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TYPE, COTYPE AND GENERALIZED RADEMACHER FUNCTIONS

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ABSTRACT. The main goal of this paper is to show that the traditional Rademacher functions can be replaced, up to a change of constants, by the generalized Rademacher functions in the definitions of type and cotype in complex Banach spaces. It is also shown that there are standard type Kahane inequalities for the generalized Rademacher functions. As an application we prove the continuity of the tensor product of certain multilinear mappings and homogeneous polynomials.

Introduction. The generalized Rademacher functions, which were introduced by Aron and Globevnik [**2**], have been used by several authors to prove new theorems and to provide simpler proofs of known results, especially in the theory of multilinear mappings and homogeneous polynomials between Banach spaces, e.g., [**1, 3, 8, 9, 12**] and [**14**]. An important result was obtained by Floret and Matos in [**8**]: if we replace the traditional Rademacher functions by the generalized ones in Khintchine's inequalities, the resulting inequalities are still true (we prove the same for Kahane's inequalities in Section 5). Since the notions of type and cotype in Banach spaces are usually introduced with the help of the traditional Rademacher functions, it is natural to ask what happens if we replace the Rademacher functions by the generalized ones in such definitions. The main result of this paper, Corollary 4.2, provides the answer: nothing happens. In other words, given $n \in \mathbf{N}$, $n \geq 2$, if the n-Rademacher functions take the place of the traditional Rademacher functions in the definitions of type and cotype in complex Banach spaces, the resulting definitions are equivalent to the original ones (up to a change of constants). The proof is an adaptation of the proof of the equivalence between the notions of Rademacher and Gaussian types and cotypes. The basic difficulty is the fact that, if $n > 2$, the n-Rademacher functions are no longer real-valued symmetric random variables. To solve this problem we must introduce the notion

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of n-symmetric random variables, Section 2, and consider a complex version of the contraction principle, Proposition 3.1.

An application of the main result is obtained when we consider, likewise, the case of linear operators, the tensor product of multilinear mappings and homogeneous polynomials. If F is a Banach space and $1 \leq p \leq \infty$, let Δ_p be the natural norm induced on $L_p(\mu) \otimes F$ from $L_p(\mu, F)$. With the help of the main result and the generalized Kahane's inequality, we shall prove the continuity of the tensor product of certain multilinear mappings and homogeneous polynomials with respect to Δ_p.

1. Preliminaries. Throughout this paper n is an integer not smaller than 2 and $E_1, \dots, E_n$, E and F are complex Banach spaces. The classical Rademacher functions will be denoted by $(r_j)_{j=1}^{\infty}$, that is, for $j \in \mathbf{N}$ and $t \in [0,1]$, $r_j(t) = \operatorname{sign}[\sin(2^j \pi t)]$.

Given $1 \leq p \leq 2$, we say that E has *type* p if there is a $C \geq 0$ such that, for every $k \in \mathbf{N}$ and $x_1, \dots, x_k \in E$,

$$\bigg(\int_0^1 \bigg\| \sum_{j=1}^{k} r_j(t) x_j \bigg\|^2 dt\bigg)^{1/2} \leq C \bigg(\sum_{j=1}^{k} \|x_j\|^p\bigg)^{1/p}.$$

Given $2 \leq q \leq \infty$, we say that E has *cotype* q if there is a $C \geq 0$ such that, for every $k \in \mathbf{N}$ and $x_1, \dots, x_k \in E$,

$$\bigg(\sum_{j=1}^{k} \|x_j\|^q\bigg)^{1/q} \leq C \bigg(\int_0^1 \bigg\| \sum_{j=1}^{k} r_j(t) x_j \bigg\|^2 dt\bigg)^{1/2}.$$

Of course, if $q = \infty$ we consider the sup-norm. The main properties and most important examples concerning type and cotype are summarized in [**7**, Chapter 11].

For a fixed natural number $n \geq 2$ we take the nth roots of unity $1 = \lambda_1, \lambda_2, \dots, \lambda_n$ considered in the order of their increasing arguments. The n-Rademacher functions, denoted by $(s_k^{(n)})_{k=1}^{\infty}$ are defined similarly to the traditional Rademacher functions. Each $s_k^{(n)}$ is a function from $[0,1]$ in the set $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbf{C}$. To define $s_1^{(n)}$ divide $[0,1]$ in n intervals of equal length and assign $s_1^{(n)}(t) = \lambda_j$ if t belongs to the jth interval. The definition of $s_2^{(n)}$ is established by a repetition of the previous procedure in each of the subintervals considered in the definition of $s_1^{(n)}$. The process continues indefinitely. Explicit descriptions can be found in [**3**] and [**8**], and the most important properties of these functions are:

(i) for every $k \in \mathbf{N}$ and $j = 1, \dots, n$, the set $\{t \in [0,1] : s_k^{(n)}(t) = \lambda_j\}$ is a union of a finite number of intervals and its measure is $1/n$. Of course, $|s_k^{(n)}(t)| = 1$ for every $k \in \mathbf{N}$ and $t \in [0,1]$.

(ii) Multiorthogonality.

$$\int_0^1 s_{i_1}^{(n)}(t) \cdots s_{i_n}^{(n)}(t)\, dt = \begin{cases} 1 & \text{if } i_1 = \cdots = i_n \\ 0 & \text{otherwise.} \end{cases}$$

(iii) The sequence $(s_k^{(n)})_{k=1}^{\infty}$ is independent.

If $p \in [1,\infty)$, $l_p(E)$ denotes the Banach space of all sequences $(x_j)_{j\in\mathbf{N}}$ in E which are absolutely p-summable with the norm $\|(x_j)_{j=1}^{\infty}\|_p = (\sum_{j=1}^{\infty} \|x_j\|^p)^{1/p}$.

If (Ω,μ) is a finite measure space, $L_p(\mu, E)$ is the Banach space of (classes of) strongly μ-measurable functions $f : \Omega \to E$ such that

$$\|f\|_p = \left(\int_\Omega \|f\|^p \, d\mu \right)^{1/p} < \infty.$$

For $p = \infty$ we make the usual modifications to define $l_\infty(E)$ and $L_\infty(\mu, E)$. If μ is the Lebesgue measure on the closed interval $[0,1]$, we use the simplified notation $L_p(E)$.

Assume now that (Ω,μ) is a probability space. We denote by $(g_k)_{k=1}^{\infty}$ a sequence of independent standard Gaussian complex random variables on (Ω,μ), i.e., each $g_k : \Omega \to \mathbf{C}$ has distribution $\mu(g_k \in B) = \gamma(B)$ for every Borel subset $B \subset \mathbf{C}$, where γ is the standard Gaussian measure on the complex plane. It is well known that, for every $0 < p < \infty$, the absolute moments

$$m_p = \left(\int_{\mathbf{C}} |z|^p \, d\gamma(z) \right)^{1/p}$$

exist and are finite (for precise values, see [**17**, p. 14]). Moreover, $m_p = \|g_k\|_{L_p}$ for all $k \in \mathbf{N}$.

2. ***N*-symmetric random variables.** Let (Ω, P) be a fixed probability space. Given a sequence $(f_j)_{j=1}^{\infty}$ of complex random variables on Ω, define

$$(f_1, \dots, f_j, \dots)(\omega) = (f_1(\omega), \dots, f_j(\omega), \dots) \quad \text{for every } \omega \in \Omega.$$

$\beta(\mathbf{C})$ denotes the Borel σ-algebra on $\mathbf{C}$ and $\mathcal{F}$ is its product σ-algebra on $\mathbf{C}^{\infty}$:

$$\mathcal{F} = \beta(\mathbf{C}) \otimes \beta(\mathbf{C}) \otimes \cdots .$$

The joint distribution of $(f_j)_{j=1}^{\infty}$, denote by $P_{(f_1, \dots, f_j, \dots)}$, is the following probability measure on $\mathcal{F}$:

$$P_{(f_1, \dots, f_j, \dots)}(B) = P((f_1, \dots, f_j, \dots)^{-1}(B)), \quad \forall\, B \in \mathcal{F}.$$

The sequence $(f_j)_{j=1}^{\infty}$ is said to be *n-symmetric* if $P_{(f_1, \dots, f_j, \dots)} = P_{(\theta_1 f_1, \dots, \theta_j f_j, \dots)}$ for every sequence $(\theta_j)_{j \in \mathbf{N}}$ of nth roots of unity, i.e., for every $j \in \mathbf{N}$, $\theta_j \in \{\lambda_1, \dots, \lambda_n\}$. The case $n = 2$ gives the classical notion of symmetric sequences, see [**10**].

Examples. It is not difficult to see that, for every $n \in \mathbf{N}$, the n-Rademacher functions $(s_k^{(n)})_{k=1}^{\infty}$ and the independent Gaussian variables $(g_k)_{k=1}^{\infty}$ on the probability space (Ω, P) are n-symmetric (since both of these sequences are independent, it is sufficient to check that $P_{(f_j)} = P_{(\theta_j f_j)}$ for every j). Now it is easy to prove the following result:

Proposition 2.1. *If E is a normed space, $x_1, \dots, x_k \in E$, $p \in \mathbf{R}$, $n \in \mathbf{N}$ and $\theta_1, \dots, \theta_k \in \{\lambda_1, \dots, \lambda_n\}$, then*

$$\int_0^1 \left\| \sum_{j=1}^{k} \theta_j s_j^{(n)}(t) x_j \right\|^p dt = \int_0^1 \left\| \sum_{j=1}^{k} s_j^{(n)}(t) x_j \right\|^p dt$$

and

$$\int_{\Omega} \left\| \sum_{j=1}^{k} \theta_j g_j(w) x_j \right\|^p dP(w) = \int_{\Omega} \left\| \sum_{j=1}^{k} g_j(w) x_j \right\|^p dP(w).$$

3. Preliminary results. Our purpose in this section is to study the comparisons between the Gaussian and the n-Rademacher averages. The results we shall prove show that the situation is identical to the case $n = 2$, but the proofs are slightly different. First we need the following complex version of the contraction principle:

Proposition 3.1. *Let $X_1, \dots, X_k$ be independent E-valued p-integrable random variables, where $1 \le p < \infty$. If $\int X_j = 0$ for all $j = 1, \dots, k$, then*

$$\Big\| \sum_{j=1}^{k} a_j X_j \Big\|_{L_p(E)} \le 4 \Big\| \sum_{j=1}^{k} X_j \Big\|_{L_p(E)}$$

for all $a_1, \dots, a_k \in \mathbf{C}$ such that $|a_j| \le 1$.

Proof. The result follows from a simple adaptation of the proof of [**10**, Lemma 4.1]. We have only to consider the sets $L_k = \{(a_1, \dots, a_k) \in \mathbf{C}^k : |a_j| \le 1 \text{ for } j = 1, \dots, k\}$ and $K_{2k} = \{(b_1, \beta_1, \dots, b_k, \beta_k) \in \mathbf{R}^{2k} : |b_j| \le 1 \text{ and } |\beta_j| \le 1 \text{ for } j = 1, \dots, k\}$. The identification $a_j = (b_j, \beta_j)$ shows that $L_k \subset K_{2k}$. Now the result follows from a careful repetition of the arguments used in the proof of [**10**, Lemma 4.1]. It should be noted that, while the result we need is a corollary of the proof of Hoffmann-Jørgensen's, it is not a corollary of the lemma itself; if we split each X_j into its real and imaginary parts, say $X_j = Y_j + iZ_j$, the new collection of real random variables $Y_1, \dots, Y_k, Z_1, \dots, Z_k$ has no need to be independent. □

Proposition 3.2. *Let $n \in \mathbf{N}$, $n > 1$ and $x_1, \dots, x_k \in E$. Then*

$$\Big\| \sum_{j=1}^{k} s_j^{(n)} x_j \Big\|_{L_2(E)} \le \frac{4}{m_1} \Big\| \sum_{j=1}^{k} g_j x_j \Big\|_{L_2(E)}.$$

Proof. We know that $|g_j(w)| = g_j(w)/\exp(i \arg(g_j(w)))$ for each $w \in \Omega$ and $j = 1, \dots, k$. Putting $a_j = 1/\exp(i \arg(g_j(w)))$ and $X_j(t) = s_j^{(n)}(t) g_j(w) x_j$, from Proposition 3.1 (with $p = 2$), we have that, for every $w \in \Omega$,

$$\begin{aligned}\int_0^1 \Big\| \sum_{j=1}^k s_j^{(n)}(t)|g_j(w)|x_j \Big\|^2 dt \\ &= \int_0^1 \Big\| \sum_{j=1}^k \frac{1}{\exp(i \arg(g_j(w)))} s_j^{(n)}(t) g_j(w) x_j \Big\|^2 dt \\ &\le 4^2 \int_0^1 \Big\| \sum_{j=1}^k s_j^{(n)}(t) g_j(w) x_j \Big\|^2 dt.\end{aligned}$$

Now we use the inequality above, Proposition 2.1 and Fubini's theorem to obtain

$$\begin{aligned}\left(\int_0^1 \Big\| \sum_{j=1}^k s_j^{(n)}(t) x_j \Big\|^2 dt \right)^{1/2} \\ &= \frac{1}{m_1} \left(\int_0^1 \Big\| \sum_{j=1}^k s_j^{(n)}(t) \Big[\int_\Omega |g_j(w)|\, dw \Big] x_j \Big\|^2 dt \right)^{1/2} \\ &\le \frac{1}{m_1} \left(\int_\Omega \int_0^1 \Big\| \sum_{j=1}^k s_j^{(n)}(t) |g_j(w)| x_j \Big\|^2 dt\, dw \right)^{1/2} \\ &\le \frac{4}{m_1} \left(\int_\Omega \int_0^1 \Big\| \sum_{j=1}^k s_j^{(n)}(t) g_j(w) x_j \Big\|^2 dt\, dw \right)^{1/2} \\ &= \frac{4}{m_1} \left(\int_0^1 \left(\int_\Omega \Big\| \sum_{j=1}^k s_j^{(n)}(t) g_j(w) x_j \Big\|^2 dw \right) dt \right)^{1/2} \\ &= \frac{4}{m_1} \left(\int_\Omega \Big\| \sum_{j=1}^k g_j(w) x_j \Big\|^2 dw \right)^{1/2}. \qquad \square\end{aligned}$$

In order to study the converse estimate we must deal with the concept of finite representability: the Banach space E is *finitely representable* in the Banach space F if, no matter how we choose $\varepsilon > 0$, for each finite dimensional subspace E_0 of E we can find a finite dimensional subspace F_0 of F and an isomorphism $T\colon E_0 \to F_0$ such that $\|T\| \cdot \|T^{-1}\| \le 1+\varepsilon$.

Proposition 3.3. *Let $n \in \mathbf{N}$, $n > 1$. If l_∞ is not finitely representable in E, then there is a constant $C \geq 0$ such that*

$$\Big\| \sum_{j=1}^{k} g_j x_j \Big\|_{L_2(E)} \leq C \Big\| \sum_{j=1}^{k} s_j^{(n)} x_j \Big\|_{L_2(E)},$$

for every $x_1, \dots, x_k \in E$.

Sketch of the proof. Since the proof is an adaptation of the proof of a theorem of Maurey and Pisier [**13**, Corollaire 1.3], we will only sketch the proof, indicating the points where the differences arise. Since l_∞ is not finitely representable in E, combining some results of [**13**, Théorème 1.2, Remarque 1.4 and Corollaire 1.2] we know that there is a $2 < q < \infty$ such that every linear operator from c_0 to $L_2(E)$ is q-summing. Hence there is a constant $K \geq 0$ such that, for every linear operator $u : c_0 \to L_2(E)$, if $\pi_q(u)$ denotes the q-summing of u, then

$$\pi_q(u) \leq K \cdot \|u\|.$$

Given $x_1, \dots, x_k \in E$, consider the operator

$$u : c_0 \to L_2(E), u((c_j)_{j=1}^{\infty}) = \sum_{j=1}^{k} c_j x_j s_j^{(n)}(t).$$

Since u is bounded, it is also q-summing. A suitable use of Pietsch's factorization theorem yields the existence of positive numbers $\alpha_1, \dots, \alpha_k$ such that $\alpha_1 + \cdots + \alpha_k = 1$ and

$$\|u((c_j))\| \leq \pi_q(u) \bigg(\sum_{j=1}^{k} \alpha_j |c_j|^q \bigg)^{1/q}.$$

Therefore, for every $w \in \Omega$,

$$\begin{aligned} \bigg(\int_0^1 \Big\| \sum_{j=1}^{k} g_j(w) x_j s_j^{(n)}(t) \Big\|^2 dt \bigg)^{1/2} &= \|u((g_j(w))\| \\ &\leq K \|u\| \bigg(\sum_{j=1}^{k} \alpha_j |g_j(w)|^q \bigg)^{1/q}. \end{aligned}$$

An appeal to the last inequality, Fubini's theorem and Proposition 2.1 yields (remember that $q > 2$)

$$\begin{aligned}\bigg(\int_\Omega \bigg\|\sum_{j=1}^k g_j(w)x_j\bigg\|^2 dw\bigg)^{1/2} &\\ &= \bigg(\int_0^1\int_\Omega \bigg\|\sum_{j=1}^k g_j(w)x_j s_j^{(n)}(t)\bigg\|^2 dw\,dt\bigg)^{1/2}\\ &\le K\|u\|\bigg(\int_\Omega \bigg(\sum_{j=1}^k \alpha_j|g_j(w)|^q\bigg)^{2/q} dw\bigg)^{1/2}\\ &\le K\|u\|\bigg(\int_\Omega \bigg(\sum_{j=1}^k \alpha_j|g_j(w)|^q\bigg) dw\bigg)^{1/q}\\ &= K\|u\|\bigg(\int_\Omega |g_1(w)|^q\,dw\bigg)^{1/q}\\ &= K\|u\|m_q.\end{aligned}$$

To complete the proof we use Proposition 3.1 with $p = 2$ and $X_j(t) = s_j^{(n)}(t)x_j$ for $j = 1, \dots, k$, to establish that

$$\begin{aligned}\|u\|^2 &= \sup\bigg\{\int_0^1 \bigg\|\sum_{j=1}^k c_j s_j^{(n)}(t)x_j\bigg\|^2 dt : |c_j| \le 1\bigg\}\\ &\le 4\int_0^1 \bigg\|\sum_{j=1}^k s_j^{(n)}(t)x_j\bigg\|^2 dt.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1 \bigg\|\sum_{j=1}^k s_j^{(n)}(t)x_j\bigg\|^2 dt &\ge \frac{1}{4}\|u\|^2\\ &\ge \frac{1}{4K^2m_q^2}\int_\Omega \bigg\|\sum_{j=1}^k g_j(w)x_j\bigg\|^2 dw. \qquad \square\end{aligned}$$

4. Main result. Let $1 \le p \le 2 \le q \le \infty$. A complex Banach space E has *n-type* p, respectively *n-cotype* q, if there is a constant $C \ge 0$

such that

$$\Big\| \sum_{j=1}^{k} s_j^{(n)} x_j \Big\|_{L_2(E)} \le C \|(x_j)_{j=1}^k\|_p$$

respectively,

$$\|(x_j)_{j=1}^k\|_q \le C \Big\| \sum_{j=1}^{k} s_j^{(n)} x_j \Big\|_{L_2(E)},$$

for every $x_1, \dots, x_k \in E$.

E has *Gaussian type* p, respectively *Gaussian cotype* q, if there is a $C \ge 0$ such that

$$\Big\| \sum_{j=1}^{k} g_j x_j \Big\|_{L_2(E)} \le C \|(x_j)_{j=1}^k\|_p$$

respectively,

$$\|(x_j)_{j=1}^k\|_q \le C \Big\| \sum_{j=1}^{k} g_j x_j \Big\|_{L_2(E)},$$

for every $x_1, \dots, x_k \in E$.

