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CROSSED PRODUCT DUALITY FOR PARTIAL C^*-AUTOMORPHISMS

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ABSTRACT. For partial automorphisms of C^*-algebras, Takai-Takesaki crossed product duality tends to fail, in proportion to the extent to which the partial automorphism is not an automorphism.

1. Introduction. Recently Exel [**1**] introduced the notion of a crossed product of a C^*-algebra by a partial automorphism (an isomorphism between ideals), in order to better understand circle actions. This generalizes crossed products by automorphisms (equivalently, integer actions), and some of the usual theory of crossed products by actions carries over to this new context [**1, 2, 5, 8**]. It seems natural to ask about the Takai-Takesaki crossed product duality [**9**]. In this paper we show that, perhaps unsurprisingly (since partial automorphisms, being partially defined, miss some of the information of the C^*-algebra), crossed product duality tends to fail for partial automorphisms. Indeed, crossed product duality seems to fail more miserably the more "partial" the partial automorphism is.

To be more precise, from experience with Takai-Takesaki duality for crossed products by abelian groups, we expect a dual action of the circle group $\mathbf{T}$ on a crossed product by a partial action, and indeed Exel [**1**] constructs such a thing. We apologize, but for our purposes we find it more convenient to work with the corresponding coaction of the integer group $\mathbf{Z}$. For abelian locally compact groups, statements about coactions are just Fourier transforms of statements about actions of the dual groups. However, at a certain point, we need a representation of the circle group $\mathbf{T}$, which is more easily dealt with as a representation of $c_0(\mathbf{Z})$. For the reader's convenience, in Section 2, after reviewing the elementary theory of partial automorphisms, we give a rough guide to

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what we need from coactions, specialized to $\mathbf{Z}$; suitable references are, e.g., [**4**] and [**7**].

In Section 3 we prove what are surely expected results concerning invariant ideals, and concerning tensor products with the identity automorphism.

In Section 4 we obtain a kind of "Wold decomposition" of a partial automorphism, showing that there is a largest subalgebra, which turns out to be an ideal, on which we have an actual automorphism, and the quotient partial automorphism is as far as possible from an automorphism ("completely nonautomorphic"). For these completely nonautomorphic partial automorphisms, crossed product duality fails most dramatically, at least under a mild condition that certain projections be multipliers.

In Section 5 we study how the behavior of the partial action α depends upon the distribution of the domains and ranges of the powers α^n. Along the way see a more fundamental reason why crossed product duality fails in general for partial automorphisms.

2. Partial automorphisms and the dual coaction. We begin with a review and embellishment of some of Exel's work [**1**]. A *partial automorphism* of a C^*-algebra is an isomorphism between two (closed, two-sided) ideals. Let $\mathrm{PAut}A$ denote the set of all partial automorphisms of A. Since an ideal of an ideal of A is an ideal of A, $\mathrm{PAut}A$ is closed under composition, where the domain of $\alpha \circ \beta$ is taken to be all elements in the domain of β which β maps into the domain of α. For $\alpha \in \mathrm{PAut}A$ we let α^0 be the identity automorphism of A, and for $n > 0$ we let α^{-n} be the nth power of the inverse α^{-1}. We denote the range of the partial automorphism α^n by D_n (so that the domain is D_{-n}). We have

$$\cdots D_{-2} \subset D_{-1} \subset A = D_0 \supset D_1 \supset D_2 \cdots$$

and

$$\alpha^n(D_k D_{-n}) = D_{n+k} D_n \quad \text{for all } n, k \in \mathbf{Z}. \tag{2.1}$$

We refer to the D_n as the ideals of α and write $D_n(\alpha)$ when there are more than one partial automorphism around. Note that $D_n(\alpha^{-1}) =$

$D_{-n}(\alpha)$. Thus, many properties of the D_n for $n < 0$ follow from the corresponding properties for $n > 0$ by replacing α by α^{-1}; when we want to invoke this rule, we will just say "by symmetry."

The *crossed product* of A by α is the C^*-completion, denoted $A \times_\alpha \mathbf{Z}$, of the *algebraic* direct sum (i.e., finite sums) $\oplus_n D_n$ equipped with *-algebra structure

$$(xy)_n = \sum_{k \in \mathbf{Z}} \alpha^k(\alpha^{-k}(x_k)y_{n-k})$$
$$(x^*)_n = \alpha^n(x^*_{-n}).$$

Definition 2.1. p_n denotes the identity element of the double-dual D_n^{**} when the latter is canonically embedded as a weak* closed ideal of A^{**}.

Thus, the p_n are central projections in A^{**}, and $D_n^{**} = A^{**}p_n$. By (2.1) we have

$$\alpha^n(p_k p_{-n}) = p_{n+k}p_n, \tag{2.2}$$

where α has been canonically extended to a partial automorphism of A^{**}.

A *covariant representation* of (A, α) on a Hilbert space $\mathcal{H}$ is a pair (π, u), where π is a nondegenerate representation of A on $\mathcal{H}$ and $u \in \mathcal{B}(\mathcal{H})$ satisfies

$$uu^* = \pi(p_1) \quad \text{and} \quad u^*u = \pi(p_{-1}); \tag{2.3}$$
$$\mathrm{Ad}u \circ \pi(a) = \pi \circ \alpha(a) \quad \text{for} \quad a \in D_{-1}. \tag{2.4}$$

Thus, u is a partial isometry with range and domain projections $\pi(p_1)$ and $\pi(p_{-1})$, respectively. Similarly to what we did for partial automorphisms, we let $u^0 = 1$, and for $n > 0$ we let u^{-n} be the nth power of the adjoint u^*.

