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SOME REMARKABLE CONGRUENCES ON COMPLETELY REGULAR SEMIGROUPS

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ABSTRACT. We express a completely regular semigroup S as $(Y; S_\alpha)$, that is, a semilattice of completely simple semigroups. For each pair $\alpha > \beta$, we consider the congruence $\kappa_{\alpha,\beta}$ on S generated by the set of pairs (a, b) where $a \in S_\alpha$, $b \in S_\beta$ and $a > b$. These congruences play an important role in finding conditions which ensure that the kernel relation K on the congruence lattice of S be a congruence. In particular, the meet and the join of these congruences provide interesting congruences in this context. Another class of congruences, constructed as follows, occurs naturally in this study. Given a congruence ρ on S and ideals $I \subseteq J$ of S, we generalize the Rees congruence relative to I by constructing a congruence which involves ρ, I and J; here ρ must saturate I and I or J may be empty.

1. Introduction and summary. The consideration of necessary and sufficient conditions on a completely regular semigroup S in order that the kernel relation K on the congruence lattice $\mathcal{C}(S)$ be a congruence in [**5**] gives rise to the following class of congruences. We write $S = (Y; S_\alpha)$ thereby indicating that S is a semilattice Y of completely simple semigroups S_α. For each pair $\alpha, \beta \in Y$ such that $\alpha > \beta$, let $\kappa_{\alpha,\beta}$ be the congruence on S generated by the pairs (a, b) such that $a \in S_\alpha$, $b \in S_\beta$, $a > b$. These congruences play a crucial role in the above evoked study. Besides the conditions on S which ensure that K be a congruence, it is of interest to find some lattices Λ of congruences on an arbitrary completely regular semigroup S with the property that $K|_\Lambda$ is a congruence.

Section 2 contains the minimum of necessary preliminaries. We establish in Section 3 that K restricted to the filter of $\mathcal{C}(S)$ generated by the join of congruences $\kappa_{\alpha,\beta}$ is a congruence and the corresponding

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quotient is a modular lattice. The main result in Section 4 asserts that, when Y has at least three elements and the restriction of K to the filter generated by the intersection of congruences $\kappa_{\alpha,\beta}$ is a congruence, then K is a congruence on all of $\mathcal{C}(S)$. Several other results in the section supplement this statement. Section 5 has a different flavor. We introduce a generalization of Rees congruences by involving two ideals of S and a congruence on S. For a fixed congruence, this produces a lattice of congruences on S with several interesting properties.

2. Preliminaries. Throughout the paper we fix an arbitrary completely regular semigroup S. When the need arises, we assume implicitly that $S = (Y; S_\alpha)$, that is, S is a semilattice Y of completely simple semigroups S_α. For $a \in S$, we denote by a^0 the identity of the maximal subgroup of S containing a. The set of idempotents of S is denoted by $E(S)$. The natural partial order on S is given by

$$a \leq b \iff a = eb = bf \quad \text{for some} \quad e, f \in E(S).$$

The lattice of all congruences on S is denoted by $\mathcal{C}(S)$. Its greatest and least elements are denoted by ω and ε, respectively. We shall also use the latter notation for the universal and equality relations on any set. A set A saturates a congruence ρ if A is the union of some ρ-classes. For $\rho \in \mathcal{C}(S)$,

$$\ker \rho = \{a \in S \mid a\rho e \text{ for some } e \in E(S)\}$$

is the kernel of ρ. The kernel relation K on $\mathcal{C}(S)$ is given by

$$\lambda K \rho \iff \ker \lambda = \ker \rho \quad (\lambda, \rho \in \mathcal{C}(S)).$$

In a lattice L, for $\alpha \in L$ let $[\alpha) = \{\beta \in L \mid \beta \geq \alpha\}$, the filter of L generated by α. For any sets A and B, $A \backslash B = \{a \in A \mid a \notin B\}$. The cardinality of a set X is denoted by $|X|$.

If I is an ideal of a semigroup T, then T is an (ideal) extension of I by the quotient semigroup T/I. If also there exists a retraction ψ of T onto I, then T is a retract extension of I determined by the partial homomorphism $\psi|_{T\backslash I}$. If T has an identity, we write $T = T^1$; otherwise, T^1 is the semigroup T with an identity adjoined.

3. The join of congruences $\kappa_{\alpha,\beta}$. For $S = (Y; S_\alpha)$ and $\alpha > \beta$, we define $\kappa_{\alpha,\beta}$ as the congruence generated by the set

$$\{(a, b) \mid a \in S_\alpha, b \in S_\beta, a > b\}.$$

That this set is not empty is guaranteed by [**4**, Lemma 2.1(ii)].

