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PROPERTY (u) IN $JH\tilde{\otimes}_\varepsilon JH$

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ABSTRACT. It is shown that the tensor product $JH\tilde{\otimes}_\varepsilon JH$ fails Pełczńyski's property (u). The proof uses a result of Kwapień and Pełczńyski on the main triangle projection in matrix spaces.

The Banach space JH constructed by Hagler [**1**] has a number of interesting properties. For instance, it is known that JH contains no isomorph of ℓ^1 and has property (S): every normalized weakly null sequence has a subsequence equivalent to the c_0-basis. This easily implies that JH is c_0-saturated, i.e., every infinite dimensional closed subspace contains an isomorph of c_0. In answer to a question raised originally in [**1**], Knaust and Odell [**2**] showed that every Banach space which has property (S) also has Pełczńyski's property (u). In [**4**], the author showed that the Banach space $JH\tilde{\otimes}_\varepsilon JH$ is c_0-saturated. It is thus natural to ask whether $JH\tilde{\otimes}_\varepsilon JH$ also has the related properties (S) and/or (u). In this note we show that $JH\tilde{\otimes}_\varepsilon JH$ fails property (u) (and hence property (S) as well). Our proof makes use of a result, due to Kwapień and Pełczńyski, that the main triangle projection is unbounded in certain matrix spaces.

We use standard Banach space notation as may be found in [**5**]. Recall that a series $\sum x_n$ in a Banach space E is called *weakly unconditionally Cauchy* (wuC) if there is a constant $K < \infty$ such that $\| \sum_{n=1}^k \varepsilon_n x_n\| \leq K$ for all choices of signs $\varepsilon_n = \pm 1$ and all $k \in \mathbf{N}$. A Banach space E has *property* (u) if whenever (x_n) is a weakly Cauchy sequence in E, there is a wuC series $\sum y_k$ in E such that $x_n - \sum_{k=1}^n y_k \to 0$ weakly as $n \to \infty$. If E and F are Banach spaces, and $L(E', F)$ is the space of all bounded linear operators from E' into F endowed with the operator norm, then the tensor product $E\tilde{\otimes}_\varepsilon F$ is the closed subspace of $L(E', F)$ generated by the weak*-weakly continuous operators of finite rank. In particular, for any $x \in E$ and $y \in F$, one obtains an element $x \otimes y \in E\tilde{\otimes}_\varepsilon F$ defined by $(x \otimes y)x' = \langle x, x'\rangle y$ for all $x' \in E'$.

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Let us also recall the definition of the space JH, as well as fix some terms and notation. Let $T = \cup_{n=0}^{\infty}\{0,1\}^n$ be the dyadic tree. The elements of T are called *nodes*. If ϕ is a node of the form $(\varepsilon_i)_{i=1}^n$, we say that ϕ has *length* n and write $|\phi| = n$. The length of the empty node is defined to be zero. For $\phi, \psi \in T$ with $\phi = (\varepsilon_i)_{i=1}^n$ and $\psi = (\delta_i)_{i=1}^m$, we say that $\phi \leq \psi$ if $n \leq m$ and $\varepsilon_i = \delta_i$ for $1 \leq i \leq n$. The empty node is $\leq \phi$ for all $\phi \in T$. Two nodes ϕ and ψ are *incomparable* if neither $\phi \leq \psi$ nor $\psi \leq \phi$ hold. If $\phi \leq \psi$, we say that ψ is a *descendant* of ϕ, and we set

$$S(\phi, \psi) = \{\xi : \phi \leq \xi \leq \psi\}.$$

A set of the form $S(\phi,\psi)$ is called a *segment*, or more specifically, an *m-n-segment* provided $|\phi| = m$ and $|\psi| = n$. A *branch* is a maximal totally ordered subset of T. The set of all branches is denoted by Γ. A branch γ, respectively a segment S, is said to *pass through* a node ϕ if $\phi \in \gamma$, respectively $\phi \in S$. If $x : T \to \mathbf{R}$ is a finitely supported function and S is a segment, we define (with slight abuse of notation) $Sx = \sum_{\phi \in S} x(\phi)$. In case $S = \{\phi\}$ is a singleton, we write simply ϕx for Sx. Similarly, if $\gamma \in \Gamma$, we define $\gamma(x) = \sum_{\phi\in\gamma} x(\phi)$. A set of segments $\{S_1, \dots, S_r\}$ is *admissible* if they are pairwise disjoint, and there are $m, n \in \mathbf{N} \cup \{0\}$ such that each S_i is an m-n-segment. The James Hagler space JH is defined as the completion of the set of all finitely supported functions $x : T \to \mathbf{R}$ under the norm:

$$\|x\| = \sup\left\{\sum_{i=1}^{r} |S_i x| : S_1, \dots, S_r \text{ is an admissible set of segments}\right\}.$$

Clearly, all S and γ extend to norm 1 functionals on JH. It is known that the set T of all node functionals, and the set Γ of all branch functionals together span a dense subspace of JH', cf., [**1**, p. 301]. Finally, if $x : T \to \mathbf{R}$ is finitely supported and $n \geq 0$, let $P_n x : T \to \mathbf{R}$ be defined by

$$(P_n x)(\phi) = \begin{cases} x(\phi) & \text{if } |\phi| \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, P_n extends uniquely to a norm 1 projection on JH, which we denote again by P_n. The proof of the following lemma is left to the reader. We thank the referee for the succinct formulation.

Lemma 1. *For any $n \in \mathbf{N}$, construct a sequence $(\pi(1), \pi(2), \dots, \pi(n))$ by writing the odd integers in the set $\{1, \dots, n\}$ in increasing order, fol-*

lowed by the even integers in decreasing order. Then

$$(-1)^{\min(\pi(i),\pi(j))+1} = 1 \quad \Longleftrightarrow \quad i+j \le n+1.$$

For any $n \in \mathbf{N}$ and $n \times n$ real matrix $M = [M(i,j)]_{i,j=1}^n$, let $E(M)$ be the matrix $[(-1)^{\min(i,j)+1}M(i,j)]$. Denote by $\sigma(M)$ the norm of M considered as a linear map from $\ell^\infty(n)$ into $\ell^1(n)$, i.e.,

$$\sigma(M) = \sup\bigg\{ \sum_{i,j=1}^n a_i b_j M(i,j) : \sup_{1\le i,j\le n} \{|a_i|, |b_j|\} \le 1 \bigg\}.$$

Lemma 2. *There is a constant $C > 0$ such that, for every $n \in \mathbf{N}$, there is an $n\times n$ real matrix M_n such that $\sigma(M_n) = 1$ and $\sigma(E(M_n)) \ge C \log n$.*

Proof. It follows easily from [**3**, Proposition 1.2] that there are a constant $C > 0$ and real $n \times n$ matrices $N_n = [N_n(i,j)]$ for every n such that $\sigma(N_n) = 1$, and $\sigma([\varepsilon(i,j)N_n(i,j)]) \ge C \log n$, where

$$\varepsilon(i,j) = \begin{cases} 1 & \text{if } i+j \le n+1, \\ -1 & \text{otherwise.} \end{cases}$$

Let π be the permutation in Lemma 1. Define $M_n(i,j) = N_n(\pi^{-1}(i), \pi^{-1}(j))$, $1 \le i,\ j \le n$, and let $M_n = [M_n(i,j)]$. Clearly, $\sigma(M_n) = \sigma(N_n) = 1$ for all n. Also,

$$\begin{aligned} \sigma(E(M_n)) &= \sigma([(-1)^{\min(\pi(i),\pi(j))+1} M_n(\pi(i),\pi(j))]) \\ &= \sigma([\varepsilon(i,j)N_n(i,j)]) \ge C \log n, \end{aligned}$$

