closure}\,(V_\beta)\}$. Note that in this proof H and G are finite-valued mappings. $\square$

Proof of Theorem 1.6. We denote by $\mathcal{A} > \mathcal{B}$ the fact that the open covering $\mathcal{B}$ refines the open covering $\mathcal{A}$. We denote by $\mathcal{A} \overset{*}{>} \mathcal{B}$ the fact that $\mathcal{B}$ is a star-refinement of $\mathcal{A}$. Also, for any two F-coverings $\mathcal{A}$ and $\mathcal{B}$ we denote by $\mathcal{A} \overset{n}{>} \mathcal{B}$ the fact that for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$

such that the inclusion $B \subset A$ is null-homotopic in all dimensions $\leq n$. Note that the assumption $F(x) \in UV^n(Y)$, $x \in X$, implies that for every F-covering $\mathcal{A}$ there exists an F-covering $\mathcal{B}$ such that $\mathcal{A} \overset{n}{>} \mathcal{B}$. We shall now define a sequence

$$\hat{\Lambda}_{n+1} \overset{n}{>} \Lambda_n > \hat{\Lambda}_n \overset{n}{>} \Lambda_{n-1} > \cdots \overset{n}{>} \Lambda_1 > \hat{\Lambda}_0$$

of F-coverings in the space Y and a sequence

$$\mathcal{V}_{n+1} > \mathcal{U}_n \overset{*}{>} \mathcal{V}_n > \mathcal{U}_{n-1} \overset{*}{>} \cdots > \mathcal{U}_1 \overset{*}{>} \mathcal{V}_0 \overset{*}{>} \mathcal{W}$$

of open coverings of the paracompact space X.

We define the F-covering $\hat{\Lambda}_{n+1}$ as $\{\mathrm{St}\,(F(\omega), \Lambda) \mid \omega \in \Omega\}$ and $\mathcal{V}_{n+1}$ as the covering of X consisting of all nonempty intersections of Ω and $F_{-1}(\hat{\Lambda}_{n+1})$. Then we find an arbitrary F-covering Λ_n such that $\hat{\Lambda}_{n+1} \overset{n}{>} \Lambda_n$, and we let $\mathcal{U}_n$ be a covering of X which refines $\mathcal{V}_{n+1}$ and $F_{-1}(\Lambda_n)$ simultaneously.

Due to the paracompactness of X, we can choose a star-refinement Ω_n of the covering $\mathcal{U}_n$ and use $*$-paracompactness of the mapping F to find an F-covering $\hat{\Lambda}_n$ such that Ω_n refines $F_{-1}(\hat{\Lambda}_n)$ and $F_{-1}(\hat{\Lambda}_n) = \mathcal{V}_n$ is a star-refinement of $\mathcal{U}_n$. Next, we repeat this procedure, starting from $\hat{\Lambda}_n$. At the end of this procedure we find an open covering $\mathcal{W}$ of X which is a locally finite star-refinement of $\mathcal{V}_0$ of order $n+2$. Let $\mathcal{W} = \{W_\alpha\}_{\alpha \in A}$ and $\mathcal{N} = \mathcal{N}(\mathcal{W})$ be the nerve of the covering $\mathcal{W}$. We define an $(\Omega \times \Lambda)$-approximation $f : X \to Y$ of the mapping F as the composition $g \circ p$ of the canonical mapping $p : X \to \mathcal{N}(\mathcal{W})$ and some suitable mapping $g : \mathcal{N}(\mathcal{W}) \to Y$. Let $\mathcal{N}^i = \mathcal{N}^i(\mathcal{W})$ be the i-skeleton of $\mathcal{N}$. By induction on $i \in \{0, 1, \ldots, n+1\}$, we shall define the mappings $g_i : N^i \to Y$ such that g_{i+1} is an extension of g_i and such that, for every i-dimensional simplex $\Delta \in \mathcal{N}^i$ with vertices $W_{\alpha_0}, W_{\alpha_1}, \ldots, W_{\alpha_i}$, there exists an element $\hat{L}^i_\Delta \in \hat{\Lambda}_{i+1}$ such that

(a_i) $\cup_{j=0}^{i} W_{\alpha_j} \subset F_{-1}(\hat{L}^i_\Delta)$ and

(b_i) $g^i(\Delta) \subset \hat{L}^i_\Delta$.

To begin the inductive proof, let $i = 0$. Here $\mathcal{N}^0 = A$ and, for every $\alpha \in A$ we simply define $g^0(\alpha)$ to be any element of $F(W_\alpha)$. We have $\mathcal{V}_0 > \mathcal{W}$. Thus, $W_\alpha \subset V^0_\alpha$ for some $V^0_\alpha \in \mathcal{V}_0$. By construction, $V^0_\alpha = F_{-1}(\hat{L}^0_\alpha)$, for some $\hat{L}^0_\alpha \in \hat{\Lambda}_0$. Hence, we obtain that

(a_0) $W_\alpha \subset V^0_\alpha = F_{-1}(\hat{L}^0_\alpha)$ and

(b_0) $g^0(\alpha) \in F(W_\alpha) \subset F(V^0_\alpha) \subset \hat{L}^0_\alpha$.