Lemma 2.2. *If (π, u) is a covariant representation, then $u^n u^{-n} = \pi(p_n)$ and $\mathrm{Ad}u^n \circ \pi(a) = \pi \circ \alpha^n(a)$ for all $n \in \mathbf{Z}$, $a \in D_{-n}$.*

Proof. This follows from the definitions and (2.2). □

In particular, the u^n are all partial isometries.

Definition 2.3. For a covariant representation (π, u), we write

$$C^*(\pi, u) = \overline{\sum_{n \in \mathbf{Z}} \pi(D_n) u^n}.$$

A quick calculation shows that, for $n, k \in \mathbf{Z}$, $a \in D_n$, and $b \in D_k$,

$$\pi(a) u^n \pi(b) u^k = \pi \circ \alpha^n (\alpha^{-n}(a) b) u^{n+k}$$

and

$$(\pi(a) u^n)^* = \pi \circ \alpha^{-n}(a^*) u^{-n},$$

so $C^*(\pi, u)$ is a C^*-algebra. For every covariant representation (π, u) of (A, α) there is a unique nondegenerate representation $\pi \times u$ of $A \times_\alpha \mathbf{Z}$ determined by

$$\pi \times u(a) = \sum_n \pi(a_n) u^n \quad \text{for} \quad a \in \bigoplus_n D_n,$$

and conversely every nondegenerate representation of $A \times_\alpha \mathbf{Z}$ is associated in this way with a unique covariant representation.

A is faithfully and nondegenerately embedded in $A \times_\alpha \mathbf{Z}$, hence in $(A \times_\alpha \mathbf{Z})^{**}$; let $i \colon A \to (A \times_\alpha \mathbf{Z})^{**}$ be this embedding. Then the universal representation of $A \times_\alpha \mathbf{Z}$ in $(A \times_\alpha \mathbf{Z})^{**}$ is of the form $i \times m$ for a unique partial isometry $m \in (A \times_\alpha \mathbf{Z})^{**}$. Moreover, $\alpha = \mathrm{Ad} m$ and

$$A \times_\alpha \mathbf{Z} = C^*(i, m) = \overline{\sum_n D_n m^n}.$$

Definition 2.4. We refer to m as the *canonical partial isometry implementing* α *in* $(A \times_\alpha \mathbf{Z})^{**}$.

We now briefly review the elementary theory of coactions, specialized to $\mathbf{Z}$. A *coaction* of $\mathbf{Z}$ on a C^*-algebra B is a nondegenerate injection $\delta\colon B \to B \otimes C^*(\mathbf{Z})$ satisfying the *coaction identity*

$$(\delta \otimes \iota) \circ \delta = (\iota \otimes \delta_{\mathbf{Z}}) \circ \delta,$$

where ι always denotes the identity automorphism, and $\delta_{\mathbf{Z}}\colon C^*(\mathbf{Z}) \to C^*(\mathbf{Z}) \otimes C^*(\mathbf{Z})$ is the homomorphism determined by $\delta(n) = n \otimes n$ for $n \in \mathbf{Z}$. Coactions of $\mathbf{Z}$ correspond bijectively to actions of $\hat{\mathbf{Z}} = \mathbf{T}$, and if β is the action of $\mathbf{T}$ associated to the coaction δ of $\mathbf{Z}$, then the coaction identity for δ says exactly that $\beta_s\beta_t = \beta_{st}$ for $s, t \in \mathbf{T}$. For example, if β is the action of $\mathbf{T}$ on $C(\mathbf{T})$ given by

$$\beta_s(f)(t) = f(st),$$

then the associated coaction δ of $\mathbf{Z}$ is given on monomials by

$$\delta(z^n) = z^n \otimes n \quad \text{for} \quad z \in \mathbf{Z}.$$

If δ is a coaction on B, the *spectral subspaces* of B are

$$B_n = \{b \in B \mid \delta(b) = b \otimes n\} \quad \text{for} \quad n \in \mathbf{Z}.$$

The disjoint union of the B_n forms a C^*-algebraic bundle over $\mathbf{Z}$:

$$B_n B_k \subset B_{n+k} \quad \text{and} \quad B_n^* = B_{-n}.$$

A *covariant representation* of $(B, \mathbf{Z}, \delta)$ is a pair (π, μ), where π and μ are nondegenerate representations of B and $c_0(\mathbf{Z})$, respectively, satisfying a certain covariance condition. In this case it is convenient to recast the covariance as a relation between the spectral subspaces B_n and the associated partition of unity

$$q_n = \mu(\chi_{\{n\}}).$$

By [**7**, Lemma 2.2], the covariance condition becomes

$$\pi(b)q_k = q_{n+k}\pi(b) \quad \text{for} \quad n, k \in \mathbf{Z}, b \in B_n.$$

If every covariant representation factors through (π, μ), then

$$\overline{\pi(B)\mu(c_0(\mathbf{Z}))}$$

is called the *crossed product* $B \times_\delta \mathbf{Z}$, and it is unique up to isomorphism.

The coaction δ is called *inner* if there is a nondegenerate homomorphism of $c_0(\mathbf{Z})$ into $M(B)$ which is covariant for the identity map of B, i.e., there is a partition of unity q_n in $M(B)$ such that $\lim_{n\to\pm\infty} q_n = 0$ strictly and

$$bq_k = q_{n+k}b \quad \text{for} \quad n, k \in \mathbf{Z}, b \in B_n.$$

In this case by [**6**, Theorem 6.9] or [**4**, Theorem 2.9] we have

$$B \times_\delta \mathbf{Z} \cong B \otimes c_0(\mathbf{Z}).$$

In particular, this holds for the *trivial* coaction $b \mapsto b \otimes 1$ of $\mathbf{Z}$ on B (take $q_n = 1$ for $n = 0$ and 0 otherwise).