as required. □

Let ψ_1 denote the node (0) and $\psi_n = (\overbrace{1\ldots 1}^{n-1}0)$ for $n \ge 2$. For convenience, define $s_0 = 0$ and $s_k = \sum_{i=1}^k i$ for $k \ge 1$. Now choose a strictly increasing sequence (n_k) in $\mathbf{N}$ and a sequence of pairwise distinct nodes (ϕ_i) such that ϕ_i is a descendant of ψ_k having length n_k whenever $s_{k-1} < i \leq s_k$, $k \in \mathbf{N}$. For any $i \in \mathbf{N}$, choose a branch γ_i which passes through ϕ_i. If $i \leq s_k$, denote by $\phi(i,k)$ the node of length n_k which belongs to γ_i. Finally, let $R_k = [R_k(i,j)]_{i,j=s_{k-1}+1}^{s_k}$ be $k \times k$ real matrices such that $\sum_k \sigma(R_k) < \infty$. Then define a sequence of elements in $JH\tilde{\otimes}_\varepsilon JH$ as follows:

$$U_l = \sum_{k=1}^{l} \sum_{i,j=s_{k-1}+1}^{s_k} R_k(i,j) e_{\phi(i,l)} \otimes e_{\phi(j,l)}$$

for $l \in \mathbf{N}$. Here $e_\phi \in JH$ is the characteristic function of the singleton set $\{\phi\}$. Since the sequence $(e_{\phi(i,l)})_{i=s_{k-1}+1}^{s_k}$ is isometrically equivalent to the $\ell^1(k)$-basis whenever $k \leq l$,

$$\left\| \sum_{i,j=s_{k-1}+1}^{s_k} R_k(i,j) e_{\phi(i,l)} \otimes e_{\phi(j,l)} \right\| = \sigma(R_k),$$

and thus $\|U_l\| \leq \sum_k \sigma(R_k) < \infty$ for any l.

Lemma 3. *The sequence (U_l) is a weakly Cauchy sequence in $JH\tilde{\otimes}_\varepsilon JH$.*

Proof. It is well known that a bounded sequence (W_n) in a tensor product $E\tilde{\otimes}_\varepsilon F$ is weakly Cauchy if and only if $(W_n x')$ is weakly Cauchy in F for all $x' \in E'$. Since (U_l) is a bounded sequence and $[T \cup \Gamma] = JH'$, it suffices to show that $(U_l x')$ is weakly Cauchy in JH for every x' in $T \cup \Gamma$. Now, for all $\phi \in T$, we clearly have $U_l \phi = 0$ for all large enough l. Next, consider any $\gamma \in \Gamma$. If γ does not pass through any ψ_k, then it cannot pass through any $\phi(i,k)$ either. So $U_l \gamma = 0$ for all l. Otherwise, due to the pairwise incomparability of (ψ_k), there is a unique k_0 such that $\psi_{k_0} \in \gamma$. If γ is distinct from γ_i for all $s_{k_0-1} < i \leq s_{k_0}$, then again $U_l \gamma = 0$ for all sufficiently large l. Now suppose $\gamma = \gamma_{i_0}$, where $s_{k_0-1} < i_0 \leq s_{k_0}$. Then, for $l \geq k_0$,

$$U_l \gamma = \sum_{j=s_{k_0-1}+1}^{s_{k_0}} R_{k_0}(i_0, j) e_{\phi(j,l)}.$$

Since each sequence $(e_{\phi(j,l)})_{l=k_0}^{\infty}$ is weakly Cauchy in JH, so is $(U_l\gamma)$. □

Now if $JH\tilde{\otimes}_\varepsilon JH$ has property (u), then it is easy to observe that there must be a block sequence of convex combinations (V_r) of (U_l) such that $\sum(V_r - V_{r+1})$ is a wuC series. Write $V_r = \sum_{l=l_{r-1}+1}^{l_r} a_l U_l$ (convex combination), where (l_r) is a strictly increasing sequence in $\mathbf{N}$. Fix $r \in \mathbf{N}$. For $s_{r-1} < i \leq s_r$, let ξ_i be a branch such that $\phi(i, l_{r+i-s_{r-1}})$ is the node of maximal length which it shares with γ_i. Then if $s_{r-1} < i,\ j \leq s_r$ and $r \leq l$,

$$\langle e_{\phi(i,l)}, \xi_j\rangle = 1 \quad \Longleftrightarrow \quad i = j \quad \text{and} \quad l \leq l_{r+j-s_{r-1}}.$$

Hence, if $r \leq l$,

$$\langle U_l\xi_i, \xi_j\rangle = \begin{cases} R_r(i,j) & \text{if } l \leq \min(l_{r+i-s_{r-1}}, l_{r+j-s_{r-1}}), \\ 0 & \text{otherwise.}\end{cases}$$

Thus, if $s_{r-1} < i, j, k \leq s_r$,

$$\langle V_{r+k-s_{r-1}}\xi_i, \xi_j\rangle = \begin{cases} R_r(i,j) & \text{if } k \leq \min(i,j), \\ 0 & \text{otherwise.}\end{cases}$$

It follows that

$$\left\langle \left\{ \sum_{k=s_{r-1}+1}^{s_r} (-1)^{k+1-s_{r-1}} (V_{r+k-s_{r-1}} - V_{r+k+1-s_{r-1}}) \right\} \xi_i, \xi_j \right\rangle = (-1)^{\min(i-s_{r-1}, j-s_{r-1})+1} R_r(i,j).$$

Notice that $k > s_{r-1}$ implies $l_{r+k-s_{r-1}} \geq l_{r+1} \geq r$, hence

$$\langle V_{r+k-s_{r-1}} P'_{n_r}\xi_i, P'_{n_r}\xi_j\rangle = \langle V_{r+k-s_{r-1}}\xi_i, \xi_j\rangle.$$

Also, $(P'_{n_r}\xi_i)_{i=s_{r-1}+1}^{s_r}$ is isometrically equivalent to the $\ell^\infty(r)$-basis. Therefore,

$$\left\| \sum_{k=s_{r-1}+1}^{s_r} (-1)^{k+1-s_{r-1}} (V_{r+k-s_{r-1}} - V_{r+k+1-s_{r-1}}) \right\| \geq \sigma([(-1)^{\min(i-s_{r-1}, j-s_{r-1})+1} R_r(i,j)]) = \sigma(E(R_r)).$$

But, since $\sum(V_r - V_{r+1})$ is a wuC series, there is a constant $K < \infty$, which may depend on the sequence (R_k), such that

$$\left\| \sum_{k=s_{r-1}+1}^{s_r} (-1)^{k+1-s_{r-1}} (V_{r+k-s_{r-1}} - V_{r+k+1-s_{r-1}}) \right\| \leq K$$

for any r. Consequently, $\sup_r \sigma(E(R_r)) \leq K$.

Now choose a strictly increasing sequence (r_m) such that $\lim_m 2^{-m} \log r_m = \infty$. Then, let

$$R_k = \begin{cases} M_{r_m}/2^m & \text{if } k = r_m \text{ for some } m, \\ 0 & \text{otherwise,} \end{cases}$$

where M_{r_m} is the matrix given by Lemma 2. Then $\sum_k \sigma(R_k) = \sum_m 2^{-m}\sigma(M_{r_m}) = 1$. So the preceding argument yields a finite constant K such that

$$K \geq \sup_m \frac{\sigma(E(M_{r_m}))}{2^m} \geq C \sup_m \frac{\log r_m}{2^m},$$

contrary to the choice of (r_m). We have thus proved the following result.

Theorem 4. *The Banach space $JH\tilde{\otimes}_\varepsilon JH$ fails property* (u).

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