We establish here some simple properties of the join of all congruences $\kappa_{\alpha,\beta}$; in the next section we shall consider their meet.

Proposition 3.1. *The relation $\theta = \vee_{\alpha>\beta}\kappa_{\alpha,\beta}$ is the least completely simple congruence on S. Let $K' = K|_{[\theta)}$. Then K' is a congruence and $[\theta)/K'$ is a modular lattice.*

Proof. That θ is the least completely simple congruence on S follows from: [**6**, Lemma 6.4], [**2**, Notation 4.8] and [**3**, Lemma 3].

It is well known that the mapping

$$\rho \longrightarrow \rho/\theta \quad (\rho \in [\theta))$$

is an isomorphism of $[\theta)$ onto $\mathcal{C}(S/\theta)$. By [**5**, Lemma 7.5(ii)], we have

$$\lambda K \rho \quad \Longleftrightarrow \quad \lambda/\theta K \rho/\theta \quad (\lambda, \rho \in [\theta)). \tag{1}$$

Let $\lambda, \rho, \sigma \in [\theta)$ with $\lambda K \rho$. By (1), we have $\lambda/\theta K \rho/\theta$ which, by [**5**, Theorem 5.1], yields $\lambda/\theta \vee \sigma/\theta K \rho/\theta \vee \sigma/\theta$ since S/θ is completely simple. Hence $(\lambda \vee \sigma)/\theta K (\rho \vee \sigma)/\theta$ which by (1) gives $\lambda \vee \sigma K \rho \vee \sigma$. Therefore K' is a congruence. It also follows from (1) that $[\theta)/K' \cong \mathcal{C}(S/\theta)/K$ which, by [**5**, Corollary 5.2], finally gives that $[\theta)/K'$ is a modular lattice. $\square$

In order to ensure that the above proposition is not vacuous, that is, that $\theta \neq \omega$ may occur, we prove the following simple statement.

Lemma 3.2. *Let S be a retract extension of a completely simple semigroup S_0 by a completely simple semigroup S_1 with a zero adjoined determined by a homomorphism $\varphi : S_1 \to S_0$. Then $\theta = \omega$ for S if and only if S_0 is trivial.*

Proof. First note that $\theta = \kappa_{1,0}$ if we consider S as a semilattice of semigroups S_0 and S_1. The corresponding retraction $\psi : S \to S_0$ is given by: $\psi|S_0 = \iota_{S_0}$, $\psi|_{S_1} = \varphi$. Let $a, b \in S_0$ be such that $a\theta b$. Then there exists a sequence

$$a = x_1 u y_1, \qquad x_1 v_1 y_1 = x_2 u_2 y_2, \quad \cdots \quad x_n v_n y_n = b,$$

for some $x_i, y_i \in S^1$ and $u_i, v_i \in S$ such that either $u_i \leq v_i$ or $v_i \leq u_i$, $i = 1, 2, \ldots, n$. Hence

$$a = x_1(u_1\psi)y_1, \qquad x_1(v_1\psi)y_1 = x_2(u_2\psi)y_2, \quad \cdots \quad x_n(v_n\psi)y_n = b,$$

and since $u_i\psi = v_i\psi$ for $i = 1, 2, \ldots, n$, we get $a = b$. Therefore $\theta|_{S_0} = \varepsilon$. It follows that, if $\theta = \omega$, we must have S_0 trivial.

Conversely, assume that S_0 is trivial. Then φ is a constant map so that the induced congruence $\bar{\varphi}$ equals ω on S_1. By [**6**, Lemma 5.4], $\theta|_{S_1} = \bar{\varphi}$ and thus $\theta|_{S_1} = \omega$. Since then any element of S_1 is θ-related to the single element in S_0, it follows that $\theta = \omega$. □

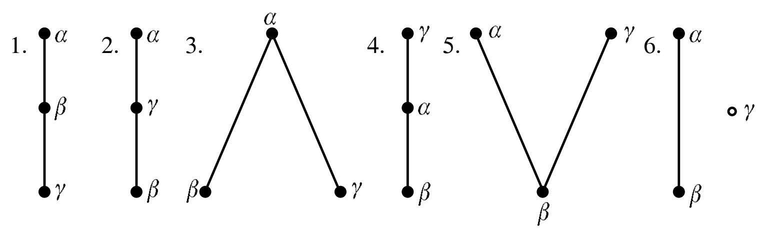
4. The meet of congruences $\kappa_{\alpha\beta}$. Besides the notation $\kappa_{\alpha,\beta}$ introduced in the preceding section, for $\alpha > \beta$ in Y, we let $\zeta_{\alpha,\beta}$ be the congruence on Y generated by the singleton $\{(\alpha, \beta)\}$. We also let

$$\kappa = \bigwedge_{\alpha>\beta} \kappa_{\alpha,\beta}, \qquad \zeta = \bigwedge_{\alpha>\beta} \zeta_{\alpha,\beta}.$$

For the main result of this section, we shall need the following simple statement of independent interest.