A nondegenerate homomorphism ρ of B to C is called *equivariant* for coactions δ and ε if

$$\varepsilon \circ \rho = (\rho \otimes \iota) \circ \delta.$$

In this case, $\rho(B)$ is an ε-*invariant* subalgebra of C, i.e., $\varepsilon(\rho(B)) \subset \rho(B) \otimes C^*(\mathbf{Z})$. If I is an invariant ideal of B, $I \times_\delta \mathbf{Z}$ is an ideal of $B \times_\delta \mathbf{Z}$, and there is a natural coaction $\tilde{\delta}$ of $\mathbf{Z}$ on B/I such that

$$(B \times_\delta \mathbf{Z})/(I \times_\delta \mathbf{Z}) \cong (B/I) \times_{\tilde{\delta}} \mathbf{Z}.$$

If α is an actual automorphism of A (so $n \mapsto \alpha^n$ is an action of $\mathbf{Z}$ on A), Takai-Takesaki duality says (in the language of coactions) that there is a coaction $\hat{\alpha}$ of $\mathbf{Z}$ on $A \times_\alpha \mathbf{Z}$ such that

$$(A \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z} \cong A \otimes \mathcal{K}(l^2(\mathbf{Z})),$$

where $\mathcal{K}$ here stands for compact operators. While the construction of the dual coaction for partial automorphisms, indicated in the following proposition, is the same as for actions, the crossed product duality is largely destroyed, as we will see in Section 4.

Proposition 2.5 [**8**]. *If α is a partial automorphism of A, then there is a unique coaction $\hat{\alpha}$ of $\mathbf{Z}$ on $A \times_\alpha \mathbf{Z}$ such that*

$$\hat{\alpha}(am^n) = am^n \otimes n \quad \textit{for} \quad n \in \mathbf{Z}, a \in D_n.$$

The spectral subspaces are

$$(A \times_\alpha \mathbf{Z})_n = D_n m^n.$$

3. Invariant ideals.

Definition 3.1. A C^*-subalgebra B of A is called α-*invariant* if

$$\alpha(B \cap D_{-1}) \subset B \quad \text{and} \quad \alpha^{-1}(B \cap D_1) \subset B. \tag{3.1}$$

Lemma 3.2. *If (and only if) B is α-invariant, then*

$$\alpha^n(B \cap D_{-n}) = B \cap D_n \quad \textit{for} \quad n \in \mathbf{Z}. \tag{3.2}$$

Proof. Equation (3.2) is trivial for $n = 0$. Since α and α^{-1} are inverses, (3.2) for $n = \pm 1$ follows from (3.1). Inductively, assume $k > 1$ and (3.2) holds for $|n| < k$. Then

$$\begin{aligned}\alpha^k(B \cap D_{-k}) &= \alpha(\alpha^{k-1}(B \cap D_{-k})) \subset \alpha(B \cap D_{k-1} \cap D_{-1}) \\ &\subset B \cap D_1 \cap D_k = B \cap D_k.\end{aligned}$$

Symmetrically,

$$\alpha^{-k}(B \cap D_k) \subset B \cap D_{-k}.$$

Since α^k and α^{-k} are inverses, we must have (3.2) for $n = \pm k$. □

Thus, if B is α-invariant, then α restricts to a partial automorphism, which we also denote by α, of B, with ideals $B \cap D_n$.

McClanahan [**5**, Proposition 5.1 and Corollary 5.2] proves most of the following result for partial actions of arbitrary discrete groups; the basis for the techniques of Propositions 3.3 and 3.4 can actually be found in [**3**, Lemma 1]. Since our notation is different, and since we include the dual coaction, we give the proof for the reader's convenience.

Proposition 3.3. *Let I be an α-invariant ideal of A, let q be the identity element of I^{**} in A^{**}, and let $i\colon I \hookrightarrow A$ be the inclusion map.*

Then (i, qm) is a covariant representation of (I, α), and $i \times qm$ is an isomorphism of $I \times_\alpha \mathbf{Z}$ onto the ideal $\overline{\sum_n ID_n m^n}$ of $A \times_\alpha \mathbf{Z}$. Moreover, this isomorphism is equivariant with respect to the dual coactions.

Proof. We first show that m commutes with q in $(A \times_\alpha \mathbf{Z})^{**}$:

$$\begin{aligned} qm &= qp_1 m = \alpha(qp_{-1})m = mqp_{-1}m^*m \\ &= mqp_{-1}^2 = mp_{-1}q = mq. \end{aligned}$$

Thus,

$$(qm)^n(qm)^{-n} = qm^n m^{-n} = qp_n \quad \text{for} \quad n \in \mathbf{Z},$$

and, since q is the identity of I^{**},

$$\mathrm{Ad}(qm) \circ i(a) = \mathrm{Ad}m \circ i(a) = \alpha(a) \quad \text{for} \quad a \in ID_{-1}.$$

We have

$$(i \times qm)(I \times_\alpha \mathbf{Z}) = \overline{\sum_n ID_n(qm)^n} = \overline{\sum_n ID_n m^n},$$

which is obviously an ideal of $A \times_\alpha \mathbf{Z}$. The equivariance is now obvious.

It remains to show $i \times qm$ is injective, and this is accomplished by showing that every covariant representation (π, u) of (I, α) factors through (i, qm). Let $\bar{\pi}$ be the unique representation of A extending π. We verify that $(\bar{\pi}, u)$ is a covariant representation of (A, α). First, if $\{e_i\}$ is an approximate identity of I, then, taking weak operator limits,

$$\bar{\pi}(p_n) = \lim \pi(e_i p_n) = \pi(qp_n) = u^n u^{-n}.$$

Now change $\{e_i\}$ to an approximate identity of ID_{-1}. Then $\{\alpha(e_i)\}$ is an approximate identity of ID_1, so for $a \in ID_{-1}$ we have

$$\begin{aligned} \mathrm{Ad}u \circ \bar{\pi}(a) &= \lim \mathrm{Ad}u \circ \pi(e_i a) = \lim \pi \circ \alpha(e_i a) \\ &= \lim \pi(\alpha(e_i)\alpha(a)) = \bar{\pi} \circ \alpha(a). \qquad \square \end{aligned}$$

Let I be an α-invariant ideal of A, and let $\zeta\colon A \to A/I$ be the quotient map. Then $\tilde{\alpha} \circ \zeta = \zeta \circ \alpha$ determines a partial automorphism $\tilde{\alpha}$ of A/I,

with ideals $\zeta(D_n)$. Moreover, the identity of $\zeta(D_n)^{**}$ is $\zeta(p_n)$ (more precisely, $\zeta^{**}(p_n)$).

Proposition 3.4. *Let I, ζ, and $\tilde{\alpha}$ be as above, and let $\tilde{m}$ be the canonical partial isometry implementing $\tilde{\alpha}$ in $(A/I \times_{\tilde{\alpha}} \mathbf{Z})^{**}$. Then $(\zeta, \tilde{m})$ is a covariant representation of (A, α), and $\zeta \times \tilde{m}$ is a surjection of $A \times_\alpha \mathbf{Z}$ onto $A/I \times_{\tilde{\alpha}} \mathbf{Z}$ with kernel $I \times_\alpha \mathbf{Z}$. Moreover, this surjection is equivariant with respect to the dual coactions.*

Proof. We have

$$\tilde{m}^n \tilde{m}^{-n} = \zeta(p_n) \quad \text{for} \quad n \in \mathbf{Z},$$

and

$$\mathrm{Ad}\tilde{m} \circ \zeta = \tilde{\alpha} \circ \zeta = \zeta \circ \alpha.$$

Clearly $\zeta \times \tilde{m}$ is a surjection of $A \times_\alpha \mathbf{Z}$ onto $A/I \times_{\tilde{\alpha}} \mathbf{Z}$.

Since $\zeta \times \tilde{m}$ vanishes on $ID_n m^n$ for every n, $\ker(\zeta \times \tilde{m}) \supset I \times_\alpha \mathbf{Z}$. For the opposite containment, let (π, u) be a covariant representation of (A, α) with $\ker(\pi \times u) = I \times_\alpha \mathbf{Z}$. Then $\ker \pi \supset I$ since $I \times_\alpha \mathbf{Z} \supset I$ and $(\pi \times u)|I = \pi|I$. So, there is a representation $\tilde{\pi}$ of A/I such that $\pi = \tilde{\pi} \circ \zeta$. Then $(\tilde{\pi}, u)$ is a covariant representation of $(A/I, \tilde{\alpha})$, and

$$\pi \times u = (\tilde{\pi} \times u) \circ (\zeta \times \tilde{m}).$$

Hence

$$\ker(\zeta \times \tilde{m}) \subset \ker(\pi \times u) = I \times_\alpha \mathbf{Z}.$$

For the equivariance, if $a \in D_n$ we have

$$\begin{aligned} \hat{\tilde{\alpha}} \circ (\zeta \times \tilde{m})(am^n) &= \hat{\tilde{\alpha}}(\zeta(a)\tilde{m}^n) = \zeta(a)\tilde{m}^n \otimes n \\ &= ((\zeta \times \tilde{m}) \otimes \iota)(am^n \otimes n) \\ &= ((\zeta \times \tilde{m}) \otimes \iota) \circ \hat{\alpha}(am^n). \qquad \square \end{aligned}$$

We will need the following elementary result on tensor products of partial automorphisms.

Proposition 3.5. *Let $\alpha \in \mathrm{PAut}A$, and let B be a C^*-algebra. Then $\iota \otimes \alpha \in \mathrm{PAut}B \otimes A$, with domain $B \otimes D_{-1}$, and*

$$(B \otimes A) \times_{\iota\otimes\alpha} \mathbf{Z} \cong B \otimes (A \times_\alpha \mathbf{Z}),$$

where the minimal tensor product is used throughout.

Proof. Crossed products by partial automorphisms of $\mathbf{Z}$ are automatically reduced [**1**, Theorem 5.2; **5**, Proposition 4.2]. If B and A are faithfully and nondegenerately represented on Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, respectively, then $B \otimes A$ is so represented on $\mathcal{K} \otimes \mathcal{H}$. So, $(B \otimes A) \times_{\iota\otimes\alpha,r} \mathbf{Z}$ and $B \otimes (A \times_{\alpha,r} \mathbf{Z})$ are both represented on $\mathcal{K} \otimes \mathcal{H} \otimes l^2(G)$, and a mildly careful examination of these representations yields the fruit that these C^*-algebras are in fact equal. □

4. Duality. In this section we begin to examine the extent to which Takai-Takesaki crossed product duality fails for partial automorphisms.

Definition 4.1. Let

$$D_\infty = \bigcap_{n>0} D_n \quad \text{and} \quad D_{-\infty} = \bigcap_{n<0} D_n.$$

Lemma 4.2. *D_∞ and $D_{-\infty}$ are α-invariant.*

Proof. Let $a \in D_\infty D_{-1}$. Then $a \in D_n D_{-1}$ for all $n > 0$, so $\alpha(a) \in D_{n+1}$ for all $n > 0$, hence $\alpha(a) \in D_\infty$. If $a \in D_\infty$, then $a \in D_{n+1}$ for all $n > 0$, so $\alpha^{-1}(a) \in D_n$ for all $n > 0$, so $\alpha^{-1}(a) \in D_\infty$. The invariance of $D_{-\infty}$ follows by symmetry. □

Proposition 4.3. *α restricts to an automorphism of $D_\infty D_{-\infty}$.*

Proof. Since both D_∞ and $D_{-\infty}$ are α-invariant, so is $D_\infty D_{-\infty}$. Since $D_\infty D_{-\infty} \subset D_1 D_{-1}$, we are done. □

The next result shows that $D_\infty D_{-\infty}$ is a kind of "automorphic core" of α.

