ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1998

A KAPLANSKY THEOREM FOR JB*-ALGEBRAS

S. HEJAZIAN AND A. NIKNAM

ABSTRACT. We provide a new proof of a previously known result, namely every (not necessarily complete) algebra norm on a JB*-algebra generates a topology stronger than the one of the JB*-norm. As a consequence, if θ is a homomorphism of a JB*-algebra A into a Banach Jordan algebra B, then

(i) the range of θ is closed in B if θ is continuous,

(ii) θ is injective if and only if it is bounded below.

Introduction. A Jordan algebra is a nonassociative algebra A over the complex or real field in which the product satisfies $ab = ba$ and $(ab)a^2 = a(ba^2)$, $a, b \in A$. The Jordan triple product $\{abc\}$ is defined to be $(ab)c + a(bc) - (ac)b$, and for a in A, L_a denotes the operator of left multiplication by a.

A Banach Jordan algebra is a Jordan algebra A equipped with a complete norm $\|\cdot\|$, such that $\|ab\| \le \|a\|\|b\|$, $a, b \in A$. A complex Banach Jordan algebra A with an involution $*$, such that $\|\{aa^*a\}\| = \|a\|^3$ for all a in A is called a JB*-algebra. It has been shown in [**18**] that in a JB*-algebra A the involution $*$ is an isometry, and every closed associative $*$-subalgebra of A is a C^*-algebra. this shows that the class of JB*-algebras coincides with the class of Jordan C^*-algebras introduced by Kaplansky in 1976, see [**17**]. For a JB*-algebra A, we denote by $C^*(a)$ the C^*-subalgebra of A generated by a self-adjoint element $a \in A$. If A is a C^*-algebra we define the Jordan product of two elements a, b in A by $a.b = (ab + ba)/2$. In terms of this product, A becomes a JB*-algebra. A closed linear $*$-subspace of a C^*-algebra B which is closed under the Jordan product is called a JC*-algebra. The theory of JB*-algebra is of capital importance in the theory of JB*-triples, and the classification of bounded symmetric domains in the complex Banach spaces, see [**6**] and [**9**].

Received by the editors on August 1, 1995, and in revised form on May 3, 1996.
1991 AMS *Mathematics Subject Classification.* 46L70.
Key words and phrases. Continuity of homomorphism, JB*-algebra, separating space.
Research supported by a grant from the University of Manitoba, Canada.

Kaplansky [**8**] proved that any algebra norm on $C(X)$, the C^*-algebra of real or complex valued continuous functions vanishing at infinity on a locally compact Hausdorff space X, dominates the usual supremum norm. This result was improved by S. Cleveland [**4**], showing that every (not necessarily complete) algebra norm on a C^*-algebra generates a topology stronger than the one of the C^*-norm. Recently, the subharmonic methods in [**1**] and [**14**] have been applied in [**5**] and [**15**], see also [**10**, Theorem 6.1.16] to give distinct but closely related proofs of Cleveland's result. Very recently the arguments in [**5**] and [**15**] have been applied in [**3**] and [**12**] to extend Cleveland's result to JB*-algebra. In this paper we prove Cleveland's theorem for JB*-algebra by purely classical methods, avoiding the application of subharmonic methods.

Suppose θ is a homomorphism of a Banach Jordan algebra A into a Banach Jordan algebra B. The range of θ is denoted by $R(\theta)$ and we define the separating space $\sigma_B(\theta)$ of θ in B by

$$\sigma_B(\theta) = \{b \in B \mid \exists \{a_n\} \subseteq A, a_n \to 0, \text{ and } \theta(a_n) \to b\}$$

and the separating space $\sigma_A(\theta)$ of θ in A is defined by $\sigma_A(\theta) = \theta^{-1}(\sigma_B(\theta))$. $\sigma_B(\theta)$ and $\sigma_A(\theta)$ are closed linear subspaces of B and A, respectively. $\sigma_A(\theta)$ is an ideal in A and $\sigma_B(\theta)$ is an ideal in $\overline{R(\theta)}$, the closure of $R(\theta)$ in B. By the closed graph theorem θ is continuous if and only if $\sigma_B(\theta) = \{0\}$. The same argument as in [**4**, page 1099] shows that the main boundedness theorem is valid for nonassociative complete normed algebras, that is, if A and B are nonassociative complete normed algebras, if θ is a homomorphism of A into B, and if $\{x_n\}$ and $\{y_n\}$ are sequences in A such that $x_n y_m = 0$, $n \neq m$, then

$$\operatorname{Sup}\left\{\frac{\|\theta(x_n y_n)\|}{\|x_n\|\|y_n\|} : n \in N\right\} < \infty.$$

2. The results.

Lemma 1. *Let A be a* JB**-algebra, and let a and b be positive elements in A such that $ab = 0$. Then $L_{a^n} L_{b^m} = L_{b^m} L_{a^n}$, $m, n \in N$.*

