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BALL SEPARATION PROPERTIES IN BANACH SPACES

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Dedicated to Professor Ky Fan on the occasion of his 85th birthday

ABSTRACT. Various ball separation properties related to Mazur intersection property in Banach spaces are studied.

Mazur [**16**] was the first to consider the following ball separation property, called property (I), in Banach spaces.

(I) Every bounded closed convex set is an intersection of (closed) balls.

For finite dimensional Banach space X, Phelps [**17**] showed that X has the property (I) if and only if the set of extreme points of the unit ball $B(X^*)$ of the dual space X^* is norm dense in the unit sphere $S(X^*)$ of X^*.

Giles, Gregory and Sims [**10**] showed that a Banach space X has the property (I) if and only if the set of weak* denting point of $B(X^*)$ is norm dense in $S(X^*)$. They raised a question whether every Banach space with the property (I) is an Asplund space. In 1995, Sevilla and Moreno [**20**] exhibit a class of non-Asplund spaces admitting an equivalent norm with property (I). It has been proved recently by Jimenez and Moreno [**14**] that Kunen space is an Asplund space with no equivalent norm with property (I).

Whitefield and Zizler [**21**] studied the following ball separation property, called (CI).

(CI) Every compact convex set is an intersection of balls.

They proved that a Banach space X has the property (CI) if the cone generated by the extreme points of $B(X^*)$ is τ_X dense in X^*

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where τ_X is the topology of uniform convergence on compact subsets of X. Sersouri [**18**] showed that this is also a necessary condition for the space to have the property (CI).

In [**19**], Sersouri gave a characterization for a Banach space to have the following property, called (I_n).

(I_n) Every compact convex set with dimension less than or equal to n is an intersection of balls.

Various stability properties of (I), (CI) and (I_n) are given in [**8, 9, 13, 18, 19, 21, 22**].

In Section 1 of the paper, we generalize the concept of weak* denting points that include both the weak* denting points and extreme points of a weak* closed convex set in dual space. We obtain a local characterization of weak* denting point by showing that an element f is a weak* denting point of $B(X^*)$ if and only if for every bounded set A in X such that $\inf f(A) > 0$, then there exists a ball B in X such that $B \supset A$ and $B \cap H = \varnothing$ where H is the kernel of f in X. We prove a result in Theorem 1.3 that yields the sufficient condition for the above characterization of (I), (CI) and (I_n). Theorem 1.3 also yields a new proof that every weakly compact convex set in Banach spaces is ball-generated. In Section 2 we introduce a geometric condition on ball separation that yields a proof for the above characterization of (I) and (CI). In Section 3 we give a ball separation characterization of Hahn-Banach smoothness of Banach spaces. In Section 4 we study the corresponding ball separation properties related to the weak* point of continuity $(w^* - pc)$ of $B(X^*)$. In Section 5 we give a stability result on weak* denting points, $w^* - pc$, weak*-weak points of continuity $(w^* - w\ pc)$ and extreme points by showing that, under the Hausdorff metric, in the family of all equivalent norms on X, there exists a dense G_δ-set $\mathcal{B}$ such that, if f is a weak* denting point, respectively $w^* - pc$, $w^* - w\ pc$, extreme point, of $B(X^*)$, then for every norm $\|\cdot\|_B$ in $\mathcal{B}$, f is a weak* denting, respectively $w^* - pc$, $w^* - w\ pc$, extreme point, of the ball in X^* under $\|\cdot\|_B$ with the center at the origin and radius $\|f\|_B$.

For a Banach space X, let $B(X) = \{x : x \in X,\ \|x\| \leqq 1\}$ and $S(X) = \{x : x \in X,\ \|x\| = 1\}$. For a set K in X, let $\overline{\text{co}}K$ be the closed convex hull of K. If K is a subset in the dual space X^*, let $\overline{\text{co}}^*K$ be the weak* closed convex hull of K and let $\overline{K}^{w^*}$ be the weak* closure of K in X^*. A weak* slice of K is a set $S(K,x,\delta) = \{f : f \in K, f(x) > \sup_{g\in K} g(x) - \delta\}$ where $x \in X$ and $\delta > 0$. f is called a weak* denting point, w^* denting point, of K if, for every $\varepsilon > 0$, there exists a weak* slice $S(x,K,\delta)$ of K such that $f \in S(x,K,\delta)$ and $\operatorname{diam} S(x,K,\delta) < \varepsilon$. f is called a weak*, respectively weak*-weak, point of continuity if the identity mapping $Id : (K, \text{weak}^*) \to (K, \|\cdot\|)$, respectively $(K, \text{weak}^*) \to (K, \text{weak})$) is continuous at f. The duality mapping D of X is defined by $D(x) = \{f : f \in S(X),\ f(x) = \|x\|\}$, $x \neq 0$ in X. All balls in this paper are closed balls, that is, sets of the form $B(x,r) = \{y : y \in X,\ \|x-y\| \leq r\}$. For balls in dual space X^*, we use the notation $B^*(f,r) = \{g : g \in X^*,\ \|f-g\| \leq r\}$.

The main tool in the paper is the following consequence of Phelps' lemma [**17**].

Lemma. *For a normed space X, let f and g be elements in $S(X^*)$, and let $A = \{x \in B(X) : f(x) > \varepsilon/2\}$ where $0 < \varepsilon < 1$. If $\inf g(A) > 0$, then $\|f-g\| < \varepsilon$.*

Proof. Since $\inf g(A) > 0$, it follows that $g^{-1}(0) \cap A = \varnothing$. Hence $\sup f(g^{-1}(0) \cap B(X)) < \varepsilon/2$. By Phelps' lemma, either $\|f-g\| < \varepsilon$ or $\|f+g\| < \varepsilon$. However,

$$\begin{aligned}\|f+g\| &= \sup(f+g)(B(X)) \geqq \sup(f+g)(A)\\ &\geqq \sup f(A) = 1.\end{aligned}$$

Hence $\|f-g\| < \varepsilon$. □

1. Let X be a normed space. For any bounded subset A in X, define

$$\|f\|_A = \sup\{|f(x)| : x \in A\}, \quad f \in X^*.$$

Then $\|\cdot\|_A$ is a semi-norm on X^*.

For a subset K of X^*, denote the diameter of K under the semi-norm $\|\cdot\|_A$ by $\operatorname{diam}_A K = \sup\{\|f-g\|_A : f,g \in K\}$.

Definition. Let $\mathcal{A}$ be a collection of bounded subsets in X. We say $f \in S(X^*)$ to be an $\mathcal{A}$-*denting point*, respectively $\mathcal{A}$-pc, of $B(X^*)$

if, for each $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists a weak* slice S of $B(X^*)$, respectively weak*-neighborhood S, such that $f \in S$ and $\mathrm{diam}_A S < \varepsilon$.

Examples. If $\mathcal{A}$ consists of all bounded subsets of X, then it is easy to see that an $\mathcal{A}$-denting point of $B(X^*)$ is just a w^*-denting point of $B(X^*)$.

If $\mathcal{A}$ is formed by all compact subsets of X, then an $\mathcal{A}$-denting of $B(X^*)$ is just an extreme point of $B(X^*)$. A proof of this can be found in the proof of Lemma 2, in [**21**].

We say that $\mathcal{A}$ is a compatible collection of bounded subsets in X if

1. If $A \in \mathcal{A}$ and $C \subset A$, then $C \in \mathcal{A}$.
2. For each $A \in \mathcal{A}$, $x \in X$, $A + x \in \mathcal{A}$ and $A \cup \{x\} \in \mathcal{A}$.
3. For each $A \in \mathcal{A}$, the closed absolutely convex hull of A is in $\mathcal{A}$.

Lemma 1.1. *Let $A \subset B(X)$, $x \in S(X)$, $\delta > 0$, and $\varepsilon > 0$. If*

$$\mathrm{diam}_A S(B(X^*), x, \delta) \le \varepsilon,$$

then

$$\sup_{y \in A} \frac{\|x + (\delta/2)y\| + \|x - (\delta/2)y\| - 2}{\delta/2} \le \varepsilon.$$

Proof. For any $y \in A$,

$$\begin{aligned}\left\| \frac{x + (\delta/2)y}{\|x + (\delta/2)y\|} - x \right\| &\le \left\| \frac{x + (\delta/2)y}{\|x + (\delta/2)y\|} - \left(x + \frac{\delta}{2} y \right) \right\| + \left\| \frac{\delta}{2} y \right\| \\ &= \left| \left\| x + \frac{\delta}{2} y \right\| - 1 \right| + \left\| \frac{\delta}{2} y \right\| \le \|\delta y\| \le \delta.\end{aligned}$$

Then, for any $f_0 \in D((x + (\delta/2)y)/\|x + (\delta/2)y\|)$,

$$f_0(x) = 1 - f_0\left(\frac{x + \delta/2}{\|x + (\delta/2)y\|} - x \right) \ge 1 - \delta.$$

Similarly, for any $g_0 \in D((x - (\delta/2))y/\|x - (\delta/2)y\|)$, $g_0(x) = 1 - g_0((x - \delta/2)/\|x - (\delta/2)y\|) - x) \ge 1 - \delta$. Thus $f_0, g_0 \in S(B(X^*), x, \delta)$ and so $\|f_0 - g_0\|_A \le \varepsilon$.

Hence

$$\frac{\|x+(\delta/2)y\|+\|x-(\delta/2)y\|-2}{\delta/2} \le f_0(y)-g_0(y)$$
$$\le \|f_0-g_0\|_A \le \varepsilon. \qquad \square$$

Lemma 1.2. *Suppose A is a bounded subset of a normed space X and $x \in S(X)$. If*

$$\sup_{y\in A} \frac{\|x+(1/n)y\|+\|x-(1/n)y\|-2}{1/n} \le \varepsilon,$$

then $\operatorname{diam}_A(S(B(X^*),x,\varepsilon/n)) \le 3\varepsilon$.

Proof. Suppose there exist $f,g \in S(B(X^*),x,\varepsilon/n)$ such that $\|f-g\|_A > 3\varepsilon$. Choose $y \in A$ such that $(f-g)(y) > 3\varepsilon$, then

$$\begin{aligned}\left\|x+\frac{1}{n}y\right\|+\left\|x-\frac{1}{n}y\right\| &\ge f\left(x+\frac{1}{n}y\right)+g\left(x-\frac{1}{n}y\right)\\ &> 1-\frac{\varepsilon}{n}+1-\frac{\varepsilon}{n}+\frac{1}{n}(f-g)(y)\\ &> 2-\frac{2\varepsilon}{n}+\frac{3\varepsilon}{n}=2+\frac{\varepsilon}{n}.\end{aligned}$$

Therefore

$$\frac{\|x+(1/n)y\|+\|x-(1/n)y\|-2}{1/n} > \varepsilon. \qquad \square$$

Theorem 1.3. *Let X be a normed space, and $\mathcal{A}$ be a compatible collection of bounded subsets in X. If $f_0 \in S(X^*)$, then the following are equivalent.*

(i) *f_0 is an $\mathcal{A}$-denting point of $B(X^*)$.*

(ii) *For all $A \in \mathcal{A}$, if* $\inf f_0(A) > 0$, *then there exists a ball B in X such that $A \subset B$ and* $\inf f_0(B) > 0$.

(iii) *For all $A \in \mathcal{A}$, if* $\inf f_0(A) > \alpha$ *for some real number α then there exists a ball B in X such that $A \subset B$ and* $\inf f_0(B) > \alpha$.

Proof. (ii) $\Leftrightarrow$ (iii). This is clear since the family $\mathcal{A}$ is compatible.

(ii) $\Rightarrow$ (i). For any $A \in \mathcal{A}$ and $\varepsilon > 0$, choose $x_0 \in X$ such that

$$\|x_0\| = \sup\{\|x\| : x \in A\} + 2\varepsilon \quad \text{and} \quad f_0(x_0) > \|x_0\| - \varepsilon. \tag{1.1}$$

Let K be the closed absolutely convex hull of $A \cup \{x_0\}$ and $K_\varepsilon = \{x \in K : f_0(x) \geq \varepsilon\}$.