Lemma 4.1. *Let Y be a semilattice with at least three elements. Then $\zeta = \varepsilon$.*

Proof. Let $\alpha, \beta, \gamma \in Y$ be such that $\alpha > \beta$, $\gamma \neq \alpha$ and $\gamma \neq \beta$. Then exactly one of the following occurs: $\alpha > \gamma$, $\alpha < \gamma$ or α and γ are incomparable; the same type of situation occurs with β versus γ. Now, pairing these cases, we arrive at the following possibilities:



Let θ be the congruence on Y with classes $[\alpha)$ and $Y\backslash[\alpha)$. Then α and β are not θ-related. By the cases enunciated above, we have

$$1.\ \zeta_{\beta,\gamma} \subseteq \theta; \qquad 2.\ \zeta_{\gamma,\beta} \subseteq \theta; \qquad 3.\ \zeta_{\beta,\beta\gamma} \subseteq \theta;$$
$$4.\ \zeta_{\gamma,\alpha} \subseteq \theta; \qquad 5.\ \zeta_{\gamma,\beta} \subseteq \theta; \qquad 6.\ \zeta_{\gamma,\beta\gamma} \subseteq \theta.$$

Since α and β are not θ-related, this shows that in all cases there exists $\zeta_{\delta,\eta}$ such that α and β are not $\zeta_{\delta,\eta}$-related. It follows that α and β are not ζ-related.

Now let $\alpha, \beta \in Y$ with $\alpha \neq \beta$. If they are comparable, by the above, they are not ζ-related. If they are not comparable, then $\alpha > \alpha\beta$ and thus α and $\alpha\beta$ are not ζ-related. But this obviously implies that also α and β are not ζ-related. Therefore $\zeta = \varepsilon$. □

Theorem 4.2. *Let $S = (Y; S_\alpha)$ be a completely regular semigroup and Y have at least three elements. Assume that K restricted to $[\kappa)$ is a congruence. Then K is a congruence on all of $\mathcal{C}(S)$.*

Proof. According to [**5**, Theorem 5.1], it suffices to show that, for any $\alpha > \beta$ in Y, we have $S_\alpha \subseteq \ker \kappa_{\alpha,\beta}$. We represent $\kappa_{\alpha,\beta}$ by means of its congruence aggregate as in [**4**], to wit $\kappa_{\alpha,b} \sim (\zeta_{\alpha,\beta}; \eta_\gamma)$ in view of [**5**, Lemma 4.4] which asserts that $\kappa_{\alpha,\beta}$ induces on Y the congruence $\zeta_{\alpha,\beta}$ for some $\eta_\gamma \in \mathcal{C}(S_\gamma)$ for each $\gamma \in Y$. By [**4**, Corollary 5.5(i)], the mapping $\kappa_{\alpha,\beta} \to \zeta_{\alpha,\beta}$ is a complete homomorphism. Since κ has its congruence aggregate of the form $\wedge_{\alpha>\beta}(\zeta_{\alpha,\beta};\ )$, it follows that $\kappa \sim (\zeta;\ )$. But Lemma 4.1 gives that $\zeta = \varepsilon$. Therefore $\kappa \subseteq \mathcal{D}$.

Now fix $\alpha > \beta$, and let $\rho = \kappa_{\alpha,\beta} \wedge \mathcal{D}$. Then

$$\ker \kappa_{\alpha,\beta} = \ker \kappa_{\alpha,\beta} \cap \ker \mathcal{D} = \ker(\kappa_{\alpha,\beta} \wedge \mathcal{D}) = \ker \rho$$

and, by the preceding paragraph, we have $\kappa \subseteq \rho$. Define a relation λ on S by

$$x\lambda y \iff x, y \in S_\gamma$$
$$\text{for some } \gamma \in Y \quad \text{and} \quad x\kappa_{\alpha,\beta}y \quad \text{if } \gamma \not\leq \beta.$$

Clearly λ is an equivalence relation. Let $x\lambda y$ with $x, y \in S_\gamma$ and $a \in S_\delta$. If $\gamma\delta \leq \beta$, then $xa\mathcal{D}ya$ implies that $xa\lambda ya$. If $\gamma\delta \not\leq \beta$, then $\gamma \not\leq \beta$, and thus $x\kappa_{\alpha,\beta}y$ which implies that $xa\kappa_{\alpha,\beta}ya$ which, together with $xa\mathcal{D}ya$ yields $xa\lambda ya$. Similarly $ax\lambda ay$ in all cases. Therefore $\lambda \in \mathcal{C}(S)$ and, in fact, $\kappa \subseteq \rho \subseteq \lambda$. Since $\kappa_{\alpha,\beta}K\rho$, the hypothesis implies that $\kappa_{\alpha,\beta} \vee \lambda K \rho \vee \lambda$.

