ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1998

ASYMPTOTIC FORMULAE OF LIOUVILLE-GREEN TYPE FOR A GENERAL FOURTH-ORDER DIFFERENTIAL EQUATION

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ABSTRACT. As asymptotic form of solutions of Liouville-Green type for a general fourth-order differential equation are given under general conditions on the coefficients for large x.

1. Introduction. In this paper we consider the asymptotic form of four linearly independent solutions of a general fourth-order differential equation

$$(1.1)\quad (p_0y'')'' + (p_1y')' + \frac{1}{2}\sum_{j=0}^{1}[\{q_{2-j}\cdot y^{(j)}\}^{(j+1)} + \{q_{2-j}\cdot y^{(j+1)}\}^{(j)}] - p_2y = 0$$

as $x \to \infty$, where x is the independent variable and the prime denotes d/dx. The functions p_j, $1 \leq j \leq 3$, and q_j, $j = 1, 2$, are defined on an interval $[a, \infty)$ and are not necessarily real-valued, while p_0 is nowhere zero in this interval. We shall consider the case where the three functions $q_1(p_2/p_0)^{3/4}$, $p_1(p_2/p_0)^{1/2}$ and $q_2(p_2/p_0)^{1/4}$ are all small compared to p_2 as $x \to \infty$.

In this case the solutions all have a similar exponential factor as given below in Theorem 4.1.

In the case where $p_1 = q_1 = q_2 = 0$, (1.1) reduces to

$$(1.2)\qquad (p_0y'')'' - p_2y = 0$$

which is the case $n = 4$ of the nth order equation considered by Hinton [**9**] and Eastham [**4**], and they showed that, subject to certain conditions in the coefficients p_0 and p_2, (1.2) has solutions

$$(1.3)\qquad y_k(x) \sim p_0^{-1/8}(x)p_2^{-3/8}(x)\exp\left(\omega_k\int_a^x\left(\frac{p_2}{p_0}\right)^{1/4}(t)\,dt\right)$$

Received by the editors on February 15, 1996, and in revised form on September 20, 1996.
AMS *Mathematics Subject Classification.* 34E05.

where ω_k, $1 \leq k \leq 4$, are the fourth roots of (1).

This form is Liouville-Green asymptotic form for the fourth-order equation (1.2). As we shall see under our case Theorem 4.1, we obtain the solutions of (1.1) which extended those of (1.2). We shall use the asymptotic theorem of Eastham [**6**, Section 2], [**7**] to get our main results for (1.1).

2. The general method. We write (1.1) in the standard way [**9**] as a first-order system

$$Y' = AY \tag{2.1}$$

where the first component of Y is y and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -(1/2)q_1 p_0^{-1} & p_0^{-1} & 0 \\ -(1/2)q_2 & -p_1 + (1/4)q_1^2 p_0^{-1} & -(1/2)p_0^{-1}q_1 & 1 \\ p_2 & -(1/2)q_2 & 0 & 0 \end{bmatrix}. \tag{2.2}$$

As in [**1, 2**], we express A in its diagonal form

$$T^{-1}AT = \Lambda, \tag{2.3}$$

and we therefore require the eigenvalues λ_j and eigenvectors v_j, $1 \leq j \leq 4$, of A.

The characteristic equation of A is given by

$$p_0\lambda^4 + q_1\lambda^3 + p_1\lambda^2 + q_2\lambda - p_2 = 0. \tag{2.4}$$

An eigenvector v_j of A corresponding to λ_j is

$$v_j = \left(1, \lambda_j, p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j, -\frac{1}{2}q_2 + p_2\lambda_j^{-1}\right)^t \tag{2.5}$$

where superscript t denotes the transpose. We assume at this stage that the λ_j are distinct, and we define the matrix T in (2.3) by

$$T = (\, v_1 \quad v_2 \quad v_3 \quad v_4 \,). \tag{2.6}$$

Now if we write

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{2.7}$$

then by (2.2) EA coincides with its own transpose.

Hence, by [**5**, Section 2], the v_j have the orthogonality property

$$(Ev_k)^t v_j = 0, \qquad k \neq j. \tag{2.8}$$

As in [**1**], we define the scalars m_j, $1 \leq j \leq 4$, by

$$m_j = (Ev_j)^t v_j, \tag{2.9}$$

and the row vectors

$$r_j = (Ev_j)^t. \tag{2.10}$$

Then, by [**5**, Section 2], we can define T^{-1} by

$$T^{-1} = (\, m_1^{-1} r_1 \quad m_2^{-1} r_2 \quad m_3^{-1} r_3 \quad m_4^{-1} r_4 \,)^t \tag{2.11}$$

and the

$$\begin{aligned} m_j &= \left[\frac{\partial}{\partial \lambda} \det(\lambda I - A) \right]_{\lambda = \lambda_j} \\ &= 4p_0 \lambda_j^3 + 3q_1 \lambda_j^2 + 2p_1 \lambda_j + q_2. \end{aligned} \tag{2.12}$$

By (2.3), the transformation

$$Y = TZ \tag{2.13}$$

takes (2.1)

$$Z' = (\Lambda - T^{-1} T') Z, \tag{2.14}$$

where

$$\Lambda = \mathrm{dg}\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4). \tag{2.15}$$

Now if we write

$$T^{-1}T' = (t_{jk}), \qquad 1 \le j, k \le 4$$

then by (2.11) and (2.6)

$$t_{jk} = m_j^{-1} r_j v_k'. \tag{2.16}$$

Hence, for the diagonal elements, we consider $k = j$; (2.16) gives

$$m_j t_{jj} = r_j v_j' \tag{2.17}$$

now by (2.9) $m_j = (Ev_j)^t v_j$. Differentiating the m_j, we get

$$m_j' = 2 r_j v_j'. \tag{2.18}$$

Hence, by (2.18), (2.17) gives

$$t_{ij} = \frac{1}{2}\frac{m_j'}{m_j}, \qquad 1 \le j \le 4. \tag{2.19}$$

Now by (2.5) and (2.10), (2.16) gives, for $j \ne k$, $1 \le j$, $k \le 4$,

$$t_{jk} = m_k^{-1}\left\{\lambda_k'\left(p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j\right) + \lambda_j\left(p_0\lambda_k^2 + \frac{1}{2}q_1\lambda_k\right)' - \frac{1}{2}q_2' + (p\lambda_k^{-1})'\right\}. \tag{2.20}$$

Now we need to work out (2.19) and (2.20) in terms of p_j, $0 \le j \le 2$, and q_j, $j = 1, 2$, to determine the form (2.14) and then make progress towards (1.1).

3. The system $Z' = (\Lambda + R + S)Z$. In our analysis, we impose basic conditions on the coefficients, as follows:

(i) p_0 and p_2 are nowhere zero in some interval $[a, \infty)$, and

$$q_1 = o(p_2^{1/4} p_0^{3/4}), \qquad x \to \infty, \tag{3.1}$$

and we write

$$\delta = \frac{p_1}{p_2^{1/4} p_0^{3/4}} = o(1). \tag{3.2}$$

Also

$$p_1 = o(p_2^{1/2} p_0^{1/2}), \qquad x \to \infty, \tag{3.3}$$

and we write

$$\gamma = \frac{q_1}{p_2^{1/2} p_0^{1/2}} = o(1). \tag{3.4}$$

Finally, let

$$q_2 = o(p_2^{3/4} p_0^{1/4}), \qquad x \to \infty, \tag{3.5}$$

and we write

$$\eta = \frac{q_2}{p_2^{3/4} p_0^{1/4}} = o(1). \tag{3.6}$$

Now, as in [**1, 2**], we can solve the characteristic equation (2.4) asymptotically as $x \to \infty$ using (3.1), (3.3) and (3.5) to get the distinct eigenvalues λ_j as

$$\lambda_j = \omega_j \left(\frac{p_2}{p_0}\right)^{1/4} (1 + \delta_j), \qquad 1 \le j \le 4, \tag{3.7}$$

where

$$\omega_1 = 1, \qquad \omega_2 = -1, \qquad \omega_3 = \bar{\omega}_4 = i, \tag{3.8}$$

and

$$\delta_j = O(\delta) + O(\gamma) + O(\eta), \qquad 1 \le j \le 4. \tag{3.9}$$

Now, by (2.12), (3.2), (3.4), (3.6) and (3.7),

$$m_j = 4\omega_j^3 p_0^{1/4} p_2^{3/4} \{1 + O(\delta) + O(\gamma) + O(\eta)\}, \qquad 1 \le j \le 4. \tag{3.10}$$

Also, on substituting (3.7) into (2.12) and differentiating, we obtain

$$m_j' = 4\omega_j^3 p_0^{1/4} p_2^{3/4}\left\{\frac{1}{4}\frac{p_0'}{p_0} + \frac{3}{4}\frac{p_2'}{p_0}\right\} + p_0^{1/4}p_2^{3/4}\{O(\gamma') + O(\gamma') + O(\eta')\}. \tag{3.11}$$

At this stage we also require the following condition

(ii) $\delta(p_2'/p_2)$, $\delta(p_0'/p_0)$, $\gamma(p_2'/p_2)$, $\gamma(p_0'/p_0)$, $\eta(p_2'/p_2)$, $\eta(p_0'/p_0)$, $q_1'/(p_2^{1/4}p_0^{3/4})$, $p_1'/(p_2^{1/2}p_0^{1/2})$, $q_2'/(p_2^{3/4}p_0^{1/4})$ are all $L(a,\infty)$.