Proof. First we show that $L_a L_b = L_b L_a$. Since $ab = 0$ and $a \geq 0$, we have $(a^{1/2})^2 b = 0$; therefore, by [**2**, Lemma 3-2], $a^{1/2} b = 0$ and by [**7**, Equation 2-42] for each $x, y \in A$, we have

$$\{xa^{1/2}y\}b = \{(xb)a^{1/2}y\} + \{xa^{1/2}(yb)\} - \{x(a^{1/2}b)y\}.$$

Taking $y = a^{1/2}$, it follows that $L_aL_b = L_bL_a$. Now for $m \in N$, by [**2**, Corollary 3-3(iii)], $a^mb = 0$; hence, $L_{a^m}L_b = L_bL_{a^m}$ since a^m and b are positive. The above argument, replacing a and b by b and a^m, respectively, shows that $b^na^m = 0$ for all $n \in N$. Therefore, $L_{a^m}L_{b^n} = L_{b^n}L_{a^m}$, $m, n \in N$. □

As a consequence, if a and b are as above, and if $f \in C^*(a)$ and $g \in C^*(b)$, then $L_fL_g = L_gL_f$ and $fg = 0$.

Lemma 2. *If θ is a homomorphism of a* JB*-*algebra A into a Banach Jordan algebra B and $\{x_n\}$ is a sequence of positive elements in $\sigma_A(\theta)$ such that $x_nx_m = 0$, $n \neq m$, then, except for a finite number of values of n, we have $\theta(x_n)^4 = 0$.*

Proof. Suppose that $\theta(x_n)^4 \neq 0$ for infinitely many n. Replacing by a subsequence, if necessary, we may assume $\theta(x_n)^4 \neq 0$ for all positive integers n. Since $x_n \in \sigma_A(\theta)$, there is a sequence $\{a_{nk}\}$ in A such that $\lim_{k\to\infty} a_{nk} = 0$ and $\lim_{k\to\infty} \theta(a_{nk}) = \theta(x_n)$. Thus, $\lim_{k\to\infty}\{x_na_{nk}x_n\} = 0$, $n \in N$. We have

$$\begin{aligned}\lim_{k\to\infty} \theta(x_n\{x_na_{nk}x_n\}) &= \lim_{k\to\infty} \theta(x_n)\{\theta(x_n)\theta(a_{nk})\theta(x_n)\}\\ &= \theta(x_n)\{\theta(x_n)\theta(x_n)\theta(x_n)\}\\ &= \theta(x_n)^4 \neq 0.\end{aligned}$$

Therefore, $\lim_{k\to\infty} \|\theta(x_n\{x_na_{nk}x_n\})\|/\|\{x_na_{nk}x_n\}\| = \infty$, $n \in N$. For each positive integer n, pick $l(n)$ such that

$$\frac{\|\theta(x_n)\{x_na_{nl(n)}x_n\}\|}{\|\{x_na_{nl(n)}x_n\}\|} > n\|x_n\|.$$

Put $y_n = \{x_na_{nl(n)}x_n\}$. Since $x_n, x_m \geq 0$ and $x_nx_m = 0$, $n \neq m$, then by Lemma 1, we have $L_{x_n}L_{x_m} = L_{x_m}L_{x_n}$. Therefore $x_my_n = 0$, $n \neq m$, and $\|\theta(x_ny_n)\|/\|x_n\|\|y_n\| > n$, $n \in N$. By the main boundedness theorem, this is a contradiction. □

Theorem 3. *Suppose θ is a homomorphism of a* JB**-algebra A into a Banach Jordan algebra B. Then*

(i) *$R(\theta)$ is closed in B if θ is continuous,*

(ii) *If θ is injective and $R(\theta)$ is dense in B, then the map ϕ of A into $B/\sigma_B(\theta)$ defined by $\phi(a) = \theta(a) + \sigma_B(\theta)$ is a continuous surjective homomorphism.*

Proof. (i) Set $\operatorname{Ker}\theta = \theta^{-1}(\{0\})$ and $A^0 = A/\ker(\theta)$, then $\operatorname{Ker}(\theta)$ is a closed $*$-ideal and A^0 is a JB*-algebra [**11, 17**]. Define $\theta^0 : A^0 \to B$ by $\theta^0(a + \operatorname{Ker}\theta) = \theta(a)$. Then θ^0 is an injective homomorphism, $R(\theta^0) = R(\theta)$ and $\|\theta^0\| = \|\theta\|$. Let x be an element in A^0. Consider the C^*-algebra generated by xx^*, then by [**11**, Proposition 2.2] and the Kaplansky theorem for commutative C^*-algebras [**8**], we have

$$\begin{aligned}\|x\|^2 &\le 2\|xx^*\| \le 2\|\theta^0(xx^*)\| \\ &= 2\|\theta^0(x)\theta^0(x^*)\| \\ &\le 2\|\theta^0\|\|\theta^0(x)\|\|x\|.\end{aligned}$$

Hence $\|x\| \le 2\|\theta^0\|\|\theta^0(x)\|$, $x \in A^0$. It follows that $R(\theta)$ is closed.