Theorem 4.1. *Let E be a complex Banach space, $n \in \mathbf{N}$, $n > 1$ and $1 \le p \le 2 \le q \le \infty$.*

(i) *E has n-type p if and only if E has Gaussian type p.*

(ii) *E has n-cotype q if and only if E has Gaussian cotype q.*

Proof. (i) If E has Gaussian type p, the result follows immediately from Proposition 3.2. Assume now that E has n-type p. Using Proposition 2.1, Fubini's theorem, the fact that $p \le 2$, the assumption that E has n-type p and that $m_p = \|g_j\|_{L_p}$ for every j (in this order), we have

$$\begin{aligned}\int_\Omega \Big\| \sum_{j=1}^{k} g_j(w) x_j \Big\|^p dw &= \int_0^1 \int_\Omega \Big\| \sum_{j=1}^{k} s_j^{(n)}(t) g_j(w) x_j \Big\|^p dw\, dt \\ &= \int_\Omega \Big\| \sum_{j=1}^{k} s_j^{(n)} g_j(w) x_j \Big\|_{L_p(E)}^p dw\end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega \left\| \sum_{j=1}^k s_j^{(n)} g_j(w) x_j \right\|_{L_2(E)}^p dw \\
&\leq C^p \int_\Omega \sum_{j=1}^k \|g_j(w)x_j\|^p \, dw \\
&= C^p \int_\Omega \sum_{j=1}^k |g_j(w)|^p \|x_j\|^p \, dw \\
&= C^p \cdot (m_p)^p \cdot \sum_{j=1}^k \|x_j\|^p.
\end{aligned}
$$

With the help of Kahane's inequalities for Gaussian variables [**11**, Corollary 4.8] we know that there is a constant C', depending only on p, such that

$$
\begin{aligned}
\left\| \sum_{j=1}^k g_j x_j \right\|_{L_2(E)} &\leq C' \left\| \sum_{j=1}^k g_j x_j \right\|_{L_p(E)} \\
&\leq C' \cdot C \cdot m_p \cdot \|(x_j)_{j=1}^k\|_p.
\end{aligned}
$$

(ii) If E has n-cotype q the result follows (again) immediately from Proposition 3.2. Assume now that E has Gaussian cotype q. Using the equality $\lambda_1 + \cdots + \lambda_n = 0$ and the n-symmetry of the n-Rademacher functions in the form of Proposition 2.1, the proof of the fact that every Banach space has cotype ∞ can be adapted to prove that every complex Banach space has n-cotype ∞. Thus we can assume $q < \infty$. Since l_∞ cannot be finitely representable in a Banach space having Gaussian cotype $q < \infty$, the result follows from Proposition 3.3. □

Corollary 4.2. *Let E be a complex Banach space, $m, n \in \mathbf{N}$, $m, n > 1$ and $1 \leq p \leq 2 \leq q \leq \infty$. If E has n-type p, respectively n-cotype q, then E has m-type p, respectively m-cotype q. In particular, in the definitions of type and cotype in complex Banach spaces, the traditional Rademacher functions can be replaced (up to a change of constants) by the n-Rademacher functions, $(s_k^{(n)})_{k=1}^\infty$, for every $n > 1$.*

Proof. Apply Theorem 4.1 in one direction for n and in the other direction for m. □

Remarks 4.3. (i) Let us fix the following notations:

$$\begin{aligned} T_{(p,n)}(E) &= \inf\bigg\{C : \bigg\|\sum_{j=1}^{k} s_j^{(n)} x_j\bigg\|_{L_2(E)} \\ &\qquad\qquad \leq C\|(x_j)_{j=1}^k\|_p, x_1, \dots, x_k \in E\bigg\}, \\ C_{(q,n)}(E) &= \inf\bigg\{C : \|(x_j)_{j=1}^k\|_q \\ &\qquad\qquad \leq C\bigg\|\bigg\|\sum_{j=1}^{k} s_j^{(n)} x_j\bigg\|_{L_2(E)}, x_1, \dots, x_k \in E\bigg\|\bigg\}. \end{aligned}$$

