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UNSOLVABLE CASES OF $P^3 + Q^3 + cR^3 = dPQR$

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Introduction. Consider the diophantine equation

$$P^3 + Q^3 + cR^3 = dPQR \tag{1}$$

where c, d, P, Q and R are rational integers, all $\neq 0$. The first results were stated (without proofs) by Sylvester [**11**] and later the equation was treated by several authors [**9**], some of them using the fact that a nontrivial solution of (1) implies that $y^2 = x^3 + (dx + 4c)^2$ has a (rational) solution with $x \neq 0$. This connection with elliptic curves makes it possible to use the powerful computational techniques for deciding solvability, that are based on calculation of the appropriate L-series for a given c and d. Bremner and Guy's paper [**2**] on equation (1) with $c = 1$ demonstrates well the power of these techniques for determining the rank and finding solutions in particular cases. However, classical methods still have some advantages, e.g., they often cover an infinite number of parameter values and the actual testing is usually far easier to perform. They may also give complementary insights, e.g., in this paper a decisive cubic residue condition occurs in every case. Using classical methods, noticeable progress has been made regarding the case $c = f^k$ when f is a prime number and $3 \nmid k$ [**5, 9**], but very few generic results exist when c has several different prime factors. This illustrates the difficulties encountered when using classical methods, but in this paper we show that this path is not yet fully explored and give results that, together with earlier results, for $c = 1, f$, cover nearly all unsolvable cases for $c = 1, f, fg, f^2g, f^3, f^3g$ where f and g are different prime numbers.

For convenience, we make the following

Definitions. If f is a cubic residue of h, we use the notation $f \sim cr\, h$. Similarly, $f \sim cr\, h$ is used when f is a cubic nonresidue of h. The subclass $\mathbf{S}$ of (positive rational) primes is defined by $p \in \mathbf{S}$ if and

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only if $p \equiv 1 \pmod 3$. Analogously for the subclass $\mathbf{T}$ of (rational) integers, $r \in \mathbf{T}$ if and only if $|r| = 1$ or Πp_i and for all p_i, p_i is a (positive rational) prime number $\equiv 2 \pmod 3$. $\mathbf{Z}[\rho]$ stands for the set of integers in $\mathbf{Q}(\sqrt{-3})$, i.e., the (complex) integers $a + b\rho$ with $a, b \in \mathbf{Z}$ and $\rho = (-1 + \sqrt{-3})/2$.

In the following we assume that f, g are rational primes, $h \in \mathbf{S}$, $M \in \mathbf{T}$ and also that upper/lower signs for different variables, in a given context, correspond.

The solvability of (1) can be decided if a descent can be made, eventually leading to another solution P_1, Q_1, R_1 of (1) with $|P_1Q_1R_1| < |PQR|$ for a general solution P, Q, R.

The case $c = f \cdot g$. Working in $\mathbf{Z}[\rho]$, extraneous equations of two kinds must be excluded in order to make the descent occur [**9, 10**], e.g., if $c = f \cdot g$:

$$P^3 + fQ^3 + gR^3 = dPQR \tag{2}$$

$$\alpha U^3 + \beta V^3 + \bar{\beta}\bar{V}^3 = 3dUV\bar{V}, \tag{3}$$

where $\alpha, \beta, U, V \in \mathbf{Z}[\rho]$ and $\alpha\beta\bar{\beta} = d^3 - 27fg$. It proves possible to exclude all extraneous equations (2) and (3) in a multitude of similar but distinct cases when $c = f \cdot g$.

(i) $3 \nmid d$. $d^3 - 27fg = hM$ or h^2M, $f \sim_{cr} h$ (or the same for g) and $f \neq 3$ if $g = h$.

(ii) If $3 \| d$, put $d = 3D$.

(a) $D^3 - fg = hM$ (or h^2M)$\equiv \pm 1, \pm 4 \pmod 9$ and f (or g) $\sim_{cr} h$.

(b) $D^3 - fg = M$, $f \equiv \pm 2 \pmod 9$, $g \equiv 4 \pmod 9$ and $D \equiv \pm 1 \pmod 3$.

(c) $D^3 - fg = M$, $f \equiv \pm 2, \mp 4 \pmod 9$, $g = 3$ and $D \equiv \pm 1 \pmod 3$.

(d) $D^3 - fg = hM$, $f \equiv \pm 2, \mp 4 \pmod 9$, $g = 3$, $D \equiv \mp 1 \pmod 3$ and $f \sim_{cr} h$.

(e) $D^3 - fg = 3^k \cdot M$ (with $k = 2, 3, 4$), $f \equiv -2 \pmod 9$, $g \equiv 4 \pmod 9$, $D \equiv 1 \pmod 3$ and $27 \nmid f + g + 1 - 3D$.

(f) $D^3 - fg = 3hM$ and $f \sim_{cr} h$.

(iii) If $3^2 \mid d$ ($\neq 0$) put $d = 9E$.

(a) $fg \equiv \pm 1 \pmod 9$ and $27E^3 - fg = M$ with $f \equiv \mp 2 \pmod 9$, $g \equiv \pm 4 \pmod 9$.

(b) $fg \equiv \pm 2, \pm 4 \pmod 9$, $27E^3 - fg = hM$ (or h^2M) and f (or g) $\sim cr\, h$.

(c) $fg \equiv \pm 3 \pmod 9$, $g = 3$, $9E^3 - f = hM$ (or h^2M) and $f \sim cr\, h$, or (instead of $f \sim cr\, h$) $E \equiv 1 \pmod h$ and $3 \sim cr\, h$.

To decide the solvability of (1) in these cases, it is sufficient to test whether it has any solution P, Q, R with $|PQ| \le 3$. We give a proof for the main part of the simplest case (i).

Theorem 1. *If $c = f \cdot g$, $3 \nmid d$, $d^3 - 27fg = hM$, $f \sim cr\, h$ and $f \neq 3$ in equation (1), then it is solvable if and only if it has a solution P, Q, R with $|PQ| \le 3$.*

Proof. We consider equation (1) with $c = f \cdot g$

$$P^3 + Q^3 + fgR^3 = dPQR \tag{1'}$$

and $(P,Q) = (Q,R) = (R,P) = 1$. We use the transformation (with $u, v, w \in \mathbf{Z}[\rho]$)

$$\begin{aligned} u &= 3P + 3Q + dR, \\ v &= 3\rho P + 3\bar\rho Q + dR, \\ w = \bar v &= 3\bar\rho P + 3\rho Q + dR. \end{aligned} \tag{4}$$

We get

$$(d^3 - 27fg)(u + v + w)^3 = 3^3 d^3 uvw. \tag{5}$$

Now $\gcd(u, v, w) \mid \gcd(9P, 9Q, (n+6)R)$ and two cases appear, depending on whether $3 \mid R$ or not.

We assume first that $3 \nmid R$. Then $\gcd(u, v, w) = 1$ and $u = \alpha U^3$, $v = \beta V^3$, $w = \bar\beta \bar V^3$ where $\alpha\beta\bar\beta = d^3 - 27fg = hM$, $\alpha, U \in \mathbf{Z}$ and $\beta, V \in \mathbf{Z}[\rho]$. Put $h = N(\eta)$ with $\eta \in \mathbf{Z}[\rho]$ primary. Equation (5) becomes (3)

$$\alpha U^3 + \beta V^3 + \bar\beta \bar V^3 = 3dUV\bar V.$$

The possible combinations of α and β are (with some $m \in \mathbf{T}$):

$$\begin{array}{ll} & \mathrm{I} \qquad\quad \mathrm{II} \\ \alpha = & M/m^2 \quad hM/m^2 \\ \beta = & \rho^j m\eta \quad\ \ \rho^j m. \end{array}$$

Congruences (mod 9), using the assumption that η is primary, show that $j = 0$ in both cases. The combination II gives a descent.

If $\alpha = hM/m^2$ and $\beta = m$, equation (3) becomes

$$hMU^3 + m^3(V^3 + \bar{V}^3) = 3m^2 dUV\bar{V}. \tag{3'}$$

Using the transformation, where $p_1, q_1, r_1 \in \mathbf{Z}$,

$$\begin{aligned} p_1 &= dU + m(V + \bar{V}), \\ q_1 &= dU + m(\rho V + \bar{\rho}\bar{V}), \\ r_1 &= dU + m(\bar{\rho}V + \rho\bar{V}) \end{aligned}$$

gives

$$fg(p_1 + q_1 + r_1)^3 = d^3 p_1 q_1 r_1. \tag{6}$$

Then $p_1 = \Delta P_1^3$, $q_1 = \Delta Q_1^3$, $r_1 = \Delta fg R_1^3$ for some $\Delta, P_1, Q_1, R_1 \in \mathbf{Z}$ and we have ($1'$) again

$$P_1^3 + Q_1^3 + fgR_1^3 = dP_1Q_1R_1$$

as the (extraneous) equation $P_1^3 + fQ_1^3 + gR_1^3 = dP_1Q_1R_1$ can be written as a norm equation in $\mathbf{Q}(\sqrt[3]{f})$, putting $\theta = \sqrt[3]{f}$,

$$N[3P_1\theta + 3Q_1\theta^2 + dR_1] = (d^3 - 27fg)R_1^3 \tag{7}$$

and because f is a cubic nonresidue of h, a prime factor in $d^3 - 27fg$, equation (7) has only trivial solutions.

Assuming that P, Q, R is the solution with least height, i.e., $|P_1Q_1R_1| \geq |PQR|$ for all P_1, Q_1, R_1, we now determine the possible values of $|PQ|$.

$$|P_1Q_1R_1| = \left|\frac{p_1 + q_1 + r_1}{\Delta d}\right| = \left|\frac{3dU}{\Delta d}\right| = \left|\frac{3U}{\Delta}\right|,$$

but also

$$|3dR| = |u + v + w| = 3|d| \left| \frac{uvw}{d^3 - 27fg} \right|^{1/3} = |3dUV\bar{V}|$$

or

$$|R| = |UV\bar{V}|$$

leading to

$$|P_1Q_1R_1| \le |3U| \le |3R|.$$

Now $|PQR| \le |P_1Q_1R_1| \le |3R|$, i.e., $|PQ| \le 3$.

We now exclude the combination I.

From equation $(1')$ we get $P^3 + Q^3 \equiv dPQR \pmod{\varphi}$, where $f = N(\varphi)$ and φ primary (if $f \equiv 1 \pmod 3$) or $f = -\varphi$ (if $f \equiv -1 (\mathrm{mod}\, 3)$).

As $\varphi \nmid PQ$ and $\gcd(u, v, w) = 1$, we also have $u \equiv \alpha U^3 \equiv (P + Q)^3/PQ \pmod{\varphi}$ and $v \equiv \beta V^3 \equiv (P + \rho Q)^3/PQ \pmod{\varphi}$. As $\gcd(P, Q) = 1$, φ divides at most one of $P + Q$, $P + \rho Q$ and $P + \bar{\rho}Q$, thus for some $\delta_1, \delta_2 \in \mathbf{Z}[\rho]$, we have

$$\alpha/\beta \equiv \delta_1^3 \pmod{\varphi} \quad \text{or} \quad \beta/\bar{\beta} \equiv \delta_2^3 \pmod{\varphi}. \tag{8}$$

By testing these congruences with the α, β-values of I, and by using the multiplicative property of the cubic character, we can rule out these subcases if $(\eta/\varphi)_3 \neq (\bar{\eta}/\varphi)_3$. This condition is readily deduced from the assumption $f \sim cr\, h$ by means of the cubic law of reciprocity. The choice of factor in f is arbitrary as $(\eta/\bar{\varphi})_3 = \overline{(\bar{\eta}/\varphi)_3}$ and $(\bar{\eta}/\bar{\varphi})_3 = \overline{(\eta/\varphi)_3}$.