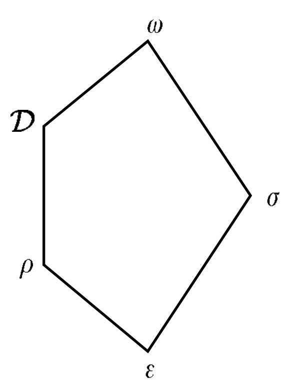
Let $a \in S_\alpha$. By [**4**, Lemma 2.1(ii)], there exists $b \in S_\beta$ such that $a > b$. Hence $a\kappa_{\alpha,\beta}b$. Also $b\lambda e$ for any $e \in E(S_\beta)$ and thus $a\kappa_{\alpha,\beta}b\lambda e$ whence $a \in \ker(\kappa_{\alpha,\beta} \vee \lambda) = \ker(\rho \vee \lambda)$. Hence there exists a sequence

$$a\rho x_1\lambda x_2\rho \cdots x_n\lambda a^0$$

for some $x_1, x_2, \ldots, x_n \in S$. Since both ρ and λ are under $\mathcal{D}$, we must have $x_1, x_2, \cdots x_n \in S_\alpha$. But then $a\kappa_{\alpha,\beta}x_1$, $x_1\kappa_{\alpha,\beta}x_2, \ldots$ by the definitions of ρ and λ, which yields $a\kappa_{\alpha,\beta}a^0$ so that $a \in \ker\kappa_{\alpha,\beta}$. We have proved that $S_\alpha \subseteq \ker\kappa_{\alpha,\beta}$, as required. ◻

Theorem 4.2 does not extend to the case when Y has only two elements.

Example 4.3. Let $S = Y_2 \times Z_2$ where $Y_2 = \{0, 1\}$ and $Z_2 = Z/(2)$. Then $= \mathcal{C}(Y)$ has the form



where $\sigma = \kappa_{\alpha,\beta}$ with $S_\alpha = \{1\} \times Z_2$, $S_\beta = \{0\} \times Z_2$ and ρ is the Rees congruence. Then $[\kappa_{\alpha,\beta}, \omega] = \{\sigma, \omega\}$ and $K|_{\{\sigma,\omega\}} = \varepsilon$ so it is a congruence. But K is not a congruence.

Theorem 4.2 is vacuous for $|Y| = 1$, for $\kappa_{\alpha,\beta}$ is not defined and K is a congruence. In general, $\kappa = \wedge_{\alpha>\beta}\kappa_{\alpha,\beta}$ is different from the equality relation as we shall see below.

A completely regular semigroup which is a chain Y of completely simple semigroups S_α in which every element acts as the zero of any element in a higher completely simple component is called the *mutually annihilating sum* (of semigroups S_α, $\alpha \in Y$), see [**1**].

Lemma 4.4. *Let S be a mutually annihilating sum of completely simple semigroups. Then K is a congruence for S.*

Proof. Let $a \in S_\alpha$ and $b \in S_\beta$ where $\alpha > \beta$. We have, by hypothesis, that $b = ab = ba$ whence $b^0 = ab^0 = b^0 a$ so that $a > b^0$. It follows, by [**4**, Lemma 2.1(iv)], that $a\kappa_{\alpha,\beta}b^0$ and thus $a \in \ker\kappa_{\alpha,\beta}$. By [**5**, Theorem 5.1], we conclude that K is a congruence for S. □

We exhibit in the following example that, in a completely simple semigroup S for which K is a congruence, $\kappa = \wedge_{\alpha>\beta}\kappa_{\alpha,\beta}$ need not be the equality relation.

Lemma 4.5. *Let S be a mutually annihilating sum of the completely simple semigroups S_α, S_β and S_γ where $\alpha > \beta > \gamma$. Then $\kappa \subseteq \mathcal{D}$, $\kappa|_{S_\alpha} = \varepsilon$, $\kappa|_{S_\beta}$ is a group congruence and $\kappa|_{S_\gamma} = \varepsilon$.*

Proof. We have seen in the proof of Theorem 4.2 that $\kappa \subseteq \mathcal{D}$. The following verification will take care of the remaining assertions of the lemma.