Suppose now inductively that the mappings $g^0, g^1, \dots, g^i$ have already been constructed with properties $(a_0), \dots, (a_i)$, $(b_0), \dots, (b_i)$. We now define a mapping $g^{i+1} : \mathcal{N}^{i+1} \to Y$ over every $(i+1)$-dimensional simplex Δ of the nerve $\mathcal{N}$. More precisely, we define g^{i+1}_Δ as an extension of the mapping $g^i|_{\partial\Delta}$, where $\partial\Delta$ is the boundary of Δ, and we set $g^{i+1}|_\Delta = g^{i+1}_\Delta$, for every $(i+1)$-simplex Δ.

Let $W_{\alpha_0}, \dots, W_{\alpha_{i+1}}$ be all the vertices of Δ, and let ∇_j be the face of Δ with vertices $\{W_{\alpha_0}, \dots, W_{\alpha_{i+1}}\} \backslash \{W_{\alpha_j}\}$. Applying (a_i) to each ∇_j, $j \in \{0, 1, \dots, i+1\}$, we conclude that

$$\varnothing \neq \bigcap_{k=0}^{i+1} W_{\alpha_k} \subset \bigcup_{\substack{k=0 \\ k\neq j}} W_{\alpha_k} \subset V^j_{\nabla_j}$$

for some $V^i_{\nabla_j} \in \mathcal{V}_i$. Hence, $\cap_{j=0}^{i+1} V^i_{\nabla_j} \neq \varnothing$. Due to the property $\mathcal{U}_{i+1} \overset{*}{>} \mathcal{V}_i$, we can find U^{i+1}_Δ which contains the union $\cup_{j=0}^{i+1} V^i_{\nabla_j}$. Applying (b_i) to each ∇_j, $j \in \{0, 1, \dots, i+1\}$, we conclude that

$$g^i(\partial\Delta) = g^i\left(\bigcup_{j=0}^{i+1} \nabla_j\right) \subset \bigcup_{j=0}^{i+1} F(V^i_{\nabla_j}) \subset F\left(\bigcup_{j=0}^{i+1} V^i_{\nabla_j}\right) \subset F(U^{i+1}_\Delta).$$

By construction, $F(U^{i+1}_\Delta)$ is a subset of an element of the covering Λ_{i+1}. Applying the assumption $\hat{\Lambda}_{i+1} \overset{n}{>} \Lambda_{i+1}$ we can find an extension $g^{i+1}_\Delta : \Delta \to \hat{L}^{i+1}_\Delta$ of the mapping $g^i|_{\partial\Delta}$ for some $\hat{L}^{i+1}_\Delta \in \hat{\Lambda}_{i+1}$, i.e., the property (b_{i+1}) holds. Finally,

$$\bigcup_{j=0}^{i+1} W_{\alpha_j} \subset \bigcup_{j=0}^{i+1} V^i_{\nabla_j} \subset U^{i+1}_\Delta \subset F_{-1}(\hat{L}^{i+1}_\Delta),$$

i.e., (a_{i+1}) holds, too.

Then the $(n+1)$th star of the point x under "W" lies in some element ω of the covering Ω and $f(x) = g^k(p(x)) \subset \hat{L}^k_\Delta \subset \hat{L}^{n+1} = \mathrm{St}\,(F(\omega), \Lambda)$. Hence, for some $x' \in \omega$ and $y' \in F(x')$ we have that $f(x)$ and y' are Λ-close. $\square$

Proof of Theorem 1.11. Here we use UV^0-property in some "centered" sense. Precisely, let $\Lambda = \{\lambda(y)\}_{y\in Y}$ and, for every $x \in X$, let $V(x) = \cup\{\lambda(y) \mid y \in F(x)\}$ be an open neighborhood of $F(x)$. Find an open neighborhood of $F(x), V(x) \supset V_0(x) \supset F(x)$ where inclusion $V(x) \supset V_0(x)$ is null-homotopic in dimension 0. Finally, let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be a locally finite open covering of X of order 1 which star-refines the covering $\{F_{-1}(V_0(x)) \cap W(x)\}_{x\in X}$. (We assume that $\Omega = \{W(x)\}_{x\in X}$.) As in the proof of the previous theorem, we construct a mapping $g : \mathcal{N}(\mathcal{U}) \to Y$ where $\mathcal{N}(U) = \mathcal{N}^{-1}(\mathcal{U})$ is the nerve of the covering $\mathcal{U}$. For $\alpha \in \mathcal{N}^0(\mathcal{U})$ we define $g(\alpha)$ to be an element of $F(U_\alpha)$. For $[\alpha,\beta] \subset \mathcal{N}^1(\mathcal{U})$ we have that $\varnothing \neq U_\alpha \cap U_\beta \subset U_\alpha \cup U_\beta \subset F_{-1}(U_0(x_{\alpha\beta})) \cap W(x_{\alpha\beta})$ for some $x_{\alpha\beta} \in X$. Hence, $\{g(\alpha), g(\beta)\} \subset V_0(x_{\alpha\beta})$, and we can find a path $g_{\alpha\beta} \subset V(x_{\alpha\beta})$ with ends $g(\alpha)$ and $g(\beta)$. So, let $f = g \circ p$ where $p : X \to \mathcal{N}^1(\mathcal{U})$ is the canonical mapping, and let $x \in X$. Then, for some $\alpha, \beta \in A$ we have that $f(x) = g(p(x)) \in g([\alpha,\beta]) = g_{\alpha\beta} \subset V(x_{\alpha\beta}) = \{\lambda(y) \mid y \in F(x_{\alpha\beta})\}$. Hence, $f(x)$ is Λ-close to a point $y \in F(x_{\alpha\beta})$. But we also have that $x \in U_\alpha \cap U_\beta \subset W(x_{\alpha\beta})$, i.e., x is Ω-close to $x_{\alpha\beta}$. □

3. Epilogue. The construction in the proof of Theorem 1.11 does not work in dimension 2 because for the 2-simplex $[\alpha,\beta,\gamma]$ in $\mathcal{N}^2(\mathcal{U})$ the paths $g_{\alpha\beta}$, $g_{\alpha\gamma}$, $g_{\beta\gamma}$ are Λ-close to values which are in general different, $F(x_{\alpha\beta})$, $F(x_{\alpha\gamma})$, $F(x_{\beta\gamma})$ and an extension from dimension 1 is thus not possible.

As a standard application of Theorem 1.6, we get the following finite-dimensional version of the Kakutani fixed-point theorem [**9**], see also [**8**, Theorem 1.2].

Corollary 3.1. *Let X be a compact metric AR with* $\dim X \leq n+1$, *and let F be an upper semi-continuous mapping of X into itself with closed UV^n values. Then there exists $x \in X$ such that $x \in F(x)$.*

The notion of quasi upper semi-continuity is derived from the notion of upper semi-continuity via the analogy with the derivative of quasi lower semi-continuity from the lower semi-continuity of multivalued mappings, see [**7, 16**].

Question 3.2. Is the paracompactness of the domain a necessary assumption in Theorem 1.3? More precisely, let X be a topological space such that each upper semi-continuous mapping $F : X \to Y$ into a topological vector space Y admits continuous single-valued $(\Omega \times V)$-approximations. Is then X always a paracompact space?

Question 3.3. Is it possible to prove a theorem which unifies the convex-valued Theorem 1.3 and the zero-dimensional Theorem 1.4 in the spirit of the union of the convex-valued and the zero-dimensional selection theorems for lower semi-continuous mappings, see [**15**]?

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REFERENCES

1. J.-P. Aubin and A. Cellina, *Differential inclusions*, Grundlehren Math. Wiss. **264**, Springer-Verlag, Berlin, 1984.

2. H. Ben-El-Mechiaekh and P. Deguire, *Approachability of fixed points for non convex set-valued maps*, J. Math. Anal. Appl. **170** (1992), 477–500.

3. Yu. G. Borisovič, B.D. Gel'man, A.D. Myškis and V.V. Obuhovskij, *Introduction into the theory of multivalued mappings*, Izdatel'stvo VGU, Voronež, 1986 (in Russian).

4. A. Cellina, *A theorem on the approximation of compact multivalued mappings*, Accad. Naz. Lincei Atti **8** (1969), 149–153.

5. L. Gorniewicz, A. Granas and W. Kryszewski, *On the homotopy method in the fixed point index theory of multivalued mappings of compact ANR's*, J. Math. Anal. Appl. **161** (1991), 457–473.

6. L. Gorniewicz and M. Lassonde, *Approximation and fixed points for compositions of R_δ-maps*, Topology Appl. **55** (1994), 239–250.

7. V. Gutev, *Unified selection and factorization theorems*, C.R. Acad. Bulgare Sci. **40** (1987), 13–15.

8. ———, *A fixed-point theorem for UV^k upper semicontinuous maps*, Proc. Amer. Math. Soc., to appear.

9. S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–459.

10. J.L. Kelley, *General topology*, Van Nostrand, Princeton, 1957.

11. R.C. Lacher, *Cell-like mappings of ANR's*, Bull. Amer. Math. Soc. **74** (1968), 933–935.

12. A. Lasota and Z. Opial, *An approximation theorem for multivalued mappings*, Podstawy Sterowania **1** (1971), 71–75.

13. E. Michael, *Continuous selections* I, Ann. Math. **63** (1956), 361–382.

14. ———, *Continuous selections* II, Ann. Math. **64** (1956), 375–390.

15. E. Michael and C. Pixley, *A unified theorem on continuous selections*, Pacific J. Math. **87** (1980), 187–188.

16. J. Przeslawski and L. Rybynski, *Michael continuous selection theorem under weak lower semicontinuity assumption*, Proc. Amer. Math. Soc. **109** (1990), 537–543.

17. S. Reich, *Approximate selections, best approximations, fixed points*, J. Math. Anal. Appl. **62** (1978), 104–113.

18. D. Repovš, P.V. Semenov and E.V. Ščepin, *On zero-dimensional Milutin maps and Michael selection theorems*, Topology Appl. **54** (1993), 77–83.

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