Let $\eta = \varepsilon^2/f_0(x_0)$, then, by (iii), there exists a ball $B = B(z,r)$ in X such that

$$K_\varepsilon \subset B \quad \text{and} \quad \inf f_0(B) > \varepsilon - \eta.$$

Since

$$f_0(z) - r = \inf f_0(B(z,r)) > \varepsilon - \eta,$$

hence

$$f_0\left(\frac{z}{\|z\|}\right) > \frac{r + \varepsilon - \eta}{\|z\|}.$$

Thus $f_0 \in S(B(X^*),\, z/\|z\|,\, 1 - (r + \varepsilon - \eta)/\|z\|)$.

Now, for every $f \in S(B(X^*),\, z/\|z\|,\, 1 - (r + \varepsilon - \eta)/\|z\|) \cap S(X^*)$, we have

$$f\left(\frac{z}{\|z\|}\right) > \frac{r + \varepsilon - \eta}{\|z\|}.$$

Hence

$$\inf f(B(z,r)) = f(z) - r > \varepsilon - \eta.$$

Notice $\varepsilon(x_0/f_0(x_0)) \in K_\varepsilon \subset B(z,r)$. Hence $f(\varepsilon(x_0/f_0(x_0))) > \varepsilon - \eta$ and

$$f(x_0) > f_0(x_0) - \frac{\eta}{\varepsilon} f_0(x_0) = f_0(x_0) - \varepsilon. \tag{1.2}$$

By (1.1) and (1.2), we have

$$\sup\{\|x\| : x \in A\} \leq \|f_0\|_K, \qquad \|f\|_K \leq \sup\{\|x\| : x \in A\} + 2\varepsilon. \tag{1.3}$$

Now

$$\inf f(K_\varepsilon) \geq \inf f(B(z,r)) > 0.$$

So

$$\inf f_0(f^{-1}(0) \cap K_\varepsilon) < \varepsilon.$$

Applying Phelps' lemma in the normed space $Y = \operatorname{span} K$ with K as the unit ball, we have

$$\left\| \frac{f_0}{\|f_0\|_K} - \frac{f}{\|f\|_K} \right\|_K < 2\frac{\varepsilon}{\|f_0\|_K}. \tag{1.4}$$

Hence, by (1.3) and (1.4), we have

$$\begin{aligned} \|f_0 - f\|_K &\le \left\| f_0 - \frac{f}{\|f\|_K}\|f_0\|_K \right\|_K + \left\| f - \frac{f}{\|f\|_K}\|f_0\|_K \right\|_K \\ &\le 2\varepsilon + |\|f\|_K - \|f_0\|_K| \le 4\varepsilon. \end{aligned}$$

It is easy to show that $S(B(X^*),\ z/\|z\|,\ 1-(r+\varepsilon-\eta)/\|z\|)$ and $S(X^*) \cap S(B(X^*),\ z/\|z\|,\ 1-(r+\varepsilon-\eta)/\|z\|)$ have the same diameter, therefore

$$\operatorname{diam}_A S\left(B(X^*), \frac{z}{\|z\|}, 1 - \frac{r+\varepsilon-\eta}{\|z\|}\right) \le 8\varepsilon.$$

This proves that f_0 is an $\mathcal{A}$-denting point of $B(X^*)$.

(i) $\Rightarrow$ (ii). Suppose $f_0 \in S(X^*)$ is an $\mathcal{A}$-denting point of $B(X^*)$, $A \in \mathcal{A}$ and $\inf f_0(A) > 0$. Without loss of generality, we may assume $A \subset B(X)$. Otherwise, choose $m > 1$ such that $(1/m)A \subset B(X)$. Then $(1/m)A \in \mathcal{A}$. If we can find a ball $B = B(z,r)$ such that $(1/m)A \subset B$ and $\inf f_0(B) > 0$, then $A \subset mB = B(mz, mr)$ and $\inf f_0(mB) > 0$.

Now assume $A \subset B(X)$ and $\inf f_0(A) = \delta > 0$. Let $\varepsilon = \delta/3$, choose $x_1 \in S(X)$, $\alpha > 0$ such that $f_0 \in S(B(X^*), x_1, \alpha)$ and

$$\operatorname{diam}_A S(B(X^*), x_1, \alpha) < \varepsilon.$$

Then

$$f_0(x_1) = 1 - \beta > 1 - \alpha$$

for some $\beta \ge 0$. Let $\beta_1 \in (\beta, \alpha)$ and $M = A \cup \{x_1\}$. Then $M \in \mathcal{A}$. So we can choose $x_2 \in S(X)$, $\gamma > 0$, $k > 0$ such that

$$\frac{1}{k} < \varepsilon\ , \quad f_0 \in S(B(X^*), x_2, \gamma)$$

and

$$\operatorname{diam}_M S(B(X^*), x_2, \gamma) < \min\left(\frac{\alpha - \beta_1}{2k}, \beta_1 - \beta\right).$$

For each $f \notin S(B(X^*), x_1, \beta_1)$,

$$\|f_0 - f\|_M \geq f_0(x_1) - f(x_1) \geq 1 - \beta - (1 - \beta_1) = \beta_1 - \beta.$$

So $f \notin S(B(X^*), x_2, \gamma)$. Thus $S(B(X^*), x_2, \gamma) \subset S(B(X^*), x_1, \beta_1)$. Let $f_{x_2} \in D(x_2)$. For each $f_1 \in B(X^*)$ and $f_1(x_1) \leq 1 - \alpha$, there exists a $\lambda \in (0, 1)$ such that

$$g = \lambda f_{x_2} + (1 - \lambda) f_1 \in \{f \in B(X^*) : f(x_2) = 1 - \gamma\}.$$

Now $\|f_{x_2} - g\|_M = (1 - \lambda)\|f_{x_2} - f_1\|_M$. Hence

$$1 - \lambda = \frac{\|f_{x_2} - g\|_M}{\|f_{x_2} - f_1\|_M} \leq \frac{(\alpha - \beta_1)/(2k)}{\|f_{x_2} - f_1\|_M} = \frac{\alpha - \beta_1}{2k\|f_{x_2} - f_1\|_M}.$$

Now

$$\|f_{x_2} - f_1\|_M \geq f_{x_2}(x_1) - f_1(x_1) \geq \alpha - \beta_1.$$

Hence

$$1 - \lambda \leq \frac{\alpha - \beta_1}{2k(\alpha - \beta_1)} = \frac{1}{2k}.$$

Thus $\lambda \geq (2k - 1)/(2k)$. Since $f_1 = (g - \lambda f_{x_2})/(1 - \lambda)$, we have

$$\begin{aligned} f_1(x_2) &= \frac{g - \lambda f_{x_2}}{1 - \lambda}(x_2) = \frac{g(x_2) - \lambda}{1 - \lambda} \\ &= \frac{1 - \gamma - \lambda}{1 - \lambda} = 1 - \frac{\gamma}{1 - \lambda} \leq 1 - 2k\gamma. \end{aligned}$$

So we have proved

$$f_0 \in S(B(X^*), x_2, \gamma) \subset S(B(X^*), x_2, 2k\gamma) \subset S(B(X^*), x_1, \alpha).$$

Thus

$$\operatorname{diam}_A S(B(X^*), x_2, 2k\gamma) \leq \operatorname{diam}_A S(B(X^*), x_1, \alpha) \leq \varepsilon.$$

By Lemma 1.1,

$$\sup_{y \in A} \frac{\|x_2 + k\gamma y\| + \|x_2 - k\gamma y\| - 2}{k\gamma} \leq \varepsilon.$$

Now we claim $A \subset B(x_2/(k\gamma), 1/(k\gamma) - \varepsilon)$. Suppose there exists $y \in A$ such that $\|x_2/(k\gamma) - y\| > 1/(k\gamma) - \varepsilon$. Then

$$\begin{aligned}\frac{\|x_2 + k\gamma y\| + \|x_2 - k\gamma y\| - 2}{k\gamma} &= \frac{\|x_2 + k\gamma y\| - 1}{k\gamma} + \frac{\|x_2 - k\gamma y\| - 1}{k\gamma} \\ &> f_{x_2}(y) + \left(\frac{1}{k\gamma} - \varepsilon - \frac{1}{k\gamma}\right) \\ &= f_{x_2}(y) - \varepsilon \\ &\geq f_0(y) - \|f_0 - f_{x_2}\|_A - \varepsilon \\ &\geq \delta - 2\varepsilon = \varepsilon.\end{aligned}$$

This contradicts Lemma 1.1. Thus $A \subset B(x_2/(k\gamma), 1/(k\gamma) - 1)$. Also

$$\begin{aligned}\inf f_0\left(B\left(\frac{x_2}{k\gamma}, \frac{1}{k\gamma} - 1\right)\right) &= f_0\left(\frac{x_2}{k\gamma}\right) - \left(\frac{1}{k\gamma} - \varepsilon\right) \\ &\geq \frac{1-\gamma}{k\gamma} - \left(\frac{1}{k\gamma} - \varepsilon\right) \\ &= \varepsilon - \frac{1}{k} > 0.\end{aligned}$$

This completes the proof. □

Corollary 1.4. *Let X be a Banach space, and let $f_0 \in S(X)$. Then the following statements are equivalent*:

(i) *f_0 is a w^*-denting point of $B(X^*)$.*

(ii) *For every bounded subset C in X^{**}, if $\inf f_0(C) > 0$, then there is a ball B in X^{**} with center in X such that $B \supset C$ and $B \cap H^{**} = \varnothing$, where H^{**} is the kernel of f_0 in X^{**}.*

(iii) *For every bounded subset C in X, if $\inf f_0(C) > 0$, then there is a ball B in X such that $B \supset C$ and $B \cap H = \varnothing$, where H is the kernel of f_0 in X.*

Proof. (i) $\Leftrightarrow$ (iii). Use Theorem 1.3.

(ii) $\Rightarrow$ (iii). It is trivial since $\overline{C}^{w^*}$ satisfies the conditions in (ii).

(iii) $\Rightarrow$ (ii). Assume $\inf f_0(C) > \delta > 0$ and $C \subset B^{**}(0, r) = \{x^{**} \in X^{**} : \|x^{**}\| \leq r\}$. Let $A = \{x \in B(0, r), f_0(x) \geq \delta\}$. Then $\overline{A}^{w^*} \supset C$.

Choose a ball B in X such that $A \subset B$ and $B \cap H = \varnothing$. Then $C \subset \overline{A}^{w^*} \subset \overline{B}^{w^*}$ and $\overline{B}^{w^*} \cap H^{**} = \varnothing$. □

Corollary 1.5. *Let X be a Banach space, and let $x_0 \in S(X)$. Then the following statements are equivalent*:

(i) *x_0 is a denting point of $B(X)$.*

(ii) *For every bounded subset C in X^*, if $\inf f_0(C) > 0$, then there is a ball B in X^* such that $B \supset C$ and $B \cap H^* = \varnothing$, where H^* is the kernel of x_0 in X^*.*

Proof. Notice that $x_0 \in S(X)$ is a denting point of $B(X)$ if and only if x_0 is a w^*-denting point of $B(X^{**})$. Then use the above corollary. □

Corollary 1.6. *Let X be a Banach space. Then the following statements are equivalent*:

(i) *Every $f \in S(X)$ is a w^*-denting point of $B(X^*)$.*

(ii) *For every bounded subset C in X^{**} and any w^*-closed hyperplane H^* in X^{**}, if $\operatorname{dist}(C, H^*) > 0$, then there exists a ball B^{**} in X^{**} with center in X such that $C \subset B^{**}$ and $B^{**} \cap H^* = \varnothing$.*

(iii) *For every bounded subset C in X and any closed hyperplane H in X, if $\operatorname{dist}(C, H) > 0$, then there exists a ball B in X such that $C \subset B$ and $B \cap H = \varnothing$.*

Corollary 1.7. *Let X be a Banach space. Then the following statements are equivalent*:

(i) *Every $x \in S(X)$ is a denting point of $B(X)$.*

(ii) *For every bounded subset C in X^* and any w^*-closed hyperplane H^* in X^*, if $\operatorname{dist}(C, H^*) > 0$ then there exists a ball B^* in X^* such that $C \subset B^*$ and $B^* \cap H^* = \varnothing$.*

Corollary 1.8. *A Banach space X has the property (I) if and only if, for any two disjoint bounded weak*-closed convex subsets A_1 and A_2 in X^{**} there exist balls B_1^{**}, B_2^{**} in X^{**} with centers in X such that*

$A_1 \subset B_1^{**}$, $A_2 \subset B_2^{**}$ and $B_1^{**} \cap B_2^{**} = \varnothing$.

Proof. $\Leftarrow$. Trivial.

$\Rightarrow$. If X has the property (I), then, by [**10**], the set of weak* denting points of $B(X^*)$ is dense on $S(X^*)$. If A_1, A_2 are disjoint weak* closed convex bounded sets, then A_1, A_2 can be separated by weak* closed hyperplane. Apply Corollary 1.4 to find balls B_1^{**} and B_2^{**}. □

Corollary 1.9 [**19**]. *For every Banach space and every natural number n, the following properties are equivalent:*

(1) *Every compact convex subset C with dimension $(C) \leq n$, is an intersection of balls.*

(2) *For every $f \in X^*$, every $(n+1)$ points $x_1, \dots, x_{(n+1)} \in X$, and every $\varepsilon > 0$, there exists $g \in \operatorname{Ext}(X^*) = \{\lambda h : \lambda > 0, h \in \operatorname{Ext} B(X^*)\}$ such that $\sup_i |(f-g)(x_i)| < \varepsilon$.*

Proof. $\Leftarrow$. Suppose that C is compact convex in X with dimension $C \leq n$. If $x \notin C$, then choose $f \in X^*$ such that $f(x) < \delta < \inf f(C)$. It is easy to see that there exist $(n+1)$ points $x_1, \dots, x_{(n+1)}$ such that

$$C \subset \overline{\text{co}}\{x_1, \dots, x_{(n+1)}\} \subset \{y \in X : f(y) > \delta\}.$$

By (2), there exists $g \in \operatorname{Ext}(X^*)$ such that $\sup_i |(f-g)(x_i - x)| < \delta - f(x)$. Then

$$\inf g(\overline{\text{co}}\{x_1, \dots, x_{(n+1)}\}) > g(x).$$

By Theorem 1.3, there exists a ball B in X such that

$$\overline{\text{co}}\{x_1, \dots, x_{(n+1)}\} \subset B \quad \text{and} \quad \inf g(B) > g(x).$$

Therefore $C \subset B$ and $x \notin B$.

$\Rightarrow$. For every $f \in X^*$, every $(n+1)$ points $x_1, \dots, x_{(n+1)} \subset X$, and every $\varepsilon > 0$, let A be the closed absolutely convex hull of $\{x_1, \dots, x_{(n+1)}\}$. If $f(A) = \{0\}$, then choose $g = 0$. If $f(A) \neq \{0\}$, let $C = \{x \in A : f(x) > \varepsilon/4\}$. Then C is compact convex and dimension

$C \leq n$. So there exists a ball $B = B(z, r)$ in X, where $r > 0$, such that $C \in B$ and $0 \notin B$. Let $g \in \operatorname{Ext} D(z/\|z\|)$, then

$$\inf g(C) \geq \inf g(B) = g(z) - r = \|z\| - r > 0.$$

It follows that $g^{-1}(0) \cap C = \varnothing$. Hence

$$\sup f(g^{-1}(0) \cap A) \leqq \frac{\varepsilon}{4}.$$

Applying the lemma to the normed space spanned by A and with A as the unit ball, we have

$$\left\| \frac{f}{\|f\|_A} - \frac{g}{\|g\|_A} \right\|_A \leq \frac{\varepsilon}{2\|f\|_A}.$$

Therefore

$$\left\| f - \frac{\|f\|_A}{\|g\|_A} g \right\|_A \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence

$$\sup_i \left| x_i \left(f - \frac{\|f\|_A}{\|g\|_A} g \right) \right| < \varepsilon. \qquad \square$$

Let $\mathcal{A}$ be a family of bounded sets in X. We use $\tau_{\mathcal{A}}$ to denote the topology on X^* generated by $\{\|\cdot\|_A : A \in \mathcal{A}\}$. Using a similar argument the following result can be proved.

Theorem 1.10. *Suppose $\mathcal{A}$ is a compatible family of bounded sets in X. Let*

(1) *The cone of $\mathcal{A}$-denting points of is $\tau_{\mathcal{A}}$-dense in X^*.*

(2) *Every closed convex set $A \in \mathcal{A}$ is the intersection of balls.*

Then (1) $\Rightarrow$ (2). Furthermore, if every w^*-slice of $B(X^*)$ contains a $\mathcal{A}$-denting point,then (1) $\Leftrightarrow$ (2).

Corollary 1.11 [**21**], [**18**]. *Let X be a Banach space. Then the following are equivalent*:

(1) *Every compact convex set is an intersection of balls.*

(2) *The cone* Ext(X^*) *is dense in* X^* *for the topology of uniform convergence on compact sets of* X.

Proof. Set $\mathcal{A}$ to be the family of all compact subsets in X and notice that the $\mathcal{A}$-denting point actually is a extreme point of $B(X^*)$. Then the conclusion follows from Theorem 1.10. □

Corollary 1.12 [**19**]. *For every Banach space, the following properties are equivalent*:

(1) *Every finite dimensional compact convex set of* X *is an intersection of balls.*

(2) *The cone* Ext X^* *is* w^**-dense in* X^*.

Proof. Let $\mathcal{A}$ be the family of all finite dimensional bounded subsets of X and apply Theorem 1.10. □

Recall that a set A in a Banach space is said to be ball-generated [**12**] if there is a family $\{F_i : i \in I\}$ such that each F_i is a finite union of balls and $A = \cap_{i\in I} F_i$.

Theorem 1.13. *Let* X *be a normed space. If the linear span of the* $\mathcal{A}$*-denting points of* $B(X^*)$ *is* $\tau_{\mathcal{A}}$*-dense in* X^*, *then every weakly closed set in* $\mathcal{A}$ *is ball-generated.*

Proof. Let A be a nonempty weakly closed set in $\mathcal{A}$, and let $x_0 \notin A$. Then there exist $\varepsilon > 0$ and $x_1^*, x_2^*, \dots, x_n^* \in X^*$ such that $A = \cup_{i=1}^n A_i$ where

$$A_i = \{x \in A : x_i^*(x) > x_i^*(x_0) + \varepsilon\}.$$

Since the linear span of the $\mathcal{A}$-denting points in τ_A dense, we can choose x_i^*, $i = 1, 2, \dots n$, such that $x_i^* = \sum_{j=1}^{m_i} \lambda_{ij} x_{ij}^*$ where x_{ij}^*, $j = 1, 2, \dots,$ m_i, $i = 1, 2, \dots, n$, are $\mathcal{A}$-denting points. Furthermore, since the set of $\mathcal{A}$-denting points of $B(X^*)$ is symmetric, we may assume that $\lambda_{ij} > 0$, $j = 1, 2, \dots, m_i$, $i = 1, 2, \dots, n$. Let

$$A_{ij} = \left\{x \in A_i : x_{ij}^*(x) > x_{ij}^*(x_0) + \frac{\varepsilon}{\lambda_i m_i}\right\},$$

$j = 1, 2, \dots, m_i$, $i = 1, 2, \dots, n$. It follows that $A = \cup_{i,j} A_{ij}$. Since x_{ij}^* is an $\mathcal{A}$-denting point of $B(X^*)$, by Theorem 1.3, there is a ball B_{ij} such that $A_{ij} \subset B_{ij}$ and $x_0 \notin B_{ij}$. Therefore $A \subset \cup_{i,j} B_{ij}$ and $x_0 \notin \cup_{i,j} B_{ij}$. This shows that A is ball-generated. □

Corollary 1.14 [**5**]. *Every weakly compact subset of a Banach space X is ball-generated.*

Proof. Suppose $K \subset X$ is weakly compact and $x \notin K$. Let A be the absolutely closed convex hull of $K \cup \{x\}$ and (Y, A), $Y = \operatorname{span} A$, be the normed space with A as its unit ball. Then (Y, A) is reflexive. Now $B(Y^*) = [B(Y)]^o$, the polar of $B(Y)$ in $(Y, A)^*$, so $B(Y^*)$ as a subset of $(Y, A)^*$ is the closure of its denting points, i.e., w^*-denting since (Y, A) is reflexive, under the norm of $(Y, A)^*$.

Let $\mathcal{A} = \{C :$ there exists a $\lambda \in R : C \subset \lambda A\}$. Then the topology $\tau_{\mathcal{A}}$ is just the norm topology of $(Y, A)^*$. By Theorem 1.3, K is ball-generated in Y. So there exists a ball $B = z + rB(Y)$, $r > 0$ such that

$$K \subset B \quad \text{and} \quad x \notin z + rB(Y).$$

Hence

$$K \subset z + rB(X) \quad \text{and} \quad x \notin z + rB(X).$$

Therefore K is ball-generated. □

Corollary 1.15 [**10**]. *Let X be a Banach space; then:*

(i) *if the w^* denting points of $B(X^*)$ are norm dense in $S(X^*)$, then X has property (I).*

(ii) *If the denting points of $B(X)$ are norm dense in $S(X)$, then X^* has property weak* (I), i.e., every weak* closed convex bounded set in X^* is an intersection of balls.*

Proof. For any bounded closed convex subset $C \subset X$ and $x \notin C$, since the w^* denting points of $B(X^*)$ are dense in $S(X^*)$, we can choose a denting point f of $B(X^*)$ such that $\inf f(C) > f(x)$. Therefore there is a ball B in X such that $C \subset B$ and $B \cap f^{-1}(f(x)) = \varnothing$, where $f^{-1}(f(x)) = \{y \in X : f(y) = f(x)\}$. In particular, $x \notin B$.

The proof of (ii) is similar to the proof of (i). ◻

Remark. We can define $\mathcal{A}$-exposed point by simply requiring the slices in the definition of $\mathcal{A}$-denting points to be parallel, and then we can get a characterization of $\mathcal{A}$-exposed points by requiring the centers of balls in Theorem 1.3 to be in the same direction.

2. In Theorem 2.1 we give a necessary and sufficient conditions for a bounded set A in X^{**} to have a ball B^{**} in X^{**} with center in X such that $A \subset B^{**}$ and $0 \notin B^{**}$. The characterization of (I) [**10**] and (CI) [**18**] follow as corollaries of Theorem 2.1. Theorem 2.1 is also used to study ball topology on Banach spaces [**4**] and B-convex sets of Banach spaces [**3**].

Theorem 2.1. *Let X be a Banach space, and let A be a bounded subset in X^{**}. Then there exists a ball B^{**} in X^{**} with center in X such that $A \subset B^{**}$ and $0 \notin B^{**}$ if and only if*

$$\overline{\mathrm{co}}^{w^*} B_1(A) \neq B(X^*),$$

where

$$B_1(A) = \{f \in B(X^*) : x^{**}(f) \leq 0 \textit{ for some } x^{**} \in A\}.$$

Proof. $\Rightarrow$. Let $x_0 \in X$, and let

$$B^{**} = B^{**}(x_0, r) \stackrel{\text{def}}{=} \{x^{**} \in X^{**} : \|x^{**} - x_0\| \leq r\}$$

be a ball in X^{**} such that

$$A \subset B^{**} \quad \text{and} \quad 0 \notin B^{**}.$$

Choose $f_0 \in S(X^*)$ such that $f_0(x_0) = \|x_0\|$. Let

$$V = \left\{ f \in B(X^*) : f\left(\frac{x_0}{\|x_0\|}\right) > \frac{r}{\|x_0\|} \right\}.$$

Then V is a w^*-neighborhood of f_0. We claim that $V \cap \mathrm{co} B_1(A) = \varnothing$.