1. For any $a \in S_\alpha$ and $e \in E(S_\beta)$, we have $e < a$ which, by [**4**, Lemma 2.1(iv)], implies that $e\kappa_{\alpha,\beta}a$ so that $a \in \ker\kappa_{\alpha,\beta}$. Therefore $\kappa_{\alpha,\beta}|_{S_\alpha} = \omega$. The same type of argument shows that $e\kappa_{\alpha,\beta}f$ for any $e, f \in E(S_\beta)$ so that $\kappa_{\alpha,\beta}|_{S_\beta}$ is a group congruence. Next let $a, b \in S_\gamma$

be such that $a\kappa_{\alpha,\beta}b$. Then there exists a sequence

$$(2) \qquad a = x_1u_1y_1, \qquad x_1v_1y_1 = x_2u_2y_2, \quad \cdots \quad x_nv_ny_n = b$$

for some $x_i, y_i \in S^1$, $u_i, v_i \in S$ such that either $u_i \in S_\alpha$, $v_i \in S_\beta$ or $u_i \in S_\beta$, $v_i \in S_\alpha$ for $i = 1, 2, \dots, n$. Since $a \in S_\gamma$ and $u_1 \in S_\alpha \cup S_\beta$, we must have either $x_1 \in S_\gamma$ or $y_1 \in S_\gamma$. This implies that $x_1v_1y_1 \in S_\gamma$ and thus, either $x_2 \in S_\gamma$ or $y_2 \in S_\gamma$. Continuing this reasoning, we conclude, from the peculiarity of the multiplication in S, that

$$a = x_1y_1, \quad x_1y_1 = x_2y_2, \quad \cdots \quad x_ny_n = b$$

so that $a = b$. Therefore $\kappa_{\alpha,\beta}|_{S_\gamma} = \varepsilon$.

2. Next $\kappa_{\beta,\gamma}|_{S_\alpha} = \varepsilon$ since the system of equations (2) with $x_i, y_i \in S^1$ and $u_i, v_i \in S_\alpha \cup S_\beta$ cannot hold if $a, b \in S_\gamma$. Similar reasoning as the one above shows that $\kappa_{\beta,\gamma}|_{S_\beta} = \omega$ and that $\kappa_{\beta,\gamma}|_{S_\gamma}$ is a group congruence.

3. Again $\kappa_{\alpha,\gamma}|_{S_\alpha} = \omega$ and $\kappa_{\alpha,\gamma}|_{S_\gamma}$ is a group congruence similarly as above. Let $a, b \in S_\beta$. For any $u \in S_\alpha$ and $v \in S_\gamma$, we have $u > v$, $a = au$, $av = bv$, $bu = b$ so that $a\kappa_{\alpha,\gamma}b$. Therefore $\kappa_{\alpha,\gamma}|_{S_\beta} = \omega$.

The desired conclusions now follow from the definition of κ, namely, $\kappa = \kappa_{\alpha,\beta} \wedge \kappa_{\beta,\gamma} \wedge \kappa_{\alpha,\gamma}$. □

5. A generalization of Rees congruence. Again S denotes an arbitrary completely regular semigroup. Let $\mathcal{I}$ be the set of all ideals of S together with the empty set ordered by inclusion.

Let $\rho \in \mathcal{C}(S)$. For $I \in \mathcal{I}$, let

$$I_\rho = \{a \in S \mid a\rho b \text{ for some } b \in I\}$$

be the saturation of I by ρ. For $I, J \in \mathcal{I}$ such that $I\rho = I \subseteq J$, define a relation $\rho_{I,J}$ on S by

$$a\rho_{I,J}b \quad \Longleftrightarrow \quad \begin{cases} \text{either} & a = b \notin J \\ \text{or} & a, b \in J\backslash I, a\rho b \\ \text{or} & a, b \in I. \end{cases}$$

It follows without difficulty that $\rho_{I,J} \in \mathcal{C}(S)$. In particular, for any ideal I of S which saturates ρ, we have that $\rho_{I,I}$ is the Rees congruence on S relative to I.

In the representation $\rho_{I,J}$ none of the ingredients ρ, I and J need be unique. We are interested in all congruences of this form for a fixed ρ. For $\rho \in \mathcal{C}(S)$, let

$$\Gamma_\rho = \{\rho_{I,J} \mid I, J \in \mathcal{I}, I = I\rho \subseteq J\}.$$

The next proposition and its corollary determine the level of uniqueness of the parameters I and J in $\rho_{I,J}$.