Further, we note that, on differentiating δ, γ and η and using (ii), we obtain

$$\delta', \gamma' \text{ and } \eta' \text{ are all } L(a,\infty). \tag{3.12}$$

Now, for the diagonal elements t_{jj}, $1 \le j \le 4$, we use (2.19); hence, by (3.10) and (3.11), (2.19) gives

$$t_{jj} = \frac{1}{8}\frac{(p_0p_2^3)'}{p_0p_2^3} + O(\delta') + O(\gamma') + O(\eta'). \tag{3.13}$$

Now, by (3.7), (3.9) and (3.10), we have

$$\begin{aligned} m_j^{-1}\lambda_k'&\left(p_0\lambda_j^2 + \frac{1}{2}q_1\lambda_j\right) \\ &= \frac{1}{16}\omega_j^{-1}\omega_k\left(\frac{p_2'}{p_2} - \frac{p_0'}{p_0}\right)\{1 + O(\delta) + O(\gamma) + O(\eta)\} \\ &\quad + O(\delta') + O(\gamma') + O(\eta'), \end{aligned} \tag{3.14}$$

$$\begin{aligned} m_j^{-1}\lambda_j&\left(p\lambda_k^2 + \frac{1}{2}q_1\lambda_k\right) \\ &= \frac{1}{8}\omega_j^{-1}\omega_k^2\left(\frac{p_2'}{p_2} + \frac{p_0'}{p_0}\right)\{1 + O(\delta) + O(\gamma) + O(\eta)\} \end{aligned} \tag{3.15}$$

$$q_2' m_j^{-1} = O\left(\frac{q_2'}{p_2^{3/4}p_0^{1/4}}\right), \tag{3.16}$$

and

$$m_j^{-1}(p\lambda_k^{-1})' = \frac{1}{16}\omega_k^{-1}\omega_j^{-3}\left(3\frac{p_2'}{p_2}+\frac{p_0'}{p_0}\right)\{1+O(\delta)+O(\gamma)+O(\eta)\} + O(\delta')+O(\gamma')+O(\eta'). \tag{3.17}$$

Similarly, as for t_{jj}, we can find t, $j \neq k$, $1 \leq j$, $k \leq 4$, by using (3.14), (3.15), (3.16), (3.17) and (2.20), and then we can write the system (2.14) as

$$Z' = (\Lambda + R + S)Z, \tag{3.18}$$

where, in this case
(3.19)

$$R=\begin{bmatrix} -(1/8)\rho_1 & -(1/8)\rho_2 & -(1/8)(1+i)\rho_3 & (1/8)(1-i/2)\rho_3-(i/16)\rho_2 \\ -(1/8)\rho_1 & -(1/8)\rho_1 & -(1/8)(1+i)\rho_3 & -(1/8)(1-i)\rho_3 \\ 0 & (i/4)\rho_3 & -(1/8)\rho_1 & (1/8)(1-i)\rho_3-(1/8)\rho_2 \\ 0 & -(i/4)\rho_3 & (1/8)(1+i)\rho_3-(1/8)\rho_2 & -(1/8)\rho_1 \end{bmatrix}$$

where

$$\rho_1 = \left[\frac{p_0'}{p_0}+3\frac{p_2'}{p_2}\right], \qquad \rho_2 = \frac{p_0'}{p_0}-\frac{p_2'}{p_2} \quad \text{and} \quad \rho_3 = \frac{p_0'}{p_0}+\frac{p_2'}{p_2} \tag{3.20}$$

and S is $L(a,\infty)$ by (3.12) and (3.13).

As in [**1, 2**], we can apply the asymptotic theorem in [**6**, Section 2] to (2.19), provided only that Λ and R satisfy the conditions in [**6**, Section 2].