(ii) It is easy to see that ϕ is a well-defined homomorphism with dense range, by [**16**, Lemma 1.3] ϕ is continuous and, by part (i), ϕ is surjective. □

Theorem 4. *Suppose θ is an injective homomorphism of a* JB**-algebra A into a Banach Jordan algebra B. Then there exists a constant $M > 0$ such that $\|x\| \le M\|\theta(x)\|$, $x \in A$.*

Proof. We may replace B by the closure of $R(\theta)$ and assume that $R(\theta)$ is dense in B. The map ϕ is a continuous surjective homomorphism by Theorem 3. It is enough to show that ϕ is injective, since then the inverse map ϕ^{-1} is continuous by the open mapping theorem and hence there is a constant M such that

$$\|x\| \le M\|\phi(x)\| = M\|\theta(x) + \sigma_B(\theta)\| \le M\|\theta(x)\|, \quad x \in A.$$

Now suppose $\phi(a) = 0$ for some nonzero element a in A. By the definition of ϕ, a lies in $\sigma_A(\theta)$ and we can assume without loss of generality that $a > 0$, since $\sigma_A(\theta)$ is a closed ideal and therefore it is self-adjoint by [**11**]. If $SP(a)$ denotes the spectrum of a, then $SP(a)$ is finite, since otherwise the same argument as in [**4**, Theorem 5.1] shows the existence of a sequence $\{x_n\}$ of nonzero positive elements of $\sigma_A(\theta)$ such that $x_n x_m = 0$, $n \neq m$, so $\theta(x_n)^4 = 0$ for all but a finite number of values of n, by Lemma 2, and so $x_n = 0$ for all but a finite number of values of n, since θ is injective and $x_n \geq 0$ for all n. From the finiteness of $SP(a)$ and the spectral theory, there is a nonzero projection $p \in \sigma_A(\theta)$. Thus, $\theta(p) \in \sigma_B(\theta)$ and $\theta(p)^2 = \theta(p) \neq 0$. This is a contradiction since $\sigma_B(\theta)$ contains no nonzero idempotents [**13**]. Therefore ϕ is injective. □

Acknowledgment. The authors would like to thank Professors L. Batten and F. Ghahramani for their useful conversation and hospitality in Manitoba. Thanks are also due to the referee for valuable remarks and suggestions.

REFERENCES

1. B. Aupetit, *The uniqueness of the complete norm topology in Banach algebras and Banach Jordan algebras*, J. Funct. Anal. **47** (1982), 7–25.

2. M. Bataglia, *Annihilators in* JB-*algebras*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 317–323.

3. A. Bensebah, *Weakness of the topology of a* JB*-*algebra*, Canad. Math. Bull. **35** (1992), 449–454.

4. S.B. Cleveland, *Homomorphisms of non-commutative $*$-algebras*, Pacific J. Math. **13** (1963), 1097–1109.

5. H.G. Dales, *On norms on algebras*, Proc. Centre Math. Anal. Austral. Nat. Univ. **21** (1989), 61–69.

6. Y. Friedman and B. Russo, *The Gelfand-Naimark theorem for* JB*-*triples*, Duke Math. J. **53** (1986), 139–148.

7. H. Hanche Olsen and E. Stormer, *Jordan operator algebras*, Pitman, 1984.

8. I. Kaplansky, *Normed algebras*, Duke Math. J. **16** (1949), 399–418.

9. W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503–529.

10. T.W. Palmer, *Banach algebras and the general theory of $*$-algebras*, Vol. I. *Algebras and Banach algebras*, Cambridge University Press, Cambridge, 1994.

11. R. Paya, J. Perez and A. Rodriguez, *Non-commutative Jordan C^*-algebras*, Manuscripta Math. **37** (1982), 87–120.

12. J. Perez, L. Rico and A. Rodriguez, *Full subalgebras of Jordan-Banach algebras and algebra norm on* JB**-algebras*, Proc. Amer. Math. Soc. **121** (1994), 1133–1143.

13. P.S. Putter and B. Yood, *Banach Jordan* $*$*-algebras*, Proc. London Math. Soc. (2) **41** (1980), 21–44.

14. T.J. Ransford, *A short proof of Johnson's uniqueness of norm theorem*, Bull. London Math. Soc. **21** (1989), 487–488.

15. A. Rodriguez, *Automatic continuity with application to* C^**-algebras*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 345–347.

16. A.M. Sinclair, *Automatic continuity of linear operators*, London Math. Soc. Lecture Note Ser. **21** (1976).

17. J.D.M. Wright, *Jordan* C^**-algebras*, Michigan Math. J. **24** (1977), 291–301.

18. M.A. Youngson, *A Vidav theorem for Banach Jordan algebras*, Math. Proc. Cambridge Philos. Soc. **84** (1978), 263–272.

Department of Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran
E-mail address: hejazian@science2.um.ac.ir

Department of Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran