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PIECEWISE WEIGHTED MEAN FUNCTIONS AND HISTOGRAMS

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ABSTRACT. Piecewise weighted mean functions are used to approximate histograms. For any histogram with uniform class intervals we determine piecewise interpolating weighted mean functions at least continuously differentiable in the approximation interval, preserving monotonicity and positivity of the given histogram, interpolating the frequencies at the middle point of each class interval, and satisfying a global area matching condition.

1. Introduction. Let $F = \{F_1, \dots, F_n\}$ be a histogram where F_i is the frequency for the class interval $[X_i, X_{i+1}]$, with $X_{i+1} - X_i = h_i > 0$, $i = 1, \dots, n$. In order to smooth the histogram F, one can be interested in the construction of a function $s(x)$, at least continuously differentiable in (X_1, X_{n+1}), which satisfies the global area matching condition

$$\int_{X_1}^{X_{n+1}} s(x)\,dx = \sum_{i=1}^{n} h_i F_i, \tag{1.1}$$

or the conditions

$$\int_{X_i}^{X_{i+1}} s(x)\,dx = h_i F_i, \quad i = 1, \dots, n. \tag{1.2}$$

In addition, it is desirable that $s(x)$ reflects the shape of the histogram, which means that properties like monotonicity and/or positivity of F should be preserved. In some recent papers histograms are approximated by using splines satisfying the area matching conditions (1.2). An algorithm which leads to a sufficient condition for the existence of positive and piecewise monotone quadratic splines as well as to their construction is derived in [**5**]. Necessary and sufficient conditions,

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under which rational quadratic splines preserve monotonicity and/or positivity of the data F, are given in [**6**].

In the present paper we show that piecewise weighted mean functions [**3**] can be used for smoothing histograms by preserving monotonicity and positivity of the data F. Although the proposed approximant satisfies only the global area matching condition (1.1), it is computationally advantageous, because it doesn't require the solution of linear systems of equations, as required in [**5**], or systems of inequalities, as in [**6**]. Moreover, no condition on the data is necessary for the existence of the approximant.

In Section 2 we generalize a result, already proved in [**3**], on the continuity class of piecewise weighted mean functions in the approximation interval.

In Section 3 we show that the proposed approximant satisfies (1.1), preserves monotonicity and positivity of the data F and interpolates F_i, $i = 1, \dots, n$, at the middle point of each class interval.

2. Piecewise interpolating mean functions. A wide class of weighted mean functions, with interpolation property, is defined in [**1**]. For a given set of real values $f_i = f(x_i)$, $i = 1, \dots, m$, and distinct nodes x_i, $i = 1, \dots, m$, arbitrarily distributed in $I \subset R$, the interpolating mean is given by

$$u_m(x) = \sum_{i=1}^{m} f_i p_i(x; m) \tag{2.1}$$

where the weight functions $p_i(x; m)$, $i = 1, \dots, m$, satisfy the conditions

$$\begin{cases} p_i(x; m) \geq 0 \\ \sum_{i=1}^{m} p_i(x; m) = 1 \\ p_i(x_j; m) = \delta_{ij}, \qquad i, j = 1, \dots, m, \end{cases} \tag{2.2}$$

and δ_{ij} denotes the Kronecker delta. We consider in particular the weight functions $p_i(x; m)$ which can be represented by the general formula

$$p_i(x; m) = \frac{\left|\prod_{k=1, k\neq i}^{m} [\varphi(x) - \varphi(x_k)]\right|^{\alpha}}{\sum_{j=1}^{m} \left|\prod_{k=1, k\neq j}^{m} [\varphi(x) - \varphi(x_k)]\right|^{\alpha}}, \quad i = 1, \dots, m, \tag{2.3}$$

where $\alpha > 0$ and $\varphi(x)$ is a function strictly monotone in I which belongs to $C^{\lfloor\alpha\rfloor}(I)$, where for any $\lambda \in R$, $\lfloor\lambda\rfloor = \max\{\text{integers } i : i < \lambda\}$. If $\varphi(x) = x$ then (2.1) becomes the Shepard interpolation formula [**2**].