Next we assume that $3 \mid R$ and more details are necessary as $\gcd(u, v, w) = \rho - \bar{\rho}$. We either get equation (3) or

$$3\alpha U^3 + \frac{\beta V^3 - \bar{\beta}\bar{V}^3}{\bar{\rho} - \rho} = 3dUV\bar{V} \tag{9}$$

with $u = 3^2\alpha U^3$, $v = (\rho - \bar{\rho})\beta V^3$, $\bar{v} = (\bar{\rho} - \rho)\bar{\beta}\bar{V}^3$ and $\alpha\beta\bar{\beta} = d^3 - 27fg$.

Also here there are two combinations of α and β and, to make the descent, we use the transformation

$$\begin{aligned} p_1 &= dU + m(V - \bar{V})/(\rho - \bar{\rho}), \\ q_1 &= DU + m(\rho V - \bar{\rho}\bar{V})/(\rho - \bar{\rho}), \\ r_1 &= dU + m(\bar{\rho}V - \rho\bar{V})/(\rho - \bar{\rho}). \end{aligned}$$

The further steps of the deduction can be made in full analogy with the case $3 \nmid R$ above and we get the same condition $|PQ| \leq 3$ for the solution with least height. $\square$

The proofs for all subcases of ii and iii above follow the same line of reasoning and the frequent conditions (mod 9) are used to exclude equations of the type (2).

We treat the cases $c = f^2 g, f^3, f^3 g$ summarily as they can be brought back to the case $c = f \cdot g$.

The case $c = f^2 g$. It is straightforward to generalize the results for $c = f \cdot g$, e.g., if $c = 3^2 f$, $d = 3D$, $D^3 - 3^2 f = hM$ (or $h^2 M$)$\equiv \pm 1, \pm 4 (\mathrm{mod}\, 9)$, $f \equiv D \equiv 3 \pmod{h}$ and $3 \sim cr\, h$, the solvability of (1) can be decided as in Theorem 1, and this extends Craig's result [**4**].

Still, with $c = 3^2 f$, if $d = 3^2 E$, $3E^3 - f = M$ and $f \equiv \pm 2, \pm 4 (\mathrm{mod}\, 9)$, a descent can be made and the solvability of (1) can be decided. More generally, the case $c = f^2 g$ can be treated as if f^2 were a prime number, i.e., as the case $c = f \cdot g$ above, if $f \sim cr\, h$ and $g \sim cr\, h$. The only complication is that

$$fP^3 + fQ^3 + gR^3 = dPQR \tag{10}$$

occurs in the descent of $P^3 + Q^3 + f^2 g R^3 = dPQR$, and vice versa, and cannot be excluded. The transformation corresponding to (4) in the proof of Theorem 1 is modified into $u = 3fP + 3fQ + dR$, etc., for equation (10).

The case $c = f^3$. If we first transform $P^3 + Q^3 + f^3 R^3 = dPQR$ into $f(d^2 + 3df + 9f^2)(u + v + w)^3 = (d + 6f)^3 uvw$ [**5**], and then if $d^2 + 3df + 9f^2 = g$, $3^3 g$ with $g \in \mathbf{S}$, we have the case $c = f \cdot g$ and can decide the solvability if also $d - 3f = hM$, $3hM$ and $f \sim cr\, h$.

The case $c = f^3 g$. If $c = f^3 g$, $d^3 - 27 f^3 g = hM$ and $f \sim cr\, h$, two equations

$$P^3 + Q^3 + f^3 g R^3 = dPQR$$

and

$$P^3 + f^3 Q^3 + g R^3 = dPQR$$

remain for a descent and to be tested for solutions of least height, in this case with $|PQ| \leq 3f^3$. The latter equation should now be transformed (4) by using $u = 3fP + 3f^2Q + dR$, etc.

The case $c = f$. This case is treated in detail in [**5**] and a minor extension can be made to those results. When $c = f$, the case N_3 is decidable also if $D^3 - f = M \equiv \pm 4 \pmod 9$.

The case $c = 1$. Several authors have studied this case and, for historical reasons, we put $d = n$ in equation (1):

$$x^3 + y^3 + z^3 = nxyz. \tag{11}$$

We first summarize earlier unsolvability results.

The equation (11) has only trivial solutions for three sets of n-values characterized by

(1) for all n, $n^2+3n+9 = p$, p^2 where $p \in \mathbf{S}$, $n-3 \in \mathbf{T}$ and $n \neq -1, 5$ [**3, 5, 8, 9, 10**].

(2) If $n \equiv 12 \pmod{27}$, put $N = (n-3)/9$. For all n, $N \in \mathbf{T}$ and $3N^2 + 3N + 1 = p, p^2$ where $p \in \mathbf{S}$ [**5**].

(3) If $n = 3N$, for all n such that $(N-1)/h \in \mathbf{T}$, where $3 \sim cr\, h$ and $N^2 + N + 1 \in \mathbf{S}$ [**4**].

In addition to these sets, see Table 1, there were a few other n-values for which the equation (11) was known to have only trivial solutions, $n = -6, -3, 0$ [**5,9**], but recently Bremner and Guy [**2**] made a computer investigation of (11) with $c = 1$, $|n| \leq 100$ and, as the Birch and Swinnerton-Dyer conjecture [**1**] has been proved for elliptic curves which are modular [**7**], their list of unsolvable equations with n-values in the investigate range is complete if the Taniyama-Weil conjecture is assumed to be true. In any case, their results are perfectly consistent with earlier unsolvability results and the new ones given here. By making some variations of the proof technique in Theorem 1, we are able to add the following sets of n-values

(4) for all n, $n - 3 = hM$, or h^2M, $n^2 + 3n + 9 = f \cdot g$ and f, or g, $\sim cr\, h$.

For the sets (5)–(7), we put $N = (n-3)/9 \in \mathbf{Z}$.

(5) If $n \equiv 3 \pmod{27}$, for all n, $N/3^k \in \mathbf{T}$ for some $k \le 3$ and $3N^2 + 3N + 1 = f \cdot g$ where $f \equiv -2 \pmod 9$, $g \equiv 4 \pmod 9$ and $27 \nmid f + g - 3N - 2$.

(6) If $n \equiv 12 \pmod{27}$, for all n, $N = hM$, $3N^2 + 3N + 1 = f \cdot g$ and f, or g, $\sim cr\, h$.

(7) If $n \equiv 21 \pmod{27}$, for all n, $N \in \mathbf{T}$ and $3N^2 + 3N + 1 = f \cdot g$ where $f \equiv -2 \pmod 9$ and $g \equiv 4 \pmod 9$.

Probably all these sets are infinite, and they contain (together with $n = -6, -3, 0$) all unsolvable cases where $|n| \le 81$ except those where $n^2 + 3n + 9$ has three prime factors.

TABLE 1. Unsolvable case of $x^3 + y^3 + z^3 = nxyz$.

Set#	n-values in the range $\lvert n\rvert \le 81$ for sets (1)–(4), $\lvert n\rvert \le 729$ for sets (5)–(7)
(1)	-61,-52,-41,-31,-26,-20,-19,-14,-13,-8,-7,-5,-2,1,2,4,7,8,11, 23, 25,28,32,37,43,49,50,56,58
(2)	-69,-42,-15,12,39
(3)	-75,-54,-39,-18,24,42,45,60,81
(4)	-74,-71,-70,-62,-58,-23,22,34,46,52,55,59,65,68,79,80
(5)	-618,-591,-483,-456,-429,-402,-267,-78,-51,111,138,165,327, 543,624,651
(6)	-663,-582,-555,282,390,552,660,687
(7)	-357,-141,48,75,183,264,399,426,453,480,615

In [**6**] an extended list of solutions of (11), relative to that in [**5**], was given. A comparison with Bremner and Guy's list shows that four solutions were missing, $n = -80, -63, -53, 72$. The solution with $n = -53$ should have been found by my search program, but a minor error in the definition of the search domain excluded it.

The proof that equation (11) has only trivial solutions for the n-values of set (4), e.g., is essentially the same as for Theorem 1, but (11) should first be transformed with

$$\begin{aligned} p &= nx + 3y + 3z, \\ q &= 3x + ny + 3z, \\ r &= 3x + 3y + nz, \end{aligned}$$

into

$$(n^2 + 3n + 9)(p + q + r)^3 = (n + 6)^3 pqr. \tag{12}$$

Now, for some $a, b, c \in \mathbf{Z}$ with $abc = n^2 + 3n + 9 = f \cdot g$ and some $P, Q, R \in \mathbf{Z}$:

$$aP^3 + bQ^3 + cR^3 = (n + 6)PQR. \tag{13}$$

As in the proof of Theorem 1, $a = b = 1$ because $f \sim_{cr} h$ and we have equation ($1'$). Here $d^3 - 27fg = (n-3)^3 = h^3M^3$ and five combinations of α, β occur when $h \| n - 3$, both when $3 \nmid R$ and $3 \mid R$. We give the combinations when $3 \nmid R$, three of which, II, III, IV, can be excluded with cubic character arguments, while the other two, I, V, give descents:

I	II	III	IV	V
$\alpha = M^3/m^2$	M^3/m^2	hM^3/m^2	h^2M^3/m^2	h^3M^3/m^2
$\beta = \rho^j m\eta^3$	$\rho^j mh\eta$	$\rho^j m\eta^2$	$\rho^j m\eta$	$\rho^j m$

In the descent cases, e.g., for combination I, the transformation

$$\begin{aligned} p_1 &= (hM + 9)U + hm(\eta V + \bar{\eta}\bar{V}), \\ q_1 &= (hM + 9)U + hm(\rho\eta V + \bar{\rho}\bar{\eta}\bar{V}), \\ r_1 &= (hM + 9)U + hm(\bar{\rho}\eta V + \rho\bar{\eta}\bar{V}) \end{aligned}$$

should be used to eventually reach the condition $|PQ| \leq 3$ for a nontrivial solution. The further steps of the deduction can be made in full analogy with Theorem 1 above and no nontrivial solution occurs. The case $h^2 \| n - 3$ gives ten combinations of α and β but the same method of proof can be used.

An interesting fact is that the congruences corresponding to (8) in the proof of Theorem 1 are solvable for some combination of α and β, other than the ones giving a descent, if $n-3$ has two (or more) different prime factors $\in \mathbf{S}$, which indicates that (11) always has a solution for such n-values. This may also be true if $n-3=hM$ and $n^2+3n+9=f\in\mathbf{S}$ as $x^3\equiv f(\bmod h)$ has the solution $x\equiv(n+6)/3(\bmod h)$ for all such n.

It is normally easy to verify that the proof fails for (n-values of) known parametric solutions [**6, 12**], e.g., if $n=N^2+5$, $n-3=hM$ and $n^2+3n+9=f\cdot g$, $f,g\in\mathbf{S}$, then $f=N^2\pm N+7$ and the test for set (4) regarding $f \sim cr\, h$ fails as $x=\pm N-1$ solves the congruence $x^3\equiv f \pmod{n-3}$, implying that $x^3\equiv f \pmod{h}$ is solvable.

TABLE 2. Parametric solutions of $P^3+Q^3+cR^3=dPQR$.