Suppose there exists

$$f = \sum_{i=1}^{n} \lambda_i f_i \in V,$$

where

$$f_i \in B_1(A), \quad \lambda_i > 0, \quad i = 1, 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1.$$

Now

$$\sum_{i=1}^{n} \lambda_i f_i \left(\frac{x_0}{\|x_0\|} \right) = f \left(\frac{x_0}{\|x_0\|} \right) > \frac{r}{\|x_0\|}.$$

Hence there exists i_0 such that

$$f_{i_0} \left(\frac{x_0}{\|x_0\|} \right) > \frac{r}{\|x_0\|}.$$

Since for every $x^{**} \in B^{**}$,

$$f_{i_0}(x_0 - x^{**}) \leq \|x^{**} - x_0\| \leq r,$$

hence

$$f_{i_0}(x^{**}) \geq f_{i_0}(x_0) - r > 0.$$

On the other hand, since $f_{i_0} \in B_1(A)$, there exists

$$x_{i_0}^{**} \in A \subset B^{**} \quad \text{such that} \quad f_{i_0}(x_{i_0}^{**}) \leqq 0,$$

which is a contradiction. Therefore $f_0 \notin \overline{\text{co}}^{w^*} B_1(A)$ and so

$$\overline{\text{co}}^{w^*} B_1(A) \neq B(X^*).$$

$\Leftarrow$. If $\overline{\text{co}}^{w^*} B_1(A) \neq B(X^*)$, then there exists $x_0 \in S(X)$ and slice $S(B(X^*), x_0, 4\delta)$ of $B(X^*)$ such that

$$S(B(X^*), x_0, 4\delta) \cap \overline{B_1(A)}^{w^*} = \varnothing.$$

For each

$$f \in S(B(X^*), x_0, \delta),$$

we have

$$(1-\delta)f(x_0) > (1-\delta)^2 > 1-2\delta.$$

Now

$$B^*((1-\delta)f, \delta) \subset B(X^*).$$

Also

$$\inf x_0(B^*(1-\delta)f, \delta) = (1-\delta)f(x_0) - \delta > 1-3\delta.$$

Thus

$$B^*((1-\delta)f, \delta) \subset S(B(X^*), x_0, 4\delta).$$

Hence for every $x^{**} \in A$,

$$\inf x^{**}(B^*(1-\delta)f, \delta) \geq 0.$$

On the other hand,

$$\begin{aligned} \inf x^{**}(B^*(1-\delta)f, \delta) &= (1-\delta)x^{**}(f) - \delta\|x^{**}\| \\ &\leq (1-\delta)x^{**}(f) - \delta d(0, A). \end{aligned}$$

Therefore,

$$(1-\delta)x^{**}(f) - \delta d(0, A) \geq 0$$

and

$$x^{**}(f) \geq \frac{\delta d(0, A)}{1-\delta}. \tag{2.1}$$

Now, if

$$f \in S(B(X^*), x_0, \delta),$$

then by (2.1)

$$nf(x_0) - f(x^{**}) \leq n - \frac{\delta d(0, A)}{1-\delta}. \tag{2.2}$$

If $f \in B(X^*)\backslash S(B(X^*), x_0, \delta)$, then

$$\begin{aligned} f(nx_0 - x^{**}) &= nf(x_0) - f(x^{**}) \\ &\leq n(1-\delta) + \|x^{**}\| \leq n(1-\delta) + M \end{aligned} \tag{2.3}$$

where $M = \sup_{x^{**}\in A} \|x^{**}\|$.

By (2.2) and (2.3), we have

$$\|nx_0 - x^{**}\| \le \max\left\{n - \frac{\delta d(0,A)}{1-\delta}, n(1-\delta)+M\right\}.$$

So, for n large enough, we have

$$\|nx_0 - x^{**}\| \le n - \frac{\delta d(0,A)}{1-\delta}, \quad \text{for all } x^{**} \in A.$$

Hence,

$$A \subset B^{**}\left(nx_0, n - \frac{\delta d(0,A)}{1-\delta}\right),$$

and of course,

$$0 \notin B^{**}\left(nx_0, n - \frac{\delta d(0,A)}{1-\delta}\right).$$

This completes the proof. □

Corollary 2.2 [**10**]. *A Banach space X has the property (I) if and only if the w^*-denting points of $B(X^*)$ are dense in $S(X^*)$.*

Proof. $\Rightarrow$. For any $\varepsilon > 0$, $\varepsilon' > 0$ and $f \in S(X^*)$, let $\delta = \min\{\varepsilon, \varepsilon'\}$. Consider

$$A = \left\{x \in B(X) : f(x) \ge \frac{\delta}{4}\right\}.$$

Since X has the property (I), there exists a ball B in X such that $A \subset B$ and $0 \notin B$. By Theorem 2.1, there exists a w^*-slice $S(B(X^*), x_0, \eta)$ such that

$$S(B(X^*), x_0, \eta) \cap \overline{\text{co}}^{w^*} B_1(A) = \varnothing.$$

Then, for every $g \in S(B(X^*), x_0, \eta)$ and $x \in A$, $g(x) > 0$. By the lemma,

$$\|g - f\| < \frac{\delta}{2} < \varepsilon'. \tag{2.4}$$

Hence

$$\operatorname{diam} S(B(X^*), x_0, \eta) < \delta \le \varepsilon. \tag{2.5}$$

By (2.4) and (2.5), it follows that $D^*_\varepsilon = \{g \in B(X^*)$: there exists a w^*-slice S of $B(X^*)$ such that $g \in S$ and $\operatorname{diam} S < \varepsilon\}$ is dense in $S(X^*)$. By the Baire category theorem, the set of w^*-denting points of $B(X^*)$, which is $\cap_{\varepsilon>0} D^*_\varepsilon$ is dense in $S(X^*)$.

$\Leftarrow$. Let A be a bounded closed convex subset of A and $x_0 \notin A$. Without loss of generality, suppose $x_0 = 0$. Since the w^*-denting points of $B(X^*)$ are dense in $S(X^*)$, we can choose the w^*-denting point f_0 of $B(X^*)$ such that

$$\inf f_0(A) > f(0) = 0.$$

Let V be a norm neighborhood of f_0 in $B(X^*)$ such that, for every $f \in V$,

$$\inf f(A) > 0.$$

Now choose a slice $S(B(X^*),\ x_0,\ \delta)$ which contains f_0 and which is contained in V. It is clear that $S(B(X^*), x_0, \delta) \cap B_1(A) = \varnothing$. Since $S(B(X^*), x_0, \delta)$ is a w^*-slice, it follows that

$$S(B(X^*), x_0, \delta) \cap \overline{\text{co}}^{w^*} B_1(A) = \varnothing.$$

By Theorem 2.1, we conclude that there exists a ball B in X such that $A \subset B$ and $0 \notin B$. This shows that X has the property (I). □

Corollary 2.3 [**18**]. *Let X be a Banach space. Then X has the property (CI) if and only if the cone K generated by the set of all extreme points of $B(X^*)$ is dense in X^* under in the topology of uniform convergence on compact subsets in X.*

Proof. Let A be any compact convex subset of X and $x_0 \notin A$. Without loss of generality, suppose $x_0 = 0$. Then we can choose the extreme point f_0 of $B(X^*)$ such that

$$\inf f_0(A) > 0.$$

Since f_0 is an extreme point of $B(X^*)$, an argument of Lemma 2 in [**21**] proved that all slices $S(B(X^*), x, \delta)$, $x \in X$ which contain f form a base of f in $(B(X^*), \tau_X)$, where τ_X is the topology of uniform convergence on compact subsets in X.

Choose a slice $S(B(X^*),\, x_0,\, \delta)$ containing f_0 such that

$$\inf f(A) > 0, \quad \text{for all} \quad f \in S(B(X^*), x_0, \delta).$$

Then

$$S(B(X^*), x_0, \delta) \cap \overline{\text{co}}^{w^*} B_1(A) = \varnothing$$

and so

$$\overline{\text{co}}^{w^*} B_1(A) \neq B(X^*).$$

Applying Theorem 2.1, we conclude that there is a ball B in X such that $A \subset B$ and $0 \notin B$. This shows that X has the property (CI).

Conversely, suppose that X has the property (CI). Given $f \in X$, $\varepsilon > 0$ and compact set A in X. If $\|f\|_A < \varepsilon$, then take any $g \in \text{ext}\, B(X^*)$ and $\lambda \in \mathbf{R}$ sufficiently small; we have $\|f - \lambda g\|_A < \varepsilon$. Assume $\|f\|_A \geq \varepsilon$. Let K be the closed absolutely convex hull of A, and let $K_0 = \{x \in K : f(x) \geq \varepsilon/2\}$. Then K_0 is compact convex and $0 \notin K_0$. Since X has the property (CI), there exists a ball B in X such that $B \supset K_0$ and $0 \notin B$. By Theorem 2.1, we conclude that $\overline{\text{co}}^{w^*} B_1(K_0) \neq B(X^*)$. Choose $x_0 \in S(X^*)$ and $\delta > 0$ such that $S(B(X), x_0, \delta) \cap \overline{\text{co}}^{w^*} B_1(K_0) = \varnothing$. Let $g \in \text{ext}\, D(x_0) = \text{ext}\,\{h \in S(X^*) : h(x_0) = \|x_0\| = 1\}$. Then $g \in \text{ext}\, B(X^*)$ and, since $g \notin \overline{\text{co}}^* B_1(K_0)$, it follows that $\inf g(K_0) > 0$. Applying the lemma to the space spanned by K_0 with K as a unit ball, we conclude that

$$\left\| \frac{f}{\|f\|_K} - \frac{g}{\|g\|_K} \right\|_K < \frac{\varepsilon}{\|f\|_K}.$$

Since $\|\cdot\|_K = \|\cdot\|_A$, we have $\|f - \lambda g\|_A < \varepsilon$ where $\lambda = \|f\|_A/\|g\|_A$. This completes the proof. □

3. A ball separation characterization of Hahn-Banach smoothness of Banach spaces is given in this section.

Recall that a Banach space is Hahn-Banach smooth if, for every $x^* \in X^*$, there is a unique Hahn-Banach extension in X^{***}. Equivalently, every point $x^* \in S(X^*)$ is a $w^* - w$ pc of $B(X^*)$.

Theorem 3.1. *Let X be a Banach space and let $f_0 \in S(X^*)$. Then the following statements are equivalent*:

(i) *f_0 is a weak*-weak point of continuity of $B(X^*)$.*

(ii) *For any $x_0^{**} \in X^{**}$, if $x_0^{**} \notin f_0^{-1}(0) = \{x^{**} \in X^{**} : f_0(x^{**}) = 0\}$, then there exists a ball B^{**} in X^{**} with center in X such that $x_0^{**} \in B^{**}$ and $B^{**} \cap f_0^{-1}(0) = \varnothing$.*

Proof. (i) $\Rightarrow$ (ii). Let $H = f_0^{-1}(0)$ in X^{**} and $x_0^{**} \notin H$. Consider the subspace $M = \operatorname{span}\{\{x_0^{**}\} \cup X\} \subset X^{**}$, and hyperplane $H_0 = \mathit{span}\{H \cap X, x_0^{**}\}$ of M. Let $f \in M^*$ such that $\ker f$ in $M = H_0$ and $f|_X = f_0$; then we claim $\|f\| > \|f_0\| = 1$. If $\|f\| = \|f_0\| = 1$, then there exists $x^{***} \in S(X^{***})$ such that $x^{***}|_M = f$ by the extension theorem. Then $x^{***}(x_0^{**}) = f(x_0^{**}) = 0$, hence $x^{***} \neq f_0$ in X^{***} and $x^{***}|_X = f_0$. This contradicts the hypothesis that f_0 has unique Hahn-Banach extension in X^{***}.

Now $\|f\| > \|f_0\| = 1$ and $\|f|_X\| = \|f_0\| = 1$; hence, there exists $x \in X$ such that $f(x_0^{**} - x) > \|x_0^{**} - x\|$. Thus, for each $y \in H \cap X$,

$$\|y - x\| \geq f_0(y - x) = f(y - x) = f(-x) = f(x_0^{**} - x) > \|x_0^{**} - x\|.$$

Let $B^{**} = \{x^{**} \in X^{**} : \|x^{**} - x\| \leq \|x_0^{**} - x\|\}$. Then $B^{**} \cap (H \cap X) = \varnothing$. Hence either $\inf f_0(B^{**} \cap X) > 0$ or $\sup f_0(B^{**} \cap X) < 0$. Therefore $\inf f_0(B^{**}) > 0$ or $\sup f_0(B^{**}) < 0$, and so $H \cap B^{**} = \varnothing$.

(ii) $\Rightarrow$ (i). Suppose f_0 is not a $w^* - w$ pc of $B(X^*)$, then the Hahn-Banach extension of f_0 in X^{***} is not unique. Let $x^{***} \in S(X^{***})$ such that $x^{***}|_X = f_0$ and $x^{***} \neq f_0$ in X^{***}. Let $\ker(x^{***}) = \{x^{**} \in X^{**} : x^{***}(x^{**}) = 0\}$, $\ker f_0 = \{x^{**} \in X^{**} : f_0(x^{**}) = 0\}$. Then $\ker x^{***} \neq \ker f_0$. Choose $x^{**} \in \ker(x^{***}) \setminus \ker(f_0)$, then there exists a ball $B^{**} = \{y^{**} \in X^{**} : \|y^{**} - x\| \leq r\}$, where $x \in X$ such that $x^{**} \in B^{**}$ and $B^{**} \cap \ker(f_0) = \varnothing$. Hence either $\inf f_0(B^{**}) > 0$ or $\sup f_0(B^{**}) < 0$. Without loss of generality, suppose $\inf f_0(B^{**}) > 0$. Then

$$\begin{aligned} f_0(x) &= \sup_{y^{**} \in B^{**}} \{f_0(x - y^{**}) + f_0(y^{**})\} \\ &> \sup_{y^{**} \in B^{**}} \{f_0(x - y^{**})\} = r. \end{aligned}$$

Thus

$$\begin{aligned} x^{***}(x^{**}) &= x^{***}(x^{**}-x) + x^{***}(x) \\ &= x^{***}(x^{**}-x) + f_0(x) \\ &\geq -\|x^{**}-x\| + f_0(x) \\ &\geq -r + f_0(x) > -r + r = 0. \end{aligned}$$

This contradicts $x^{**} \in \ker(x^{***})$. $\square$

Theorem 3.2. *Let X be a Banach space. Then the following are equivalent*:

(i) *X is a Hahn-Banach smooth.*

(ii) *For every w^*-closed hyperplane H in X^{**} and for any $x^{**} \in X^{**}\backslash H$, there exists a ball B^{**} in X^{**} with center in X such that $x^{**} \in B^{**}$, and $B^{**} \cap H = \varnothing$.*

4. In this section we consider ball separation properties of X that are related to the weak* points of continuity (w^*-pc) of $B(X^*)$.

Definition 4.1. A Banach space X is said to have the *property* (II) if, for every bounded closed convex subset B in X, $B = \cap_{i\in I} K_i$, where for every $i \in I$, $K_i = \overline{\text{co}}\{\cup_{n=1}^{n} B_i\}$ for some balls $B_1, B_2, \dots, B_n$ in X.

Let $D_\varepsilon^* = \{f \in S(X^*)$. There exists a w^*-neighborhood V of f in $B(X^*)$ such that $\operatorname{diam} V < \varepsilon\}$.

Obviously, $\cap_{\varepsilon>0} D_\varepsilon^*$ is the set of all w^*-pc of $B(X^*)$.

Lemma 4.2. *Let X be a Banach space. Then, for every $f_0 \in S(X^*)$ and for any w^*-neighborhood V of f_0 in $B(X^*)$, there exists a w^*-neighborhood W of f_0 in $B(X^*)$ such that $W = \{f \in B(X^*) : f(y_i) > \eta_i > 0, i = 1, 2, \dots, n\}$, for some $y_1, y_2, \dots, y_n$ in X and $W \subset V$.*

Proof. Suppose $V = \{f \in B(X^*) : f(x_i) > \eta_i, x_i \in B(X^*), i = 1, 2, \dots, n\}$. We may suppose $\eta_i \geq -1$, $i = 1, 2, \dots, n$. Choose η_i' such that $f_0(x_i) > \eta_i' > \eta_i$. Then choose x_0 such that $\|x_0\| = 1$, $f_0(x_0) > \max\{1 - (\eta_i' - \eta_i), i = 1, 2, \dots, n\}$. Consider the w^*-open set of $B(X^*)$, $W = \{f \in B(X^*) : f(y_i) > f_0(x_0) + \eta_i' > 0, i = 1, 2, \dots, n\}$, where $y_i = x_0 + x_i$. Since $f_0(y_i) = f_0(x_0 + x_i) > f_0(x_0) + \eta'_i$, we have $f_0 \in W$. For every $f \in W$, $f(x_0) + f(x_i) = f(y_i) > f_0(x_0) + \eta'_i$, $i = 1, 2, \dots, n$. Hence $f(x_i) > f_0(x_0) + \eta'_i - 1 > 1 - (\eta'_i - \eta_i) + \eta'_i - 1 = \eta_i$, $i = 1, 2, \dots, n$. Therefore, $W \subset V$. □

We are ready now to prove the result corresponding to Theorem 1.3 for $\mathcal{A}$-pc.

Theorem 4.3. *Let X be a Banach space, and let $\mathcal{A}$ be a compatible family of bounded sets in X. Then for any $f_0 \in S(X^*)$ the following statements are equivalent*:

(i) *f_0 is a $\mathcal{A}$-pc of $B(X^*)$.*

(ii) *For every $A \in \mathcal{A}$, if $\inf f_0(A) > 0$, then there exist finitely many balls $B_1, B_2, \dots, B_n$ in X such that $A \subset \overline{\text{co}}\{\cup_{i=1}^n B_i\}$ and $\inf f_0(\overline{\text{co}}\{\cup_{i=1}^n B_i\}) > 0$.*

(iii) *For every $A \in \mathcal{A}$, if $\inf f_0(A) > \alpha$ for some real number α, then there exist finitely many balls $B_1, B_2, \dots, B_n$ in X such that $A \subset \overline{\text{co}}\{\cup_{i=1}^n B_i\}$ and $\inf f_0(\overline{\text{co}}\{\cup_{i=1}^n B_i\}) > \alpha$.*

Proof. (ii) ⇔ (iii). This is clear by the properties of $\mathcal{A}$.

(ii) ⇒ (i). For any $A \in \mathcal{A}$ and $\varepsilon > 0$, choose $x_0 \in X$ such that

$$\|x_0\| = \sup\{\|a\| : a \in A\} + 2\varepsilon \quad \text{and} \quad f_0(x_0) > \|x_0\| - \varepsilon. \tag{4.1}$$

Without loss of generality, we may assume that $\|x_0\| \le 1$. Let K be the closed absolutely convex hull of $A \cup \{x_0\}$ and $K_\varepsilon = \{x \in K : f_0(x) \ge \varepsilon\}$. Let $\eta = \varepsilon^2 / f_0(x_0)$, then, by (iii) there exist finitely many balls $B_1 = B(z_1, r_1), \dots, B_n = B(z_n, r_i)$ in X such that

$$K_\varepsilon \subset \overline{\text{co}}(\cup_i B_i) \quad \text{and} \quad \inf f_0(\overline{\text{co}}(\cup_i B_i)) > \varepsilon - \eta.$$

Choose $z_0 \in S(X)$ such that $f_0(z_0) > 1 - \varepsilon$. Consider $V = \{f \in B(X^*) : f(z_i) > r_i + \varepsilon - \eta,\ i = 1, \dots, n \text{ and } f(z_0) > 1 - \varepsilon\}$. Now

$$f_0(z_i) - r_i = \inf f_0(B(z_i, r_i)) > \varepsilon - \eta, \qquad \text{for all } i.$$

Hence

$$f_0(z_i) > r_i + \varepsilon - \eta, \qquad \text{for all } i$$

and so $f_0 \in V$.

Now for each $f \in V$, we have

$$\frac{f}{\|f\|}(z_i) \geqq f(z_i) > r_i + \varepsilon - \eta, \qquad \text{for all } i.$$

Hence

$$\inf \frac{f}{\|f\|}(B(z_i, r_i)) \geqq f(z_i) - r_i > \varepsilon - \eta.$$

Notice that

$$\varepsilon \frac{x_0}{f_0(x_0)} \in K_\varepsilon \subset \overline{\text{co}}\Big(\bigcup_i B(z_i, r_i)\Big).$$

So $(f/\|f\|)(\varepsilon(x_0/f_0(x_0))) > \varepsilon - \eta$. Thus

$$\frac{f}{\|f\|}(x_0) > f_0(x_0) - \frac{\eta}{\varepsilon} f_0(x) > f_0(x_0) - \varepsilon. \tag{4.2}$$

By (4.1) and (4.2), we have

$$\sup\{\|a\| : a \in A\} \leq \left\| \frac{f}{\|f\|} \right\|_K, \qquad \|f_0\|_K \leq \sup\{\|a\| : a \in A\} + 2\varepsilon.$$

Now

$$\inf \frac{f}{\|f\|}(K_\varepsilon) \geq \inf \frac{f}{\|f\|}(B_i) > \varepsilon - \eta > 0.$$

Hence

$$\inf f_0(f^{-1}(0) \cap K) < \varepsilon.$$

Applying the lemma in the normed space spanned by K with K as the unit ball, we have

$$\left\| \frac{f_0}{\|f_0\|_K} - \frac{f/\|f\|}{\|f/\|f\|\,\|_K} \right\|_K = \left\| \frac{f_0}{\|f_0\|_K} - \frac{f}{\|f\|_K} \right\|_K < 2 \frac{\varepsilon}{\|f_0\|_K}. \tag{4.3}$$

By (4.1) and (4.3) and the fact that $f(z_0) > 1-\varepsilon$, which implies that $1-\|f\| < \varepsilon$, we have

$$\begin{aligned}
\|f_0 - f\|_K &\le \left\|f_0 - \frac{\|f_0\|_K}{\|f\|_K} f\right\|_K + \left\|f - \frac{\|f_0\|_K}{\|f\|_K} f\right\|_K \\
&\le 2\varepsilon + |\|f\|_K - \|f_0\|_K| \\
&\le 2\varepsilon + \left|\left\|\frac{f}{\|f\|}\right\|_K - \|f_0\|_K\right| + \left|\|f\|_K - \left\|\frac{f}{\|f\|}\right\|_K\right| \\
&\le 2\varepsilon + 2\varepsilon + \|f\|_K\left(\frac{1}{\|f\|} - 1\right) \\
&\leqq 4\varepsilon + (1-\|f\|) \le 4\varepsilon + \frac{\varepsilon}{1-\varepsilon} \le 6\varepsilon.
\end{aligned}$$

Therefore,

$$\mathrm{diam}_A V \le 12\varepsilon.$$

This proves that f_0 is an $\mathcal{A}$-pc of $B(X^*)$.

(i) $\Rightarrow$ (ii). Suppose $A \in \mathcal{A}$ and $\inf f_0(A) = 2\alpha > 0$. Without loss of generality, assume $A \subset B(X)$. Now, using Lemma 4.2, we can choose

$$V = \{g \in B(X^*) : g(x_i) > \eta_i > 0,\ x_i \in X, i = 1, 2, \dots, k\},$$

a w^*-neighborhood of f_0 in $B(X^*)$ and $\mathrm{diam}_A V < \alpha$. Then $f_0(x_i) > \eta_i$, $i = 1, 2, \dots, k$. Choose ξ_i such that $f_0(x_i) > \xi_i > \eta_i > 0$, $i = 1, 2, \dots, k$. Choose $m > 0$ such that $\xi_i - 1/m > \eta_i$, $i = 1, 2, \dots, k$. And, finally, choose $x_0 \in X$ such that $\|x_0\| < \alpha/2$ and $f_0(x_0) > 0$. Let

$$K = \overline{\mathrm{co}}\{B(mx_1, m\xi_1) \cup \dots \cup B(mx_k, m\xi_k) \cup \{x_0\}\}.$$

Then, for each $i = 1, 2, \dots, k$,

$$\inf f_0(B(mx_i, m\xi_i)) = f_0(mx_i) - m\xi_i = m(f(x_i) - \xi_i) > 0.$$

Therefore

$$\begin{aligned}
&\inf f_0(K) \\
&\quad = \min\{\inf\{f_0(B(mx_i, m\xi_i)) : i = 1, 2, \dots, k\}, f_0(x_0)\} > 0.
\end{aligned}$$

It remains to prove $A \subset K$.

Suppose not. Choose $x \in A\backslash K$. By the separation theorem, there exists $g \in S(X^*)$ such that $\inf g(K) > g(x) \geq -1$. Hence $g(mx_i) - m\xi_i = \inf g(B(mx_i, m\xi_i)) \geq \inf g(K) > -1$, $i = 1, 2, \dots, k$. Then $g(x_i) > \xi_i - (1/m) > \eta_i$, $i = 1, 2, \dots, k$. That means $g \in V$. Thus $\|g - f_0\|_A < \alpha$. Now $\inf f_0(A) = 2\alpha$, hence

$$g(x) = f_0(x) - (f_0 - g)(x) \geq 2\alpha - \|f_0 - g\|_A > 2\alpha - \alpha = \alpha.$$

But $x_0 \in K$, so $\inf g(K) \leq g(x_0) \leq \|x_0\| < (\alpha/2)$. This contradicts with $\inf g(K) > g(x) > \alpha$. Therefore $A \subset K$ and completes the proof. □

Corollary 4.4. *Let X be a Banach space, and let $f_0 \in S(X^*)$. Then the following statements are equivalent*:

(i) *f_0 is a w^*-pc of $B(X^*)$.*

(ii) *For every bounded subset C in X, if $\inf f_0(C) > 0$, then there exist finitely many balls $B_1, B_2, \dots, B_n$ in X such that $C \subset \overline{\text{co}}\{\cup_{i=1}^n B_i\}$ and $(\overline{\text{co}}\{\cup_{i=1}^n B_i\}) \cap H = \varnothing$, where $H = \{x \in X : f_0(x) = 0\}$.*

(iii) *For every bounded subset $C \subset X^{**}$, if $\inf f_0(C) > 0$, then there exist finitely many balls $B_1, B_2, \dots, B_n$ in X^{**} with centers in X such that*

$$C \subset \overline{\text{co}}\Big\{\bigcup_{i=1}^n B_i\Big\} \quad \text{and} \quad \Big(\overline{\text{co}}\Big\{\bigcup_{i=1}^n B_i\Big\}\Big) \cap H^{**} = \varnothing,$$

where $H^{**} = \{x \in X^{**} : f_0(x) = 0\}$.

Proof. (ii) $\Leftrightarrow$ (i). By Theorem 4.3.

(ii) $\Leftrightarrow$ (iii). The proof is similar to the proof of Corollary 1.4. □

Corollary 4.5. *Let X be a Banach space. Then the following statements are equivalent*:

(i) *every element of $S(X^*)$ is a w^*-pc of $B(X^*)$;*

(ii) *for every bounded subset C of X and every closed hyperplane H in X, if $\operatorname{dist}(C, H) > 0$, then there exist finitely many balls $B_1, B_2, \dots, B_n$, such that $C \subset \overline{\text{co}}\{\cup_{i=1}^n B_i\}$ and $\overline{\text{co}}\{\cup_{i=1}^n B_i\} \cap H = \varnothing$;*

(iii) *for every bounded subset C of X^{**} and every closed hyperplane H^{**} in X^{**}, if $\operatorname{dist}(C, H^{**}) > 0$, then there exist finitely many balls*

$B_1^{**}, B_2^{**}, \dots, B_n^{**}$, *with centers in* X *such that* $C \subset \overline{\text{co}}\{\cup_{i=1}^n B_i^{**}\}$ *and* $\overline{\text{co}}\{\cup_{i=1}^n B_i^{**}\} \cap H^{**} = \varnothing$.

Theorem 4.6. *A Banach space* X *has the property* (II) *if and only if the set of* w^**-pc of* $B(X^*)$ *is norm dense in* $S(X^*)$.

Proof. $\Leftarrow$. For any bounded closed convex subset C in X and any $x \notin C$, since w^*-pc of $B(X^*)$ are dense in $S(X^*)$, there is a w^*-pc f of $B(X^*)$ such that $\inf f(C) > f(x)$. Without loss of generality, suppose $f(x) = 0$. Then by Corollary 4.4, there exist finitely many balls $B_1, B_2, \dots, B_n$, such that $C \subset \overline{\text{co}}\{\cup_{i=1}^n B_i\}$ and $(\overline{\text{co}}\{\cup_{i=1}^n B_i\}) \cap f^{-1}(0) = \varnothing$. In particular, $x \notin \overline{\text{co}}\{\cup_{i=1}^n B_i\}$.

$\Rightarrow$. It is clear that $D_\varepsilon^* = \{f \in S(X) :$ there exists a w^*-neighborhood V of f in $B(X^*)$ such that $\operatorname{diam} V < \varepsilon\}$ is an open subset of $S(X^*)$. It suffices to prove D_ε^* is dense in $S(X^*)$, then by Baire's theorem, $\cap_{\varepsilon>0} D_\varepsilon^* = \cap_{1/n} D_{1/n}^*$ is a G_δ dense set of $S(X^*)$.

Now, for any $\varepsilon > 0$ and any $\delta > 0$, take $\delta' = \min\{\varepsilon/4, \delta/3\}$. For each $f_0 \in S(X^*)$, consider the bounded closed convex subset

$$B_{\delta'} = \{x \in B(X) : f_0(x) \geq \delta'\} .$$

Then there exist finitely many balls $B(x_1, r_1)$, $B(x_2, r_2), \dots, B(x_k, r_k)$ such that $B_{\delta'} \subset \overline{\text{co}}\{B(x_1, r_1) \cup \dots \cup B(x_k, r_k)\} \overset{\text{def}}{=} A$ and $0 \notin A$. By the separation theorem, there exists $f \in S(X^*)$ such that

$$\inf f(A) > f(0) = 0. \tag{4.4}$$

Since $\inf f(B_{\delta'}) > \inf f(A) > 0$, by the lemma, $\|f - f_0\| < 2\delta' < \delta$. Let

$$V = \left\{ g \in B(X^*) : g\left(\frac{x_i}{\|x_i\|}\right) > \frac{r_i}{\|x_i\|}, i = 1, 2, \dots, k \right\}.$$

Since

$$f(x_i) - r_i = \inf f(B(x_i, r_i)) \geq \inf f(A) > 0, \quad i = 1, 2, \dots, k.$$

So for $i = 1, 2, \dots, k$, $f(x_i) > r_i$, and $f(x_i/\|x_i\|) > r/\|x_i\|$. Hence $f \in V$.

For any $g \in V$, $g(x_i/\|x_i\|) > r_i/\|x_i\|$. Hence $g(x_i) - r_i > 0$, and so

$$\inf g(B(x_i, r_i)) = g(x_i) - r_i\|g\| > 0.$$

Therefore, $\inf g(B'_\delta) \geq \inf g(A) = \min\{\inf g(B(x_i, r_i)) : i = 1, 2, \dots, k\} > 0$. By the lemma $\|g - f_0\| < 2\delta'$. By (4.2), $\|f - f_0\| < 2\delta'$. Hence $\operatorname{diam} V \leq 4\delta' = \varepsilon$. Thus $f \in D^*_\varepsilon$. This completes the proof that D^*_ε is dense in $S(X^*)$. □

The proof of the following result is similar to the one given in [**2**].

Theorem 4.7. *Let X be an Asplund space with the property (II). Then, for each closed subspace Y in X, there exists a subspace Z in X containing Y with the same density as Y and Z has the property (II).*

Proof. Since X is an Asplund space, every nonempty bounded subset of X^* is w^*-dentable. By a theorem of [**15**] there are continuous functions $f_n : (X, \|\ \|) \to (X^*, \|\ \|)$, $n \in N$, such that $f_0(x) = \lim_{n\to\infty} f_n(x)$ exists in $(X^*, \|\ \|)$ and $f_0(x) \in D(x)$, $x \in X$. Define $f(x) = \{f_1(x), f_2(x), \dots\}$, $x \in X$. Then, by [**7**], f is norm-norm lower semi-continuous and, for every subspace Z in X,

$$Z^* = \{x^*|_Z : x^* \in f(x), x \in Z\} \tag{4.5}$$

Let $Y \neq \{0\}$ be a subspace of X, and let $\alpha = \text{density}\, Y$. Let $Z_0 = Y$, then density $(f(Z_0)) \leq \alpha$. Since X has property (II), $S(X^*) = \overline{A}$ where A is the set of all w^*-pc of $B(X^*)$. Choose $A_0 \subset A$, $\overline{A}_0 \supset S([f(Z_0)])$ and $\operatorname{card} A_0 \leq \alpha$. Next, choose a subspace $Z_1 \supset Z_0$, density $Z_1 = \alpha$, $\|x^*\| = \|x^*|Z_1\|$ for all $x^* \in [A_0]$, the span of A_0 and every $x^* \in A_0$ is a Z_1-pc of $B(X^*)$, that is, for all $\varepsilon > 0$, there is a w^*-neighborhood of $B(X^*)$ containing x^* which is determined by some elements in Z_1 and has its diameter less than ε. Continuing by induction, there exist subspaces Z_n in X and subsets $A_n \subset A$ such that

(i) density $Z_n \leq \alpha$ and $|A_n| \leq \alpha$;

(ii) $Z_{n+1} \supset Z_n$, and $A_{n+1} \supset S([A_n])$;

(iii) $\overline{A_n} \supset S([f(Z_n)])$;

(iv) $\|x^*\| = \|x^*|Z_{n+1}\|$, for all $x \in [A_n]$;

(v) every $x^* \in A_n$ is a Z_{n+1}-pc of $B(X^*)$.

Set $Z = \overline{\cup_0^\infty Z_n}$, $E = [\cup_0^\infty A_n]$ and $T : E \to Z^*$ by $T(x^*) = x^*|Z$, $x^* \in E$. Then Z and Y have the same density, $E \supset [\cup_0^\infty f(Z_n)]$, $S(E) = \overline{\cup_0^\infty A_n}$. T is an isometry and every element in $T(\cup_0^\infty A_n)$ is a w^*-pc of $B(Z^*)$. Since f is norm-norm lower semi-continuous, $\overline{f(\cup_0^\infty Z_n)} \supset f(\overline{\cup_0^\infty A_n}) = f(Z)$. Hence $E \supset f(Z)$, by (4.5), $Z^* = [T(f(Z))] = T(E)$. It follows that $T(\cup A_n)$ is a dense subset of $S(Z^*)$ and S_0, the set of w^*-pc points of $B(Z^*)$, is dense in $S(Z^*)$. Therefore, Z has the property (II). □

A Banach space X is said to have the weak$^*(II)$ property if every bounded weak* closed convex set A in X^* can be represented by $A = \cap_{i\in K} K_i$ where K_i is the closed convex hull of finitely many balls in X^* for each $i \in I$. As in the proof of Theorem 4.6, it can be proved that X has the property weak$^*(II)$ if and only if the set of points of continuity of $B(X)$ is norm dense in $S(X)$.

Theorem 4.8. *If a Banach space X has the weak*(II) property and $Y \subset X$ is a infinite dimensional subspace of X, then there exists a subspace Z of X such that density $Z =$* density Y *and Z has the weak*(II) property.*

Proof. Suppose density $Y = \alpha$. Let $Z_1 = Y$. Choose $D_1 \subset S(Z_1)$ dense in $S(Z_1)$ and $|D| = \alpha$. Let A be a subset of the set of points of continuity of $B(X)$ that is dense in $S(X)$. Choose $A_1 \subset A$ such that $\overline{A_1} \supset D_1$ and $|A_1| = \alpha$. Since $\overline{A_1} \supset D_1$, $\overline{A_1} \supset \overline{D_1} = S(Z_1)$. Let $Z_2 = [Z_1 \cup A_1]$. Continuing by induction, we can find sequences $\{Z_n\}$ with the following properties:

(i) $Z_1 \subset Z_2 \subset \cdots$;

(ii) density $(Z_n) = \alpha, n = 1, 2, 3 \cdots$;

(iii) $A_n \subset A, A_n \subset Z_{n+1}$, and $\overline{A_n} \supset S(z_n), n = 1, 2, \cdots$.

Let $Z = [\overline{\cup Z_n}]$, then density $(Z) = \alpha$ and $\overline{A_n} \supset S(\cup Z_n)$. So

$$\overline{\cup A_n} \supset \overline{S(\cup Z_n)} = S([\cup Z_n]) = S(Z).$$

Now $A_n \subset Z_{n+1}$, hence $\cup A_n \subset Z$. Since $A_n \subset A \subset S(Z)$, $\overline{\cup A_n} \subset S(Z)$, so $\overline{\cup A_n} = S(Z)$. Therefore $\cup A_n$ is a dense subset of points of continuity of $S(Z)$. □

Theorem 4.9. *Let X be a Banach space. If each separable subspace of X has the weak* (II) property, then so does X.*

Proof. Suppose X does not have the weak*(II) property. As in the proof of Theorem 4.6, there exists $\varepsilon_0 > 0$ such that $D_{\varepsilon_0} = \{x : x \in S(X)$, there exists a weak neighborhood V of x in $B(X)$ such that $\operatorname{diam} V < \varepsilon_0\}$ is not dense in $S(X)$. Hence, there exists $x_0 \in S(X)$ and $\delta > 0$ such that

$$B(x_0, \delta) \cap D_{\varepsilon_0} = \varnothing. \tag{4.6}$$

Choose separable subspace $Z_1 \subset X$ and $x_0 \in Z_1$. Let $B_{Z_1}(x, r) = \{y : y \in Z_1, \|y - x\| \le r\}$ for $x \in Z_1$. Let A_1 be a countable dense subset of $B_{Z_1}(x_0, \delta) \cap S(Z_1)$. Then $A_1 \cap D_{\varepsilon_0} = \varnothing$ by (4.6). Hence for each $x \in A_1$, by the definition of D_{ε_0}, $x \in \overline{B(X) \backslash B(x, \varepsilon_0)}^w$.

By following Kaplansky's theorem:

For any subset A of a Banach space, if $x \in \overline{A}^w$, then there exists a countable subset $B \subset A$ such that $x \in \overline{B}^w$. There exists a countable subset $B_x \subset B(X) \backslash B(x_0, \varepsilon)$ such that $x \in \overline{B_x}^w$. Let $Z_2 = [Z_1 \cup \cup_{x \in A_1} B_x]$, then Z_2 is separable.

Continuing by induction, we get:

(i) Subspaces: $Z_1 \subset Z_2 \subset \cdots$.

(ii) A_n dense in $B_{Z_n}(x_0, \delta) \cap S(Z_n)$ and A_n is countable.

(iii) For $x \in A_n$, there is $B_x \subset B(Z_{n+1}) \backslash B(x_0, \varepsilon)$ such that $x \in \overline{B_x}^w$.

Let $Z = [\cup Z_n]$. Then Z is separable and $\cup_{n=1}^{\infty} A_n$ is dense in $B_z(x_0, \delta) \cap S(Z)$. By (iii),

$$\left(\bigcup_{n=1}^{\infty} A_n \right) \cap D_{\varepsilon_0}(Z) = \varnothing. \tag{4.7}$$

Hence, for each $y \in B_Z(x_0, \delta) \cap S(Z)$ and any w-neighborhood V of y, there exists $x \in \cup_{n=1}^{\infty} A_n$ and $x \in V$. By (4.7), $\operatorname{diam} V \ge \varepsilon_0$.

Hence $y \notin D_{\varepsilon_0}(Z)$. Therefore, $D_{\varepsilon_0}(Z)$ is not dense in $S(Z)$. That is a contradiction. □

Theorem 4.10. *If every separable subspace of a Banach space X has the property (II), then X also has (II).*

Proof. Suppose the Banach space X does not have the property (II); then there exists a bounded closed convex subset K in X and a point x_0 not in K such that, for any finitely many balls $B_1, B_2, \dots, B_n$, if $K \subset \overline{\text{co}}(\cup_1^n B_n)$, then $x_0 \in \overline{\text{co}}(\cup_1^n B_n)$. Take a separable subspace Y_1 such that $x_0 \in Y_1$. Let A_1 be a countable dense subset of Y_1, and let

$$\mathcal{F}_1 = \{F : x_0 \notin F,\ F = \overline{\text{co}}(\cup B_n)\},$$

where $B_i = B(x_i, r_i) = \{x \in X : \|x - x_i\| \leq r_i\}$ and $x_i \in A_1, r_1 \in Q$ where Q is the set of all rational numbers. Then $\mathcal{F}_1$ is a countable set. For each $F \in \mathcal{F}_1$, since $x_0 \notin F$, $K - F \neq \varnothing$. Choose $x_F \in K - F$. Let $Y_2 = [Y_1 \cup \{x_F : F \in \mathcal{F}_1\}]$; then Y_2 is separable. Choose $A_2 \supset A_1$ to be a countable dense subset in Y_2, and define

$$\mathcal{F}_2 = \Big\{F : x_0 \notin F, F = \overline{\text{co}}\Big(\bigcup B_n\Big)\Big\},$$

where $B_i = B(x_i, r_i) = \{x \in X : \|x - x_i\| \leq r_i\}$ and $x_i \in A_2, r_i \in Q$.

Continuing by induction, there exist separable subspaces $Y_n \subset X$, countable sets A_n and $\mathcal{F}_n$ such that

(1) $x_0 \in Y_n$;

(2) $A_n \subset A_{n+1}$;

(3) A_n is dense subset in Y_n;

(4) $\mathcal{F}_n$ is countable.

Now let $Y = [\cup Y_n]$, $K_0 = K \cap Y$. For any finitely many balls $B_i = B_Y(y_i, r_i) = \{y \in Y : \|y - y_i\| \leq r_i\}$ in Y, if $\overline{\text{co}}(\cup_{i=1}^n B_i) \supset K_0$, we claim that $x_0 \in \overline{\text{co}}(\cup_{i=1}^n B_i)$. If not, then there exists $y_0^* \in S(Y^*)$ such that $d = y_0^*(x_0) - \sup\ y_0^*(\overline{\text{co}} \cup_{i=1}^n B_i) > 0$. Now $A = \cup_1^\infty A_n$ is dense in Y. For each $i = 1, 2, \dots, n$, choose $x_i \in A$ such that $\|x_i - y_i\| < d/4$, and choose $q_i \in Q$ such that $r_i + (d/4) < q_i < r_i + d/2$. Then

$$B_Y(y_i, r_i) \subset B_Y(x_i, q_i) \subset B_Y\left(y_i, r_i + \frac{3d}{4}\right).$$

Since $\sup y_0^*(B_Y(y_i, r_i)) \leqq y_0^*(x_0) - d$, it is easy to see

$$\sup y_0^*(B_Y(x_i, q_i)) < y_0^*(x_0) - \frac{d}{4}, \qquad i = 1, 2, \dots, n.$$

Hence

$$\sup y_0^*\left[\overline{\mathrm{co}} \bigcup_{i=1}^{n} B_Y(x_i, q_i)\right] \leq y_0^*(x_0) - \frac{d}{4}.$$

Choose m large enough, such that $x_i \in A_m$, $i = 1, 2, \dots, n$. By the definition of $\mathcal{F}_m$,

$$F = \overline{\mathrm{co}} \bigcup_{i=1}^{n} B(x_i, q_i) \in \mathcal{F}_m.$$

Now $x_F \in K - F$ and $x_F \in Y_{m+1} \cap Y$. Thus $x_F \in K \cap Y = K_0$ and $x_F \notin \overline{\mathrm{co}} \cup_{i=1}^{n} B_Y(x_i, q_i)$. Then $x_F \in K_0 - \overline{\mathrm{co}} \cup_{i=1}^{n} B_Y(x_i, q_i)$. This contradicts $K_0 \subset \overline{\mathrm{co}} \cup_{i=1}^{n} B_Y(x_i, q_i)$. Therefore $x_0 \in \overline{\mathrm{co}} \cup_{i=1}^{n} B_i$. This shows that Y does not have the property (II). □

5. Let X be a Banach space, and let $\mathcal{B}$ be the family of unit balls determined by the set of equivalent norms on X. Let h be the Hausdorff metric on $\mathcal{B}$, that is, $h(B_1, B_2) = \inf\{\varepsilon > 0 : B_1 \subset B_2 + \varepsilon B_2, B_2 \subset B_1 + \varepsilon B_1\}$ for B_1, B_2 in $\mathcal{B}$. It is well known that $(\mathcal{B}, h)$ is a complete metric space. If X has the property (I) then it is proved in [**9**] that there exists a dense G_δ set $\mathcal{B}_0$ in $\mathcal{B}$ such that, for every norm $\|\cdot\|_B$ in $\mathcal{B}_0$, $(X, \|\cdot\|_B)$ has the property (I). In this section we show that, for every Banach space X, there exists a dense G_δ set $\mathcal{B}_0$ in $\mathcal{B}$ such that for any compatible family of bounded sets $\mathcal{A}$ in X, if f is an $\mathcal{A}$-denting point of $B(X^*)$, then, for every $B \in \mathcal{B}_0$, f is an $\mathcal{A}$-denting point of the ball in $(X, \|\cdot\|_B)^*$ with center at origin and radius $\|f\|_B$.

We first state two simple lemmas.

Lemma 5.1. *Let $\delta > 0$, $r > 0$, B_0, $B \in \mathcal{B}$. If $h(B_0, B) < \delta$, $z \in X$, $f \in X^*$, then $z + rB_0 \subset z + (r/(1-\delta))B$, and*

$$\inf f\left(z + \frac{r}{1-\delta} B\right) \geq \inf f(z + rB_0) - \frac{2\delta}{1-\delta} r\|f\|_{B_0}.$$

Proof. If $h(B_0, B) < \delta$, then $(1-\delta)B_0 \subset B \subset (1+\delta)B_0$. Hence $z + rB_0 \subset z + (r/(1-\delta))B$ and

$$\begin{aligned}\inf f\left(z + \frac{r}{1-\delta}B\right) &\geq \inf f\left(z + \frac{r}{1-\delta}(1+\delta)B_0\right) \\ &= f(z) - \frac{r}{1-\delta}(1+\delta)\|f\|_{B_0} \\ &= \inf f(z + rB_0) - \frac{2\delta}{1-\delta}r\|f\|_{B_0}. \qquad \square\end{aligned}$$

Lemma 5.2. *Suppose $B_0 \in \mathcal{B}$, $r > 0$, $\varepsilon > 0$, $z \in X$, $x \in B_0$, $f \in X^*$, and let $B = \overline{B_0 + \varepsilon B(X)} \in \mathcal{B}$. Then $(r/\varepsilon)B + z - (r/\varepsilon)x \supset z + rB(X)$ and*

$$\inf f\left(\frac{r}{\varepsilon}B + z - \frac{r}{\varepsilon}x\right) = \inf f(z + rB(X)) - \frac{r}{\varepsilon}(f(x) + \|f\|_{B_0}).$$

Proof.

$$\frac{r}{\varepsilon}B + z - \frac{r}{\varepsilon}x \supset \frac{r}{\varepsilon}(x + \varepsilon B(X)) + z - \frac{r}{\varepsilon}x = z + rB(X),$$

and

$$\begin{aligned}\inf f\left(\frac{r}{\varepsilon}B + z - \frac{r}{\varepsilon}x\right) &= \frac{r}{\varepsilon}(\inf f(B_0) + \varepsilon \inf f(B(X))) \\ &\quad + f(z) - \frac{r}{\varepsilon}f(x) \\ &= -\frac{r}{\varepsilon}\|f\|_{B_0} + r\inf f(B(X)) \\ &\quad + f(z) - \frac{r}{\varepsilon}f(x) \\ &= \inf f(z + rB(X)) \\ &\quad - \frac{r}{\varepsilon}(f(x) + \|f\|_{B_0}). \qquad \square\end{aligned}$$

Let X be a normed space, and let $n, k \in \mathbf{N}$. Define $\mathcal{W}_{n,k} = \{(f, A) : f \in B(X^*)$, $A \subset X^{**}$, and there exists $z_1, \ldots, z_l$ in X, $r_1, \ldots, r_l$ in

$[0,k]$ such that

$$A \subset \overline{\text{co}} \bigcup_{i=1}^{l} (z_i + r_i B(X^{**}))$$

and

$$\inf f(\overline{\text{co}} \bigcup_{i=1}^{\ell} (z_i + r_i B(X^{**}))) > \frac{1}{3n} \Big\},$$

$B_{n,k,m} = \{B \in \mathcal{B}$: there exists $\gamma \in (0,1)$ such that for all $(f,A) \in \mathcal{W}_{n,k}$ there exists $z_1, \dots, z_l \in X$, $r_1, \dots, r_l \in [0, m-\gamma]$ with $A \subset \overline{\text{co}} \cup_{i=1}^{\ell}$ $(z_i + r_i)$ and $\inf f(\overline{\text{co}}(\cup_{i=1}^{\ell}(z_i + r_i B))) > \gamma\}$, and

$$\mathcal{B}_0 = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{B}_{n,k,m}.$$

Also define $\mathcal{W}'_{n,k} = \{(f,A) : f \in B(X^*),\ A \subset X^{**}$ such that there exists $z \in X$ and $r \le k$ with $A \subset z + rB(X^{**})$ and $\inf f(z + rB(X)) > 1/(3n)\}$.

$\mathcal{B}'_{n,k,m} = \{B \in \mathcal{B}$: there exists $\gamma \in (0,1)$ such that for all $(f,A) \in \mathcal{W}'_{n,k}$, there exist $z \in X$ and $r \in [0, m-\gamma]$ with $A \subset z + rB^{**}$ and $\inf f(z + rB) > \gamma\}$ and

$$\mathcal{B}'_0 = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{B}'_{n,k,m}.$$

Theorem 5.3. *$\mathcal{B}_{\prime}$, $\mathcal{B}'_0$ are dense G_δ subsets of $\mathcal{B}$.*

Proof. We shall prove that $\mathcal{B}_0$ is a dense G_δ subset of $\mathcal{B}$, the proof for $\mathcal{B}'_0$ is similar.

Claim 1. $\mathcal{B}_{n,k,m}$ is open. Let $B_0 \in \mathcal{B}_{n,k,m}$. Then there exists $\gamma \in (0,1)$ such that for every $(f,A) \in \mathcal{W}_{n,k}$, there exist $z_1, \dots, z_l \in X$ and $r_1, \dots, r_l \in [0, m-\gamma]$ satisfying

$$A \subset \overline{\text{co}} \bigg(\bigcup_{i=1}^{\ell} (z_i + r_i B_0^{**}) \bigg),$$

and

$$\inf f\left(\overline{\mathrm{co}}\left(\bigcup_{i=1}^{\ell}(z_i + r_i B_0)\right)\right) > \gamma.$$

Let $M = \sup\{\|x\| : x \in B_0\}$. Choose $\delta > 0$ such that

$$\frac{2\delta}{1-\delta} mM \leq \frac{\gamma}{2} \quad \text{and} \quad \frac{m-r}{1-\delta} \leq m - \frac{\gamma}{2}.$$

By Lemma 5.1, for all $B \in \mathcal{B}$, $h(B_0, B) < \delta$ and $i = 1, \dots, l$, we have

$$z_i + r_i B_0 \subset z_i + \frac{r_i}{1-\delta} B,$$

and

$$\inf f\left(z_i + \frac{r_i}{1-\delta} B\right) \geq \inf f(z_i + r_i B_0) - \frac{2\delta}{1-\delta} r_i \|f\|_{B_0}$$
$$> \gamma - \frac{2\delta}{1-\delta} mM > \gamma - \frac{\gamma}{2} = \frac{\gamma}{2}.$$

Thus

$$A \subset \overline{\mathrm{co}}\left(\bigcup_{i=1}^{\ell}(z_i + r_i B_0^{**})\right) \subset \overline{\mathrm{co}}\left(\bigcup_{i=1}^{\ell}\left(z_i + \frac{r_i}{1-\delta} B^{**}\right)\right),$$

and

$$\inf f\left(\overline{\mathrm{co}}\left(\bigcup_{i=1}^{\ell}\left(z_i + \frac{r_i}{1-\delta} B\right)\right)\right) > \frac{\gamma}{2}.$$

Observe that

$$\frac{r_i}{1-\delta} \leq \frac{m-\gamma}{1-\delta} \leq m - \frac{\gamma}{2}.$$

This shows that $\mathcal{B}_{n,m,k}$ is open.

Claim 2. $\mathcal{B}_{n,k,m}$ is $(2k/m)$ dense. Let $B_0 \in \mathcal{B}$. Choose $s > 0$ large enough such that $\varepsilon = 12nsk/(12nsm - 1) < 2k/m$. Denote $B = \overline{B_0 + \varepsilon B(X)}$. Let $(f, A) \in \mathcal{W}_{n,k}$, then there exists $z_1, \dots, z_l \in X$ and $r_1, \dots, r_l \leq k$ such that

$$A \subset \overline{\mathrm{co}}\left(\bigcup_{i=1}^{\ell}(z_i + r_i B(X^{**}))\right)$$

and

$$\inf f\left(\overline{\text{co}}\bigcup_{i=1}^{\ell}(z_i + r_i B(X^{**}))\right) > \frac{1}{3n}.$$

Choose $\gamma \in (0, 1/(12ns))$ and $x \in B_0$ such that $(k/\varepsilon)(f(x) + \|f\|_{B_0}) < 1/(6n)$. By Lemma 5.2,

$$\frac{r_i}{\varepsilon}B + z_i - \frac{r_i}{\varepsilon}x \supset z_i + r_i B(X),$$

and

$$\begin{aligned}\inf f\left(\frac{r_i}{\varepsilon}B + z_i - \frac{r_i}{\varepsilon}x\right) &= \inf f(z_i + r_i B(X)) - \frac{r_i}{\varepsilon}(f(x) + \|f\|_{B_0}) \\ &> \frac{1}{3n} - \frac{k}{\varepsilon}(f(x) + \|f\|_{B_0}) \\ &> \frac{1}{3n} - \frac{1}{6n} > \gamma.\end{aligned}$$

Therefore,

$$\begin{aligned}A &\subset \overline{\text{co}}\left(\bigcup_{i=1}^{\ell}(z_i + r_i B(X^{**}))\right) \\ &\subset \overline{\text{co}}\left(\bigcup_{i=1}^{\ell}\left(\frac{r_i}{\varepsilon}B^{**} + z_i - \frac{r_i}{\varepsilon}x_i\right)\right),\end{aligned}$$

and

$$\inf f\left(\overline{\text{co}}\left(\bigcup_{i=1}^{\ell}\left(\frac{r_i}{\varepsilon}B + z_i - \frac{r_i}{\varepsilon}x_i\right)\right)\right) > \frac{1}{6n}.$$

Also

$$\frac{r_i}{\varepsilon} \leq \frac{k}{\varepsilon} = \frac{12nsm - 1}{12ns} = m - \frac{1}{12ns}.$$

Thus $B \in \mathcal{B}_{n,k,m}$ and Claim 2 is proved.

By the Baire category theorem, the set

$$\mathcal{B}_0 = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\mathcal{B}_{n,k,m},$$

is a dense G_δ-subset of $\mathcal{B}$. □

Theorem 5.4. *If $f \in X^*$ is an $\mathcal{A}$-denting point of $B(X^*)$, respectively $\mathcal{A}$-point of continuity, then, for each $B_0 \in \mathcal{B}_0$, respectively $B_0 \in \mathcal{B}'_0$, f is an $\mathcal{A}$-denting point, respectively $\mathcal{A}$-point of continuity, of the ball in $(X, \|\cdot\|_{B_0})^*$ with center at origin and radius $\|f\|_{B_0}$.*

Proof. Suppose that $f \in S(X^*)$ is an $\mathcal{A}$-denting point of $B(X^*)$. Fix $B_0 \in \mathcal{B}_0$. For each $A \in \mathcal{A}$, with $\inf f(A) > 0$, by Theorem 1.3, $(f, A) \in \mathcal{W}'_{n_0,k_0}$ for some n_0, k_0. Now

$$B_0 \in \mathcal{B}'_0 = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{B}'_{n,k,m}.$$

So $B_0 \in \mathcal{B}'_{n_0,k_0,m_0}$ for some m_0. By the definition of $\mathcal{B}'_{n,k,m}$, there is a ball $B = z + rB_0$ such that

$$A \subset B \quad \text{and} \quad \inf f(B) > 0.$$

By Theorem 1.3 again, this shows that f is an $\mathcal{A}$-denting of the ball in $(X, B_0)^*$ with center at the origin and radius $\|f\|_{B_0}$.

The proof for $\mathcal{A}$-pc is similar. □

Corollary 5.5. *If $f \in X^*$ is a w^*-denting point, respectively extreme point, of $B(X^*)$, then for each $B_0 \in \mathcal{B}_0$, f is a w^*-denting point, respectively extreme point, of the ball in $(X, \|\cdot\|_{B_0})$. With the center at origin and radius $\|f\|_{B_0}$.*

Corollary 5.6. *If $f \in X^*$ is a w^*-point of continuity of $B(X^*)$, then for each $B_0 \in \mathcal{B}'_0$, f is a w^*-point of continuity of the ball in $(X, \|\cdot\|_{B_0})^*$ with center at the origin and radius $\|f\|_{B_0}$.*

Corollary 5.7. *If $(X, B(X))^*$ has the property that every point of $S(X^*)$ is a weak* denting point of $B(X^*)$, respectively is strictly convex, then for each $B_0 \in \mathcal{B}$, $(X, B_0)^*$ has the property that every point of $S(X^*)$ is a weak* denting point of $B(X^*)$, respectively is strictly convex.*

Similarly, by using Theorem 3.1, one can prove:

Theorem 5.8. *If $f \in X^*$ is a weak*-weak point of continuity of $(X, B(X))^*$, then for each $B_0 \in \mathcal{B}_0'$, f is a weak*-weak point of continuity of a ball in $(X, \|\cdot\|_{B_0})^*$ with center at the origin and radius $\|f\|_{B_0}$. As a consequence, if X is Hahn-Banach smooth, then $(X, \|\cdot\|_{B_0})$ is Hahn-Banach smooth for every $B_0 \in \mathcal{B}_0'$.*

Corollary 5.9. *If $(X, B(X))$ has the property (I), respectively (II), then for each $B_0 \in \mathcal{B}_0'$, respectively $B_0 \in \mathcal{B}_0)$, (X, B_0) has the property (I), respectively (II).*

A Banach space X is called nicely smooth if, for all $x^{**} \neq y^{**}$ in X^{**}, there exists a ball B^{**} in X^{**} with center in X such that $x^{**} \in B^{**}$ and $y^{**} \notin B^{**}$.

Corollary 5.10. *If $(X, B(X))$ is nicely smooth, then for each $B_0 \in \mathcal{B}_0'$, (X, B_0) is nicely smooth.*

Remarks. (a) It was proved in [**9**] that if $(X, B(X))$ has the property (I), then there exists a dense G_δ subset $\mathcal{B}_0$ of $\mathcal{B}$ such that for each $B_0 \in \mathcal{B}_0$, $((X, B_0))$ has the property (I). Corollary 5.5 includes this result.

(b) If we check the proof carefully we can see that, for each $B_0 \in \mathcal{B}_0'$, respectively $B_0 \in \mathcal{B}_0$, B_0 inherits the following separation properties of $B(X)$: if one bounded set of X^{**} and one point in X^{**} or a w^*-closed hyperplane of X^{**} can be separated by a ball, respectively by a finite union of balls, of $(X, B(X))^{**}$ with center at X, they can also be separated by a ball, respectively a finite union of balls, of $(X, B_0)^{**}$ with centers in X.

(c) It is clear that we can establish the dual analogous assertions. In fact, if X is a dual space, let $\mathcal{B}$ be all equivalent dual norms of X; then Theorem 5.3 still holds true.

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