4. The asymptotic result.

Theorem 4.1. *Let the coefficients p_0 and p_2 be nowhere zero in $[a,\infty)$ and $C^{(2)}[a,\infty)$ with p_1, q_1 and q_2 $C^{(1)}[a,\infty)$. Let* (3.1), (3.3), (3.5) *and* (ii) *hold. Also, let*

$$\frac{p_0'}{p_0}\left(\frac{p_0}{p_2}\right)^{1/4} \longrightarrow 0 \quad \textit{and} \quad \frac{p_2'}{p_2}\left(\frac{p_0}{p_2}\right)^{1/4} \longrightarrow 0, \qquad x\to\infty, \tag{4.1}$$

and

$$\left(\frac{p_0}{p_2}\right)^{1/4}\left(\frac{p_0'}{p_0}\right)^2,\left(\frac{p_0}{p_2}\right)^{1/4}\left(\frac{p_2'}{p_2}\right)^2,\left(\frac{p_0}{p_2}\right)^{1/4}\frac{p_0''}{p_2},\left(\frac{p_0}{p_2}\right)^{1/4}\frac{p_2''}{p_2} \tag{4.2}$$

are all $L(a,\infty)$.

Let

$$\mathrm{Re}\,(\lambda_i-\lambda_j) \tag{4.3}$$

have one sign in $[a,\infty)$ *for each unequal pair* (i,j). *Then* (1.1) *has solutions* y_k, $1\le k\le 4$, *such that*

$$y_k(x)\sim p_0^{-1/8}(x)p_2^{-3/8}(x)\exp\left(\int_a^x \lambda_k(t)\,dt\right),\quad x\to\infty. \tag{4.4}$$

Proof. We apply the asymptotic theorem in [**6**, Section 2] to (3.19). By (3.19) and (3.20), we first require

$$\frac{p_0'}{p_0}=o\left\{\left(\frac{p_0}{p_2}\right)^{1/4}\right\},\qquad \frac{p_2'}{p_2}=o\left\{\left(\frac{p_2}{p_0}\right)^{1/4}\right\},$$

this being [**6**]; for our system this is true by (4.1). We also require that

$$\left\{(\lambda_i-\lambda_j)^{-1}\frac{p_0'}{p_0}\right\}'\in L(a,\infty),\qquad \left\{(\lambda_i-\lambda_j)^{-1}\frac{p_2'}{p_2}\right\}'\in L(a,\infty),$$

for $i\ne j$, this being [**6**] for our system.

This is true by (4.2) and (ii). As in [**1**], we also note that (4.2) implies that the simplifying condition [**6**] be satisfied. Now since all conditions hold for the asymptotic result of [**6**], it follows that, as $x\to\infty$, (3.18) has four linearly independent solutions $Z_k(x)$ such that:

$$Z_k(x)=\{e_k+o(1)\}\exp\left(\int_a^x\left\{\lambda_k(t)-\frac{1}{8}\frac{(p_0p_2^3)'}{p_0p_2^3}\right\}dt\right), \tag{4.5}$$

where e_k is the coordinate vector with kth component unit and other components zero. Now we transform back to Y by means of (2.6) and

(2.13). By taking the first component on each side of (4.5) and carrying out the integration of $-(1/8)((p_0p_2^3)'/(p_0p_2^3))$, we obtain (4.4) after an adjustment of a constant multiple in y_k.

5. Concluding remarks. (i) First we note that in (4.3) we have for convenience used a simplified form of the Levinson dichotomy condition [**10, 6**, Section 1]. As given in [**1**], the above theorem also holds if (4.3) is generalized to

$$\operatorname{Re}(\lambda_j - \lambda_k) = f + g \tag{5.1}$$

where f has one sign in $[a, \infty)$ and g is $L(a, \infty)$ [**6**]. Here we give some cases where (5.1) holds. Substituting (3.7) back into (2.4) and using (3.2), (3.4) and (3.6) as in [**1**], we obtain

$$\begin{aligned}\delta_j = -\frac{1}{4}\omega_j^3\delta - \frac{\omega_j^2}{4}\gamma - \frac{\omega_j}{4}\eta + O(\delta^2) + O(\gamma^2) + O(\eta^2)\\ + O(\delta\gamma) + O(\delta\eta) + O(\gamma\eta).\end{aligned} \tag{5.2}$$