$\{c,d\}=$	$\{P,Q,R\}=$
$\{1+\delta\varepsilon\zeta, 3+\delta(\varepsilon+\zeta)^2\}$	$\{\varepsilon,\zeta,-(\varepsilon+\zeta)\}$
$\{-\varepsilon^3-\zeta^3+d\varepsilon\zeta, d\}$	$\{\varepsilon,\zeta,1\}$
$\{1+\varepsilon(\delta^4+\delta^2+1), \delta^2+5+4\varepsilon\}$	$\{\delta^2+\delta+1, \delta^2-\delta+1, 2\}$
$\{-(1+\varepsilon\delta)(\delta^2+3\delta+1), \varepsilon(\delta+1)^2+2\delta\}$	$\{1+9\delta+32\delta^2+62\delta^3+82\delta^4$ $+86\delta^5+67\delta^6+31\delta^7+8\delta^8+\delta^9+\varepsilon(\delta+9\delta^2+32\delta^3+63\delta^4+88\delta^5$ $+101\delta^6+88\delta^7+49\delta^8+15\delta^9+2\delta^{10})+\varepsilon^2(\delta^5+6\delta^6+15\delta^7+20\delta^8+15\delta^9$ $+6\delta^{10}+\delta^{11}),\ -1-9\delta-34\delta^2-67\delta^3-66\delta^4-21\delta^5+12\delta^6+13\delta^7+5\delta^8+\delta^9$ $-\varepsilon(2\delta+18\delta^2+68\delta^3+139\delta^4+162\delta^5+103\delta^6+28\delta^7-3\delta^8-4\delta^9-\delta^{10})$ $-\varepsilon^2(\delta^2+9\delta^3+34\delta^4+71\delta^5+90\delta^6+71\delta^7+34\delta^8+9\delta^9+\delta^{10}), 3\delta+21\delta^2$ $+57\delta^3+78\delta^4+61\delta^5+27\delta^6+6\delta^7+\delta^8+\varepsilon(3\delta^2+21\delta^3+61\delta^4+95\delta^5+85\delta^6$ $+43\delta^7+11\delta^8+\delta^9)+\varepsilon^2(\delta^3+7\delta^4+21\delta^5+35\delta^6+35\delta^7+21\delta^8+7\delta^9+\delta^{10})\}$

Parametric and numeric solutions. There exist parametric solutions of (1) in two and three parameters and those with the highest density of small c- and d-values are given in Table 1. Despite appearance, the derivation of these solutions is exceedingly simple. If $P^3+Q^3+cR^3=dPQR$, then the equation with $c'=c+\delta PQ$, $d'=d+\delta R^2$ is also satisfied by (P,Q,R). The P,Q,R-values of the last solution are found by means of Sylvester's theorem of derivation [**11**], applied upon the second solution with $\varepsilon=\delta+1$, $\zeta=\delta$, $d=2\delta$, to generate small c,d-values. The ubiquitous small solutions, in P,Q,R, in Table 4 [**5**] are generated by these parametric solutions. We have found several new numeric solutions of (1) in the domain $|c|\le 27$, $|d|\le 27$ in addition to those given in [**5**], or possible to derive from those, and the solvability of equation (1) can now be decided in remarkably many, 1454 out of 1485, instances within that domain. Most of the undecided cases occur if $c = f^3$ with the additional conditions $d - 3f \in \mathbf{T}$, $d^2+3df+9f^2 = h_1h_2M$ and $f \sim cr\, h_i$, $i = 1, 2$. In some instances, when no generic case did apply, a more direct method to exclude extraneous equations that works in the real domain was used.

1. Test the equations corresponding to (2) moduli h, $h^2(h \mid d^3 - 27c)$.

2. Transform the equations corresponding to $(3')$ and (9) in the proof of Theorem 1, using

$$3U = A + B + C, \qquad 3V = A + \rho B + \bar{\rho} C, \qquad 3\bar{V} = A + \bar{\rho} B + \rho C$$

where $A, B, C \in \mathbf{Z}$. The resulting equation is of the form, with i, $C_i \in \mathbf{Z}$ for all,

$$(14) \quad C_1(A^3 + B^3 + C^3) + C_2(A^2B + B^2C + C^2A) \\ + C_3(AB^2 + BC^2 + CA^2) + C_4ABC = 0.$$

3. Test (14) moduli f, $f^2(f \mid c)$ with all combinations of $A, B, C (\mathrm{mod} f)$ or $(\mathrm{mod}\ f^2)$.

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