Proposition 5.1. *For $\rho_{I,J}$, $\rho_{K,L} \in \Gamma_\rho$, we have*

$$\begin{aligned}\rho_{I,J} \subseteq \rho_{K,L} \quad \Longleftrightarrow \quad & \rho|_{J\backslash L} = \varepsilon, \qquad (J\backslash L)\rho \cap J = J\backslash L,\\ & I \subseteq L \quad \text{if } |I| > 1, \quad I = x\rho \quad \text{for some } x \in S,\\ & I \subseteq K \quad \text{if } |I| > 1, \quad I \neq x\rho \quad \text{for all } x \in S.\end{aligned}$$

Proof. Necessity. Let $a, b \in J\backslash L$ be such that $a\rho b$. If $a \in I$, then $b \in I$ since $a\rho b$ and $I = I\rho$. If $a \notin I$, then also $b \notin I$ so that $a, b \in J\backslash I$. Thus $a\rho_{I,J}b$ whence $a\rho_{K,L}b$. Since $a, b \notin L$, we get $a = b$. Therefore $\rho|_{J\backslash L} = \varepsilon$.

Next let $a \in (J\backslash L)\rho \cap J$, say $a\rho b$ and $b \in J\backslash L$. Hence $a, b \in J$ and $a\rho b$ which implies that either $a, b \in I$ or $a, b \in J\backslash L$ since $I\rho = I$ whence $a\rho_{I,J}b$. It follows that $a\rho_{K,L}b$. Since $b \notin L$, also $a \notin L$ and $a = b$ so that $a \in J\backslash L$. Therefore $(J\backslash L)\rho \cap J \subseteq J\backslash L$ and the opposite inclusion is trivial.

Assume that $|I| > 1$ and $I = x\rho$ for some $x \in S$, and let $a \in I$. There exists $b \in I$ such that $a \neq b$. Hence $a\rho_{I,J}b$ so that $a\rho_{K,L}b$. Since $a \neq b$, we get $a, b \in L$. Therefore $I \subseteq L$. Assume that $|I| > 1$ and $I \neq x\rho$ for all $x \in S$, and let $a \in I$. There exists $b \in I$ such that a and b are not ρ-related. Hence $a\rho_{I,J}b$ whence $a_{\rho_{K,L}}b$. Since a and b are not ρ-related, it follows that $a, b \in K$. Therefore $I \subseteq K$.

Sufficiency. It suffices to consider $a, b \in S$ such that $a \neq b$ and $a\rho_{I,J}b$. Then either $a, b \in I$ or $a, b \in J\backslash I$, $a\rho b$.

Consider the case $a, b \in I$. Since $a \neq b$, we must have $|I| > 1$. If $I = x\rho$ for some $x \in S$, then $I \subseteq L$ so that $a, b \in L$ and $a\rho b$ whence either $a, b \in K$ or $a, b \in L \backslash K$, $a\rho b$ and in either case $a\rho_{K,L}b$. If $I \neq x\rho$ for all $x \in S$, then $I \subseteq K$ so that $a, b \in K$ whence $a\rho_{K,L}b$.

Finally consider the case $a, b \in J \backslash I$, $a\rho b$. By the hypothesis $\rho|_{J\backslash L} = \varepsilon$, we cannot have $a, b \in J\backslash L$. Thus, either $a, b \in L$, in which case $a, b \in K$ or $a, b \in L\backslash K$ so that $a\rho_{K,L}b$, or $a \in J\backslash L$, $b \in J \cap L$ or $b \in J\backslash L$, $a \in J \cap L$. The last two cases being symmetric, we assume that $a \in J\backslash L$ and $b \in J \cap L$. Since $a\rho b$, we get $b \in (J\backslash L)\rho \cap J$ which, by hypothesis, yields $b \in J\backslash L$. Hence $a, b \in J\backslash L$ which, as we have seen, is impossible. Therefore, this case cannot occur. □

Corollary 5.2. *For $\rho_{I,j}, \rho_{K,L} \in \Gamma_\rho$, we have*

$$
\begin{gathered}
\rho_{I,J} = \rho_{K,L} \quad \Longleftrightarrow \quad \rho|_{(J\backslash L)\cup(L\backslash J)} = \varepsilon, \\
(J\backslash L)\rho \cap J = J\backslash L, \qquad (L\backslash J)\rho \cap L = L\backslash J, \\
\textit{if } |I| > 1, \quad I = x\rho \textit{ for some } s \in S, \quad \textit{then } I \subseteq L, \\
\textit{if } |K| > 1, \quad K = x\rho \textit{ for some } x \in S, \quad \textit{then } K \subseteq J, \\
\textit{if } |I| > 1, \quad I \neq x\rho \textit{ for some } x \in S \textit{ or } |K| > 1, \\
K \neq x\rho \textit{ for all } x \in S, \quad \textit{then } I = K.
\end{gathered}
$$

Proof. Comparing this with the result in Proposition 5.1, it suffices to consider the case $|I| > 1$, $I \neq x\rho$ for all $x \in S$. With the condition in that proposition, $I \subseteq K$ so $|K| > 1$ and $K \neq x\rho$ for all $x \in S$ and thus also $K \subseteq I$ and therefore $I = K$. □

For the proof of the main result of this section we need some preparation.