Then, by (2.4) and (5.2), we get

$$\begin{aligned}\lambda_j - \lambda_k = (\omega_j - \omega_k)\left(\frac{p_2}{p_0}\right)^{1/4} - \frac{1}{4}(\omega_j^4 - \omega_k^4)\left(\frac{p_2}{p_0}\right)^{1/4}\delta\\ - \frac{1}{4}(\omega_j^3 - \omega_k^3)\left(\frac{p_2}{p_0}\right)^{1/4}\gamma\\ - \frac{1}{4}(\omega_j^2 - \omega_k^2)\left(\frac{p_2}{p_0}\right)^{1/4}\delta\\ + O\left(\left(\frac{p_2}{p_0}\right)^{1/4}\delta^2\right) + O\left(\left(\frac{p_2}{p_0}\right)^{1/4}\gamma^2\right)\\ + O\left(\left(\frac{p_2}{p_0}\right)^{1/4}\eta^2\right) + O\left(\left(\frac{p_2}{p_0}\right)^{1/4}\delta\gamma\right)\\ + O\left(\left(\frac{p_2}{p_0}\right)^{1/4}\delta\eta\right) + O\left(\left(\frac{p_2}{p_0}\right)^{1/4}\gamma\eta\right).\end{aligned} \tag{5.3}$$

Now suppose that

$$\left(\frac{p_2}{p_0}\right)^{1/4}\delta^2 \in L(a, \infty), \tag{5.4}$$

$$\left(\frac{p_2}{p_0}\right)^{1/4}\gamma^2 \in L(a, \infty), \tag{5.5}$$

and

$$\left(\frac{p_2}{p_0}\right)^{1/4}\eta^2 \in L(a,\infty). \tag{5.6}$$

Then it follows immediately from (5.3)–(5.6) that (5.1) is satisfied in the following two cases.

Case A. p_i, $0 \le i \le 2$, and q_i, $i = 0, 1$, are real-valued functions.

Case B. p_i, $0 \le i \le 2$, and q_i, $i = 0, 1$, are pure-imaginary functions.

(ii) We consider Theorem 4.1 as applied to the coefficients

$$p_0 = c_1 x^{\alpha_1}, \qquad p_1 = c_2 x^{\alpha_2}, \qquad p_2 = c_3 x^{\alpha_3},$$
$$q_1 = c_4 x^{\alpha_4}, \qquad q_2 = c_5 x^{\alpha_5},$$

where α_i, $1 \le i \le 5$, and c_i, $1 \le i \le 5$, are real constants with $c_1 \ne 0$ and $c_3 \ne 0$. Then (3.1), (3.3), (3.5) and Section 3 (ii) all hold under the three conditions:

$$\alpha_3 + 3\alpha_1 - 4\alpha_4 > 0, \tag{5.7}$$
$$\alpha_3 + \alpha_1 - 2\alpha_2 > 0, \tag{5.8}$$

and

$$\alpha_3 + \alpha_1 - 4\alpha_5 > 0. \tag{5.9}$$

Also (4.1) and (4.2) hold if

$$\alpha_1 - \alpha_3 - 4 < 0. \tag{5.10}$$

Again (5.4) holds if

$$\alpha_3 + 7\alpha_1 - 8\alpha_5 - 4 > 0, \tag{5.11}$$

and (5.5) holds if

$$3\alpha_3 + 5\alpha_1 - 8\alpha_5 - 4 > 0, \tag{5.12}$$

while (5.6) holds if

$$\alpha_1 + 3\alpha_3 - 4\alpha_5 > 0. \tag{5.13}$$

Now (5.7) is implied by (5.10) and (5.11), (5.8) is implied by (5.10) and (5.12), and (5.9) is implied by (5.10) and (5.13); hence, we need only (5.10)–(5.13).

(iii) We consider Theorem 4.1 as applied to the coefficients $p_0 = c_1 x^{\alpha_4}\exp((2/3)x^b)$, $p_1 = c_2 x^{\alpha_2}\exp(x^a)$, $p_2 = c_3 x^{\alpha_3}\exp(2x^b)$, $q_1 = c_4 x^{\alpha_4}\exp(x^a)$, $q_2 = c_5 x^{\alpha_5}\exp(x^a)$, where c_1 and c_3 are not equal to zero and α_i, $1 \le i \le 5$, a and b are real constants with $a \ge 0$ and $a < b$.

Again it is easy to check that all the conditions (3.1), (3.3), (3.5), Section 3 (ii), (4.1), (4.2), (5.1)–(5.6) are satisfied.

(iv) Now for the particular Equation (1.2) in which $p_1 = q_1 = q_2 = 0$ in (1.1), the asymptotic formula (4.4) reduces to (1.3) which agrees with the result of Eastham [**4**, Theorem 1] and Hinton [**9**].

(v) For the example of power coefficiency with $q_1 = q_2 = 0$, the result here agrees with [**11** p. 131, condition (ii)].

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