Lemma 4.4. *$D_\infty D_{-\infty} = \{0\}$ if and only if there is no nonzero C^*-subalgebra B of A such that $\alpha|B$ is an automorphism.*

Proof. If such a B exists, we have $B \subset D_n$ for all n, so

$$D_\infty D_{-\infty} \supset B \neq \{0\}.$$

The converse is trivial since $\alpha|D_\infty D_{-\infty}$ is an automorphism. □

Definition 4.5. We call α *completely nonautomorphic* if $D_\infty D_{-\infty} = \{0\}$.

Proposition 4.6. *The quotient partial automorphism on $A/(D_\infty D_{-\infty})$ is completely nonautomorphic.*

Proof. Let $\zeta\colon A \to A/(D_\infty D_{-\infty})$ be the quotient map. Recall that the ideals of the quotient partial automorphism are $\zeta(D_n)$. Since $D_\infty D_{-\infty} \subset D_n$ for all n, we have

$$\begin{aligned}\bigcap_n \zeta(D_n) &= \bigcap_n D_n/(D_\infty D_{-\infty}) \\ &= \left(\bigcap_n D_n\right)\Big/(D_\infty D_{-\infty}) \\ &= (D_\infty D_{-\infty})/(D_\infty D_{-\infty}) = \{0\}. \qquad \square\end{aligned}$$

The following result shows that crossed product duality tends to be maximally false for completely nonautomorphic partial automorphisms. Let p_∞ (respectively, $p_{-\infty}$) denote the identity of D_∞^{**} (respectively, $D_{-\infty}^{**}$) in A^{**}.

Theorem 4.7. *If α is completely nonautomorphic and $\lim_{n\to\pm\infty} p_n = p_{\pm\infty}$ strictly in $M(A)$, then the dual coaction $\hat{\alpha}$ of $\mathbf{Z}$ on $A \times_\alpha \mathbf{Z}$ is inner, so*

$$(A \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z} \cong (A \times_\alpha \mathbf{Z}) \otimes c_0(\mathbf{Z}).$$

Proof. We must produce a partition of unity $\{q_n \mid n \in \mathbf{Z}\}$ in $M(A \times_\alpha \mathbf{Z})$ such that $\lim_{n\to\pm\infty} q_n = 0$ strictly and

$$am^n q_k = q_{n+k} am^n \quad \text{for} \quad n, k \in \mathbf{Z}, a \in D_n. \tag{4.1}$$

Define

$$q_n = \begin{cases} p_\infty(p_{n+1} - p_n) & \text{if } n < 0, \\ p_n - p_{n+1} & \text{if } n \geq 0. \end{cases}$$

The q_n for $n \geq 0$ form a partition of $1 - p_\infty$, and the q_n for $n < 0$ form a partition of p_∞, since

$$0 = p_\infty p_{-\infty} = \text{weak*-} \lim_{n\to-\infty} p_\infty p_n.$$

Further, the hypotheses imply $\lim_{n\to\pm\infty} q_n = 0$ strictly in $M(A)$, hence in $M(A \times_\alpha \mathbf{Z})$.

By induction and taking adjoints, and since $m^0 = 1$ and the q_n are in the center of A^{**}, (4.1) will follow from

$$mq_k = q_{k+1} m \quad \text{for} \quad k \in \mathbf{Z}. \tag{4.2}$$

We first show

$$mp_k = p_{k+1} m \quad \text{for} \quad k \in \mathbf{Z}. \tag{4.3}$$

For $k \geq 0$

$$\begin{aligned} mp_k &= mp_{-1}p_k = mp_k p_{-1} = mm^k m^{-k} m^{-1} m \\ &= m^{k+1} m^{-k-1} m = p_{k+1} m, \end{aligned}$$

while for $k < 0$

$$\begin{aligned} mp_k &= mm^k m^{-k} = mm^{-1} m^{k+1} m^{-k-1} m = p_1 p_{k+1} m \\ &= p_{k+1} p_1 m = p_{k+1} m, \end{aligned}$$

showing (4.3). Since m commutes with p_∞, (4.2) follows for all $k \neq -1$. For the remaining case,

$$\begin{aligned} mq_{-1} &= mp_\infty(p_0 - p_{-1}) = p_\infty m(p_0 - p_{-1}) \\ &= p_\infty(mp_0 - mp_{-1}) = p_\infty(m - m) = 0 \\ &= p_0 m - p_1 m = (p_0 - p_1)m = q_0 m. \qquad \square \end{aligned}$$

Corollary 4.8. *Let α be completely nonautomorphic, and assume the projections $p_{\pm 1}$ and $p_{\pm\infty}$ are in $M(A)$. Then (assuming $A \neq \{0\}$) A contains a nonzero α-invariant ideal I such that $\hat{\alpha}$ is inner on $I \times_\alpha \mathbf{Z}$.*

Proof. First note that $m \in M(A \times_\alpha \mathbf{Z})$ by [**8**, Proposition 2.13], so for all $n \in Z$ we have $m^n \in M(A \times_\alpha \mathbf{Z})$, hence $p_n \in M(A)$. Let

$$I_1 = \overline{\sum_{n<0} p_\infty (p_{n+1} - p_n) A}$$

$$I_2 = \overline{\sum_{n>0} p_{-\infty} (p_{n-1} - p_n) A}$$

$$I_3 = \overline{\sum_{n,k>0} (1 - p_\infty - p_{-\infty})(p_{1-n} - p_{-n})(p_{k-1} - p_k) A}.$$

Then each I_j is an α-invariant ideal to which the hypotheses of Theorem 4.7 apply, and $A = \oplus_1^3 I_j$. $\square$

5. The distribution of ideals. The ideals D_n decrease in both directions from $n = 0$. In each direction, the behavior of the partial action is dramatically influenced by whether the ideals are eventually constant or strictly decreasing forever, and upon whether the intersection is $\{0\}$.