Piecewise mean functions are introduced in [**3**] for interpolating a set of real values $f_i = f(x_i)$, $i = 0, 1, \dots, n+1$, with $x_0 < x_1 < \cdots < x_{n+1}$, and are defined by applying formulas of type (2.1) and (2.3) to the pairs of nodes x_i, x_{i+1}, $i = 0, 1, \dots, n$. The resulting interpolation scheme is

$$u_2(x) = \sum_{j=i}^{i+1} f_j p_{ij}(x), \quad x \in [x_i, x_{i+1}], \tag{2.4}$$

where, for $j = i, i+1$,

$$p_{ij}(x) = \frac{\left|\prod_{k=i,k\neq j}^{i+1} \varphi(x) - \varphi(x_k)\right|^{\alpha_i}}{\sum_{k=1}^{i+1} |\varphi(x) - \varphi(x_k)|^{\alpha_i}}, \tag{2.5}$$

with $\alpha_i > 0$. For the weight functions defined by (2.5), the conditions (2.2) become

$$\begin{cases} p_{ij}(x) \geq 0 & j = i, i+1, \\ \sum_{j=i}^{i+1} p_{ij}(x) = 1, & \\ p_{ij}(x_k) = \delta_{jk} & j, k = i, i+1, \end{cases} \tag{2.6}$$

so that $u_2(x)$ interpolates the values $f_0, f_1, \dots, f_{n+1}$ and is a weighted mean of f_i and f_{i+1} in each interval $[x_i, x_{i+1}]$.

The continuity class of $u_2(x)$in (x_0, x_{n+1}) can be derived from the following proposition, which generalizes a result already proved in [**3**] for the first derivative of $u_2(x)$.

Proposition 2.1. *For $0 \leq \nu < \alpha_i$, it holds*

$$p_{ik}^{(\nu)}(x_i^+) = \begin{cases} \delta_{ki} & \nu = 0, \\ 0 & 0 < \nu < \alpha_i, \end{cases} \tag{2.7}$$

and

$$p_{ik}^{(\nu)}(x_{i+1}^-) = \begin{cases} \delta_{k,i+1} & \nu = 0, \\ 0 & 0 < \nu < \alpha_i, \end{cases} \tag{2.8}$$

with $k = i, i+1$ *and* $i = 0, 1, \dots, n$.

Proof. We set

$$p_{ij}(x) = \begin{cases} A_i(x)B_{i+1}(x) & j = i, \\ A_i(x)B_i(x) & j = i+1, \end{cases}$$

where

$$B_j(x) = |\varphi(x) - \varphi(x_j)|^{\alpha_i}, \quad \text{with} \quad j = i, i+1$$

and

$$A_i(x) = (B_i(x) + B_{i+1}(x))^{-1}.$$

By repeated differentiations of A_i and B_j because of monotonicity and regularity of $\varphi(x)$, for $\nu = 0, 1, \dots, \lfloor \alpha_i \rfloor$ and $j = i, i+1$,

$$\begin{cases} B_j^{(\nu)}(x_j) = 0, \\ A_i^{(\nu)}(x_j) \qquad \text{is bounded.} \end{cases} \tag{2.9}$$

Setting, say, $k = i$ and by using the Leibniz formula,

$$\begin{aligned} p_{ii}^{(\nu)} &= A_i^{(\nu)} B_{i+1} + \binom{\nu}{1} A_i^{(\nu-1)} B'_{i+1} + \cdots \\ &\quad + \binom{\nu}{\nu-1} A'_i B_{i+1}^{(\nu-1)} + A_i B_{i+1}^{(\nu)}. \end{aligned} \tag{2.10}$$

From (2.9), with $j = i+1$, and (2.10), we have

$$p_{ii}^{(\nu)}(x_{i+1}^-) = 0.$$

By using the Leibniz formula for $p_{i,i+1}^{(\nu)}(x)$, we obtain $p_{i,i+1}^{(\nu)}(x_i^+) = 0$ and

$$p_{ii}^{(\nu)}(x_i^+) = \begin{cases} 1 & \nu = 0, \\ 0 & \nu = 1, \dots, \lfloor \alpha_i \rfloor \end{cases}$$

since, by the second part of (2.2), for $r, s = i, i+1$ with $r \neq s$,

$$p_{ir}(x) = 1 - p_{is}(x). \tag{2.11}$$

A similar proof holds for $k = i+1$. □

Corollary 2.1. *It holds*

$$u_2(x) \in C^{\lfloor \alpha \rfloor}(x_0, x_{n+1})$$

where

$$\alpha = \min\{\alpha_0, \alpha_1, \dots, \alpha_n\}.$$

Proof. From (2.7) and (2.8) $u_2(x)$ has null derivatives up to order $\lfloor \alpha \rfloor$ at the points $x_1, x_2, \dots, x_n$. □