Lemma 5.3. *Let $\rho \in \mathcal{C}(S)$, $\rho_{I_\alpha,J_\alpha} \in \Gamma_\rho$ for $\alpha \in A, I = \cup_{\alpha\in A} I_\alpha$ and $J = \cup_{\alpha\in A} J_\alpha$. Then $\vee_{\alpha\in A}\rho_{J_\alpha,I_\alpha} = \rho_{I,J}$.*

Proof. Let $\lambda = \vee_{\alpha\in A}\rho_{I_\alpha,J_\alpha}$. First note that

$$
\begin{aligned}
I\rho &= \{x \in S \mid x\rho y \text{ for some } y \in I\} \\
&= \{x \in S \mid x\rho y \text{ for some } y \in I_\gamma \text{ for some } \gamma \in A\}
\end{aligned}
$$

$$= \bigcup_{\alpha\in A} \{x \in S \mid x\rho y \text{ for some } y \in I_\alpha\}$$
$$= \bigcup_{\alpha\in A} I_\alpha = I.$$

Now let $\beta \in A$, $a\rho_{I_\beta,J_\beta}b$ and $a \neq b$. First assume that $a\rho b$. Then $a, b \in J_\beta$ so that $a, b \in J$. Since $a\rho b$, by the above we have either $a, b \in I$ or $a, b \notin I$. In the first case $a\rho_{I,J}b$ and in the second case $a, b \in J\backslash I$ and $a\rho b$ so that again $a\rho_{I,J}b$. Next assume that a and b are not ρ-related. Then $a, b \in I_\beta$ and thus $a, b \in I$ and $a\rho_{I,J}b$. Therefore $\rho_{I_\beta,J_\beta} \subseteq \rho_{I,J}$ and $\lambda \subseteq \rho_{I,J}$.

Conversely let $a\rho_{I,J}b$ and $a \neq b$. First let $a\rho b$. Then $a, b \in J$, say $a \in J_\alpha$ and $b \in J_\beta$. Hence $a\rho a^0 b\rho b^0 a\rho b$, where

$$\begin{aligned}
&\text{either } a, a^0 b \in I_\alpha \quad \text{or} \quad a, a^0 b \in J_\alpha\backslash I_\alpha,\\
&\text{either } a^0 b, b^0 a \in I_\alpha \quad \text{or} \quad a^0 b, b^0 a \in J_\alpha\backslash I_\alpha,\\
&\text{either } b^0 a, b \in I_\beta \quad \text{or} \quad b^0 a, b \in J_\beta\backslash I_\beta
\end{aligned}$$

since both I_α and I_β are ρ-saturated. Therefore

$$a\rho_{I_\alpha,J_\alpha} a^0 b\rho_{I_\alpha,J_\alpha} b^0 a\rho_{I_\beta,J_\beta} b,$$

so that $a\lambda b$. Finally let a and b not be ρ-related. Then $a, b \in I$, say $a \in I_\alpha$ and $b \in I_\beta$. Hence $a, ab \in I_\alpha$ and $ab, b \in I_\beta$ which implies that $a\rho_{I_\alpha,J_\alpha} ab\rho_{I_\beta,J_\beta} b$. Consequently $a\lambda b$ which completes the proof that $\rho_{I,J} \subseteq \lambda$ and equality prevails. □

Lemma 5.4. *Let* $\rho \in \mathcal{C}(S)$, $\rho_{I_\alpha,J_\alpha} \in \Gamma_\rho$ *for* $\alpha \in A$, $I = \cap_{\alpha\in A} I_\alpha$ *and* $J = \cap_{\alpha\in A} J_\alpha$. *Then* $\wedge_{\alpha\in A}\rho_{I_\alpha,J_\alpha} = \rho_{I,J}$.

Proof. Let $\lambda = \wedge_{\alpha\in A}\rho_{I_\alpha,J_\alpha}$ and $a \in I\rho$. Then $a\rho b$ for some $b \in I$. Hence $b \in I_\alpha$ and thus $a \in I_\alpha\rho = I_\alpha$ for every $\alpha \in A$ so that $a \in I$.