Lemma 5.1. *For $n > 0$, the following are equivalent*:

(i) $D_n = \{0\}$;

(ii) $D_{-n} = \{0\}$;

(iii) $D_{n-1} D_{-1} = \{0\}$;

(iv) $D_{1-n} D_1 = \{0\}$.

Proof. Since α is injective, this follows from the following relations:

$$D_n = \alpha^n (D_{-n}) = \alpha(D_{n-1} D_{-1}) = \alpha^{n-1}(D_{1-n} D_1). \qquad \square$$

Definition 5.2. A partial automorphism such that $D_n = \{0\}$ for some $n > 0$ will be called *nilpotent*.

The next result concerns the most trivial nilpotent partial automorphisms.

Proposition 5.3. *If $D_1 = \{0\}$, then $A \times_\alpha \mathbf{Z} = A$ and the dual coaction $\hat{\alpha}$ is the trivial coaction $a \mapsto a \otimes 1$, so*

$$(A \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z} \cong A \otimes c_0(\mathbf{Z}).$$

Proof. This follows immediately from the definitions, since $\oplus_n D_n = A$ and $m = 0$. □

Perhaps the simplest nontrivial nilpotent partial automorphisms are given by the following example.

Example 5.4. Define $\sigma_n \in \mathrm{PAut}\mathbf{C}^n$ by

$$\sigma_n(z_1, \ldots, z_{n-1}, 0) = (0, z_1, \ldots, z_{n-1}).$$

We will refer to this as the *shift on* $\mathbf{C}^n$. Exel [**1**] shows that

$$C^n \times_{\sigma_n} \mathbf{Z} \cong M_n,$$

the algebra of $n \times n$ matrices.

The next lemma shows that all nilpotent partial automorphisms contain shifts.

Lemma 5.5. *Let $n > 1$, and suppose $D_{n-1} \neq \{0\}$, $D_n = \{0\}$, and*

$$A = D_{1-n} + D_{2-n}D_1 + D_{3-n}D_2 + \cdots + D_{n-1}.$$

Then (A, α) is isomorphic to $(D_{n-1} \otimes \mathbf{C}^n, \iota \otimes \sigma_n)$, so

$$A \times_\alpha \mathbf{Z} \cong D_{n-1} \otimes M_n.$$

Proof. It suffices to prove (A, α) is isomorphic to $(D_{1-n} \otimes \mathbf{C}^n, \iota \otimes \sigma_n)$, since composing with $\alpha^{n-1} \otimes \iota$ will then give an isomorphism with $(D_{n-1} \otimes \mathbf{C}^n, \iota \otimes \sigma_n)$. The hypothesis implies

$$D_{-1} = D_{1-n} + D_{2-n}D_1 + D_{3-n}D_2 + \cdots + D_{-1}D_{n-2}$$

and that the ideals D_{1-n}, $D_{2-n}D_1$, $D_{3-n}D_2$, ... , D_{n-1} have pairwise zero intersection. Define $\theta\colon D_{1-n} \otimes \mathbf{C}^n \to A$ by

$$\theta(a \otimes (z_1, \dots, z_n)) = \sum_1^n z_i \alpha^{i-1}(a).$$

θ is clearly an isomorphism, and

$$\begin{aligned}\theta(D_{1-n} \otimes (\mathbf{C}^{n-1} \times \{0\})) &= \sum_1^{n-1} \alpha^{i-1}(D_{1-n}) \\ &= \sum_1^{n-1} D_{i-n} D_{i-1} = D_{-1}.\end{aligned}$$

We have

$$\begin{aligned}\theta \circ (\iota \otimes \sigma_n)(a \otimes (z_1, \dots, z_{n-1}, 0)) &= \theta(a \otimes (0, z_1, \dots, z_{n-1})) \\ &= \sum_1^{n-1} z_i \alpha^i(a) \\ &= \alpha\left(\sum_1^{n-1} z_i \alpha^{i-1}(a)\right) \\ &= \alpha \circ \theta(a \otimes (z_1, \dots, z_{n-1}, 0)). \qquad \square\end{aligned}$$

The following theorem shows that A can have many subquotients (in fact, it is sometimes an inverse limit of such) on which α looks like a nilpotent shift.

Theorem 5.6. *For each $n > 1$ the ideal*

$$I_n = D_{1-n} + D_{2-n}D_1 + D_{3-n}D_2 + \cdots + D_{n-1}$$

of A is α-invariant. Moreover, $I_n \supset I_{n+1}$, and if β_n is the quotient partial automorphism of I_n/I_{n+1}, then

$$(I_n/I_{n+1}, \beta_n) \cong (D_{n-1}/(D_n + D_{n-1}D_{-1}) \otimes \mathbf{C}^n, \iota \otimes \sigma_n).$$

Consequently,

$$I_n/I_{n+1} \times_{\beta_n} \mathbf{Z} \cong D_{n-1}/(D_n + D_{n-1}D_{-1}) \otimes M_n.$$

Proof. We have

$$\begin{aligned}\alpha(I_n D_{-1}) &= \alpha(D_{1-n} + D_{2-n}D_1 + \cdots + D_{-1}D_{n-2}) \\ &= D_{2-n}D_1 + \cdots + D_{n-1} \subset I_n.\end{aligned}$$