We assume throughout the paper that $\alpha > 1$, so that $u_2(x)$ is at least $C^1(x_0, x_{n+1})$. Moreover, $u_2(x)$ satisfies the following properties

$$\min\{f_i, f_{i+1}\} \le u_2(x) \le \max\{f_i f_{i+1}\}, \quad x \in [x_i, x_{i+1}], \tag{2.12}$$

$$\text{if } f_i = f_{i+1} = c, \text{ then } u_2(x) = c, \quad x \in [x_i, x_{i+1}], \tag{2.13}$$

$$\text{if } f_0, f_1, \dots, f_{n+1} \text{ are monotonic, then } u_2(x) \text{ is monotonic in } [x_0, x_{n+1}]. \tag{2.14}$$

Properties (2.12) and (2.13) follow immediately from (2.6) and ensure that $u_2(x)$ preserves the positivity of $f_0, f_1, \dots, f_{n+1}$ and reproduces exactly the constant function. Property (2.14) is proved in [**3**].

3. Smoothing of histograms.

In order to determine piecewise interpolating means satisfying the area matching condition (1.1), we state the following

Lemma 3.1. *If, for any* $\delta \in [0, h_i/2]$ *and* $\bar{x}_i = (x_i + x_{i+1})/2$,

$$\varphi(\bar{x}_i - \delta) - \varphi(x_i) = -[\varphi(\bar{x}_i + \delta) - \varphi(x_{i+1})], \tag{3.1}$$

then

$$\int_{x_i}^{x_{i+1}} u_2(x)\, dx = h_i u_2(\bar{x}_i), \tag{3.2}$$

where

$$u_2(\bar{x}_i) = (f_i + f_{i+1})/2.$$

Proof. From (3.1) for $r, s = i, i+1$, with $r \neq s$,

$$p_{ir}(\bar{x}_i - \delta) = p_{is}(\bar{x}_i + \delta)$$

and, by using (2.11),

$$u_2(\bar{x}_i - \delta) - f_i = -[u_2(\bar{x}_i + \delta) - f_{i+1}], \tag{3.3}$$

which proves our assertion. □

For any histogram F, with uniform class intervals, we define the following sequence of interpolation points

$$\begin{gathered}(x_0 = X_1, f_0 = F_1), \quad (x_1 = \overline{X}_1, f_1 = F_1), \\ (x_2 = \overline{X}_2, f_2 = F_2), \cdots, \quad (x_n = \overline{X}_n, f_n = F_n), \\ (x_{n+1} = X_{n+1}, f_{n+1} = F_n),\end{gathered} \tag{3.4}$$

where, for $i = 1, \ldots, n$, $\overline{X}_i$ denotes the middle point of the class interval $[X_i, X_{i+1}]$. We state the following

Theorem 3.1. *Let F be any histogram with $h_i = h$, $i = 1, \ldots, n$. Assume that $u_2(x)$ is the piecewise weighted mean function defined by* (2.4) *and* (2.5), *with $\varphi(x)$ satisfying* (3.1), *for the sequence of interpolation points* (3.4). *Then $u_2(x)$ is such that*

(a) *u_2 carries over positivity and monotonicity of the sequence $F_1, \ldots, F_n$,*

(b) *$u_2(x) \in C^{\lfloor \alpha \rfloor}(X_1, X_{n+1})$, with $\alpha = \min\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$,*

(c) *$u_2(\overline{X}_i) = F_i$, $i = 1, \ldots, n$,*

(d) *the area matching condition* (1.1) *holds with $s(x) = u_2(x)$.*

Proof. Property (a) follows immediately from (2.12), (2.13) and (2.14). Property (b) holds in virtue of Corollary 2.1 and (c) holds since $u_2(x_i) = f_i$, $i = 0, 1, \ldots, n+1$.

In $[x_0, x_1]$ and in $[x_n, x_{n+1}]$, $u_2(x)$ coincides with the histogram in virtue of (2.13); moreover, from Lemma 3.1, in each interval $[x_i, x_{i+1}]$, $i = 1, \ldots, (n-1)$, we have

$$\int_{x_i}^{x_{i+1}} u_2(x)\, dx = h(f_i + f_{i+1})/2.$$

Property (d) follows from additivity of integrals. □

The graphical application of the proposed method is presented in [**4**], where some examples are given for $u_2(x)$ obtained by setting $\varphi(x) = x$ and $\alpha_i = \alpha$ in (2.5).

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