Therefore $I\rho = I$. Let $a, b \in S$. Then

$$\begin{aligned}
a\lambda b \quad &\Longleftrightarrow \quad a\rho_{I_\alpha, J_\alpha} b \text{ for all } \alpha \in A \\
&\Longleftrightarrow \quad \left.\begin{cases} \text{either } a = b \notin J_\alpha \\ \text{or } a, b \in J_\alpha \backslash I_\alpha, a\rho b \\ \text{or } a, b \in I_\alpha \end{cases}\right\} \quad \text{for all } \alpha \in A, \\
a\rho_{I,J} b \quad &\Longleftrightarrow \quad \begin{cases} \text{either } a = b \notin J \\ \text{or } a, b \in J \backslash I, a\rho b, \\ \text{or } a, b \in I. \end{cases}
\end{aligned}$$

It suffices to consider the case $a \neq b$. If $a\rho b$, then

$$\begin{aligned}
a\lambda b \quad \Longleftrightarrow \quad a, b \in J_\alpha \\
\text{for all} \quad \alpha \in A \quad \Longleftrightarrow \quad a, b \in J \quad \Longleftrightarrow \quad a\rho_{I,J} b.
\end{aligned}$$

If a and b are not ρ-related, then

$$\begin{aligned}
a\lambda b \quad \Longleftrightarrow \quad a, b \in I_\alpha \\
\text{for all } \alpha \in A \quad \Longleftrightarrow \quad a, b \in I \quad \Longleftrightarrow \quad a\rho_{I,J} b.
\end{aligned}$$

Therefore $\lambda = \rho_{I,J}$, as required. □

For any set X, denote by $\mathcal{P}(X)$ the lattice of all subsets of X.

Theorem 5.5. *Let $\rho \in \mathcal{C}(S)$ and*

$$\Gamma_\rho = \{\rho_{I,J} \mid I, J \in \mathcal{I}, I = I\rho \subseteq J\}.$$

Then Γ_ρ is a distributive complete sublattice of $\mathcal{C}(S)$ containing ρ with greatest element ω and least element ε. The mapping

$$\chi : \lambda \longrightarrow \ker \lambda \quad (\lambda \in \Gamma_\rho)$$

is a complete homomorphism of Γ_ρ into $\mathcal{P}(S)$. Hence $K|_{\Gamma_\rho}$ is a complete congruence.

Proof. Lemmas 5.3 and 5.4 show that Γ_ρ is a complete sublattice of $\mathcal{C}(S)$. Clearly $\omega = \rho_{S,S}$, $\rho = \rho_{\varnothing,S}$ and $\varepsilon = \rho_{\varnothing,\varnothing}$ so that $\omega, \rho, \varepsilon \in \Gamma_\rho$.

Next let

$$\Sigma = \{(I,J) \in \mathcal{I} \times \mathcal{I} \mid I = I\rho \subseteq J\}$$

under the operations of coordinatewise union and intersection. Now Lemmas 5.3 and 5.4 show that the mapping

$$\varphi : (I,J) \longrightarrow \rho_{I,J} \quad ((I,J) \in \Sigma)$$

is a homomorphism of σ onto Γ_ρ. Observing that the operations in $\mathcal{I}$ are set-theoretical union and intersection, we deduce that $\mathcal{I}$ is a distributive lattice and thus so is $\mathcal{I} \times \mathcal{I}$. Since Σ is a sublattice of $\mathcal{I} \times \mathcal{I}$, it also is distributive and therefore its homomorphic image Γ_ρ is distributive as well.

Now let $\{\rho_{I_\alpha,J_\alpha} \mid \alpha \in A\}$ be a subfamily of Γ_ρ. Letting $I = \cup_{\alpha\in A} I_\alpha$ and $J = \cup_{\alpha \in A} J_\alpha$, by Lemma 5.3 we obtain

$$\begin{aligned} \ker\left(\bigvee_{\alpha\in A} \rho_{I_\alpha,J_\alpha}\right) &= \ker \rho_{I,J} = I \bigcup (\ker\rho \cap J) \bigcup E(S), \\ \bigcup_{\alpha\in A} \ker \rho_{I_\alpha,J_\alpha} &= \bigcup_{\alpha\in A} (I_\alpha \cup (\ker\rho \cap J_\alpha) \cup E(S)) \\ &= I \cup \left(\bigcup_{\alpha\in A} (\ker\rho \cap J_\alpha)\right) \cup E(S) \\ &= I \cup (\ker\rho \cap J) \cup E(S). \end{aligned}$$

Since φ is always a complete $\wedge$-homomorphism, the above evidently shows that φ is a complete homomorphism of Γ_ρ into $\mathcal{P}(S)$. As a consequence, $K|_{\Gamma_\rho}$ is a complete congruence. $\square$

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