The containment $I_n \supset I_{n+1}$ follows from $D_n \supset D_{n+1}$. Noting that the ideals of β_n are

$$D_k(\beta_n) = I_n D_k / I_{n+1} D_k,$$

we see that $(I_n/I_{n+1}, \beta_n)$ satisfies the hypotheses of the above lemma. We finish by observing that

$$I_{n+1}D_{n-1} = D_n + D_{n-1}D_{-1},$$

so

$$D_{n-1}(\beta_n) = D_{n-1}/(D_n + D_{n-1}D_{-1}). \qquad \square$$

A version of the above result holds even for $n = 1$, although it (like the empty set) is best dealt with separately:

Theorem 5.7. *Let $I = D_{-1} + D_1$. Then*

$$(A \times_\alpha \mathbf{Z})/(I \times_\alpha \mathbf{Z}) \cong A/I$$

and

$$((A \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z})/((I \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z}) \cong A/I \otimes c_0(\mathbf{Z}).$$

Proof. We have

$$(A \times_\alpha \mathbf{Z})/(I \times_\alpha \mathbf{Z}) \cong A/I \times_\beta \mathbf{Z}$$

and

$$((A \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z})/((I \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z}) \cong (A/I \times_\beta \mathbf{Z}) \times_{\hat{\beta}} \mathbf{Z},$$

where β is the quotient partial automorphism. But β has domain $\{0\}$, so by Proposition 5.3,

$$A/I \times_\beta \mathbf{Z} = A/I$$

and

$$(A/I \times_\beta \mathbf{Z}) \times_{\hat{\beta}} \mathbf{Z} \cong A/I \otimes c_0(\mathbf{Z}). \qquad \square$$

We use the above results to show that crossed product duality fails in general for partial automorphisms. To be precise, crossed product duality demands a *canonical* isomorphism of $(A \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z}$ with $A \otimes \mathcal{K}(l^2(\mathbf{Z}))$—an "accidental" isomorphism is irrelevant. Without putting too fine a point on it, let us agree that "canonical" implies at least that if $I \supset J$ are α-invariant ideals of A, then the isomorphism carries $(I \times_\alpha \mathbf{Z}) \times_{\hat{\alpha}} \mathbf{Z}$ onto $I \otimes \mathcal{K}(l^2(\mathbf{Z}))$, and similarly for J. Taking quotients, we get an isomorphism of $(I/J \times_{\bar{\alpha}} \mathbf{Z}) \times_{\hat{\bar{\alpha}}} \mathbf{Z}$ with $I/J \otimes \mathcal{K}(l^2(\mathbf{Z}))$. But with $I = I_n$ and $J = I_{n+1}$, the above results would then imply

$$I/J \otimes \mathcal{K}(l^2(\mathbf{Z})) \cong I/J \otimes M_n \otimes c_0(\mathbf{Z}),$$

which is false except in very special examples. This failure of crossed product duality has an advantage over Corollary 4.8, since it does not require any projections to be multipliers. However, this does not handle all cases: the following example shows that even when $D_\infty = D_{-\infty} = \{0\}$, we can have $I_n = A$ for all $n > 0$.

Example 5.8. This example was invented by Nándor Sieben. Here A will be $C_0(\mathbf{R})$ and α will be translation by π:

$$\alpha(f)(t) = f(t - \pi),$$

with domain

$$D_{-1} = \{f \in A \mid f(t) = 0 \text{ for } t \in S\},$$

where

$$S = \{0\} \cup \left\{ \pm \sum_1^n \frac{1}{k} \mid n \in \mathbf{N} \right\}.$$

The reader can check that

$$\bigcup_{n>0} (S + n\pi) \quad \text{and} \quad \bigcup_{n<0} (S + n\pi)$$

are both dense in $\mathbf{R}$, so

$$D_\infty = D_{-\infty} = \{0\},$$

and moreover, for each $n > 0$,

$$\bigcup_{k=1}^{n} (S + k\pi) \cap \bigcup_{k=0}^{n-1} (S - k\pi) = \varnothing,$$

so

$$I_{n+1} \supset D_n + D_{-n} = A.$$

We saw in the preceding section that D_∞ and $D_{-\infty}$ are α-invariant. What can we say about the restricted and quotient partial automorphisms?

Definition 5.9. α is a *forward shift* (respectively, a *backward shift*) if $D_n = A$ for all $n \leq 0$ and $D_\infty = \{0\}$ (respectively, $D_n = A$ for all $n \geq 0$ and $D_{-\infty} = \{0\}$).

Of course, α is a forward shift if and only if α^{-1} is a backward shift. The simplest nonnilpotent forward shift is

$$\sigma\colon (x_1, x_2, \dots) \longmapsto (0, x_1, x_2, \dots)$$

on c_0, whose crossed product is the compact operators. If we adjoin an identity to c_0, the crossed product becomes the Toeplitz algebra generated by a nonunitary isometry [**1**]. The following result shows that nonnilpotent forward shifts tend to look like the preceding example on a certain ideal.

Proposition 5.10. *Let α be a forward shift, and assume $p_1 \in M(A)$. Then*

$$I = \overline{\sum_{n>0} (p_{n-1} - p_n)A}$$

is an α-invariant ideal, and

$$(I, \alpha|I) \cong ((1 - p_1)A \otimes c_0, \iota \otimes \sigma),$$

where σ is the forward shift on c_0. A similar result holds for backward shifts.

Proof. The hypotheses imply

$$I = \bigoplus_{n \geq 0} \alpha^n((1 - p_1)A),$$

and the proposition follows easily. □

Proposition 5.11. *If α is completely nonautomorphic, then $\alpha|D_{-\infty}$ is a forward shift and $\alpha|D_\infty$ is a backward shift.*

Proof. The ideals for $\alpha|D_{-\infty}$ are $D_n D_{-\infty}$, which coincide with $D_{-\infty}$ for $n \leq 0$, and we have

$$\bigcap_{n>0} D_n D_{-\infty} = D_\infty D_{-\infty} = \{0\}.$$

The other statement follows by symmetry. □

Note that it is possible for $D_\infty D_{-\infty} = \{0\}$ while neither D_∞ nor $D_{-\infty}$ is $\{0\}$, e.g., the direct sum of a forward shift and a backward shift. In fact, this is almost typical, as we will discuss after the next result.

Proposition 5.12. *If $\tilde{\alpha}$ is the quotient partial automorphism on A/D_∞ (respectively, $A/D_{-\infty}$, $A/(D_\infty + D_{-\infty})$), then $D_\infty(\tilde{\alpha}) = \{0\}$ (respectively, $D_{-\infty}(\tilde{\alpha}) = \{0\}$, $D_\infty(\tilde{\alpha}) = D_{-\infty}(\tilde{\alpha}) = \{0\}$).*

Proof. The first part follows from

$$(D_n + D_\infty)/D_\infty = D_n/D_\infty \quad \text{for} \quad n > 0,$$

and the other parts are shown similarly. □

In particular, a completely nonautomorphic partial automorphism falls naturally into three pieces: a forward shift on $D_{-\infty}$, a backward

shift on D_∞, and a quotient partial automorphism on $A/(D_\infty + D_{-\infty})$ satisfying $D_\infty = D_{-\infty} = \{0\}$. The simplest nonnilpotent illustration of the latter phenomenon is $\oplus_{n>0}(\mathbf{C}^n, \sigma^n)$. This can also be visualized as the partial automorphism on $c_0(\mathbf{N}^2)$ with domain $\{x \mid x_{n,1} = 0 \text{ for all } n \in \mathbf{N}\}$ and which takes such an x to y, where

$$y_{n,k} = \begin{cases} x_{n-1,k+1} & \text{if } n > 1, \\ 0 & \text{if } n = 1. \end{cases}$$

Note that in this case

$$I_n = \{x_{k,l} \mid k + l \geq n + 1\}.$$

The following result shows that partial automorphisms with $D_\infty = D_{-\infty} = \{0\}$ tend to look like the preceding example on a certain ideal.

Proposition 5.13. *Let α satisfy $D_\infty = D_{-\infty} = \{0\}$, and assume $p_1, p_{-1} \in M(A)$. Then*

$$I = \overline{\sum_{n,k>0} (p_{1-n} - p_{-n})(p_{k-1} - p_k)A}$$

is an α-invariant ideal, and

$$(I, \alpha|I) \cong \bigoplus_{n>0} ((p_{1-n} - p_{-n})(1 - p_1)A \otimes \mathbf{C}^n, \iota \otimes \sigma_n).$$

Proof. The proof is almost as easy as in Proposition 5.10, noting that

$$\alpha((p_{-k} - p_{-k-1})(p_{l-1} - p_l)) = (p_{1-k} - p_{-k})(p_l - p_{l+1}) \quad \text{for} \quad k, l > 0$$

and the hypotheses imply

$$I = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \alpha^k((p_{1-n} - p_{-n})(1 - p_1)A). \qquad \square$$

Note that the summands in the above proposition are the subquotients of Theorems 5.6 and 5.7.

The nilpotent partial automorphisms studied in the preceding section gave trivial examples of the D_n being eventually constant. We finish by examining the general case.

Lemma 5.14. *If $n \leq 0$ and $D_n = D_{n-1}$, then*

$$D_n = D_k \quad \textit{for all} \quad k \leq n, \tag{5.1}$$

and similarly for $n \geq 0$.

Proof. The hypothesis implies

$$D_{n-1} = \alpha^{-1}(D_n D_1) = \alpha^{-1}(D_{n-1} D_1) = D_{n-2},$$

giving (5.1) by induction. The other part follows by symmetry. □

Thus, if $n \leq 0$ and $D_n = D_{n-1}$, then $D_n = D_{-\infty}$ is α-invariant. Curiously, a partial converse holds:

Proposition 5.15. *If $n < 0$ and $\alpha(D_n) \subset D_n$, then $D_n = D_{n-1}$.*

Proof. We have

$$\begin{aligned} D_n D_{n-1} &= \alpha^{-1}(D_{n+1} D_n D_1) = \alpha^{-1}(\alpha(D_n) D_n) \\ &= \alpha^{-1} \circ \alpha(D_n) = D_n, \end{aligned}$$

so $D_n \subset D_{n-1}$, whence $D_n = D_{n-1}$. □

What can we say about $\alpha|D_n$? It depends on the D_k for $k \geq 0$:

Lemma 5.16. *If*

$$\begin{aligned} n &= \max\{j \leq 0 \mid D_j = D_{j-1}\} \quad \textit{and} \\ k &= \min\{j \geq 0 \mid D_j = D_{j+1}\} \end{aligned}$$

are both finite, then $n = -k$ and α restricts to an automorphism of D_n.

Proof. Assuming without loss of generality that $n \geq -k$, it suffices to show $\alpha(D_n) = D_n$:

$$\begin{aligned}\alpha(D_n) &= \alpha(D_{n-1}) = D_n D_1 = D_{-k} D_1 \\ &= \alpha^{-k}(D_k D_{k+1}) = \alpha^{-k}(D_k) \\ &= D_{-k} = D_n. \qquad \square\end{aligned}$$

Thus, if the D_n are eventually constant in both directions, then this behavior starts at the same place forward and backward, generalizing Lemma 5.1. Moreover, when $D_n = D_{n-1} = D_k = D_{k+1}$ for $n = -k \leq 0$, then D_n is the automorphic core $D_\infty D_{-\infty}$. On the other hand, if $n \leq 0$ and $D_n = D_{n-1}$, but $D_k \neq D_{k+1}$ for all $k > 0$, then in any event $D_\infty \subset D_n$ and the quotient partial automorphism on A/D_n is nilpotent. As usual, similar statements hold for $n \geq 0$.

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