It might be useful to know, especially in infinite dimensional holomorphy, that if E, p and q are fixed, then the sequences $(T_{(p,n)}(E))_{n\in\mathbf{N}}$ and $(C_{(q,n)}(E))_{n\in\mathbf{N}}$ are bounded. It is easy to see that $T_{(1,n)}(E) = 1 = C_{(\infty,n)}(E)$ for every n. If E has type $p > 1$, respectively cotype $q < \infty$, then from the proof of Proposition 3.2, respectively Proposition 3.3, we have

$$T_{(p,n)}(E) \leq (4/m_1)\tilde{T}_p(E)$$

respectively,

$$C_{(q,n)}(E) \leq K_E \tilde{C}_q(E),$$

for every n, where $\tilde{T}_p(E)$ is the Gaussian type p constant of E, $\tilde{C}_q(E)$ is the Gaussian cotype q constant of E and K_E is a constant depending only on E. In fact, $K_E = 2Km_r$ where $r > 2$ is such that every operator from c_0 to $L_2(E)$ is r-summing and K is such that $\pi_r(u) \leq K \cdot \|u\|$ for every $u : c_0 \to L_2(E)$.

(ii) We have been working with the notions of type and cotype for spaces, not for operators. Not everything we did can be extended to the case of operators. Given $1 \leq p \leq 2 \leq q \leq \infty$, $n \in \mathbf{N}$, $n > 1$ and $T : E \to F$ a bounded linear operator between complex Banach spaces, we say that T has *n-type* p, respectively *n-cotype* q, if there is a $C \geq 0$ such that

$$\bigg\|\sum_{j=1}^{k} s_j^{(n)} T(x_j)\bigg\|_{L_2(F)} \leq C\|(x_j)_{j=1}^k\|_p$$

respectively,

$$\|(T(x_j))_{j=1}^k\|_q \leq C\Big\|\sum_{j=1}^k s_j^{(n)} x_j\Big\|_{L_2(E)},$$

for every $x_1, \dots, x_k \in E$.

The case $n = 2$ gives the classical operator ideals of all operators of type p, respectively all operators of cotype q, (see [**17**, Section 25]). If $s_j^{(n)}$ is replaced by g_j for every $j \in \mathbf{N}$, we say that T has *Gaussian type* p, respectively *Gaussian cotype* q. The proof of Theorem 4.1 can be easily adapted to prove that:

(a) T has n-type p if and only if T has Gaussian type p.

(b) If T has n-cotype q then T has Gaussian cotype q.

(c) If l_∞ is not finitely representable in E and T has Gaussian cotype q, then T has n-cotype q.

Therefore, for every $n \in \mathbf{N}$, $n > 1$, the n-Rademacher functions can replace the traditional Rademacher functions in the definition of operators of type p; and if l_∞ is not finitely representable in E then the same occurs with the definition of operators of cotype q.

5. Generalized Kahane's inequality. We have already mentioned that Floret and Matos showed in [**8**] that there are Khintchine inequalities for the generalized Rademacher functions. In this section we prove that the same occurs with Kahane's inequalities (this result will be needed in the next section). The proof is an adaptation of the proof of [**18**, Theorem III.A.18]. In that fashion we first have to prove the following lemma.

Lemma 5.1. *Let E be a complex Banach space and $\{x_1, \dots, x_k\}$ be a finite sequence of elements in E. If $n \in \mathbf{N}$, $n > 1$, μ is the Lebesgue measure on the closed interval $[0,1]$ and $V(t) = \|\sum_{j=1}^k s_j^{(n)}(t) x_j\|$ for every $t \in [0,1]$, then for every $\alpha > 0$,*

$$\mu(\{t : V(t) > 2\alpha\}) \leq n^2 \mu(\{t : V(t) > \alpha\})^2.$$

Proof. The proof of [**18**, Proposition III.A.19] still works if 2 is replaced by n. We shall only sketch the steps that should be followed.

The functions $V_r(t)$ for $r = 1, \ldots, k$, and the sets A_m and C_m for $m = 1, \ldots, k$; A, B and C are defined analogously to the original proof. The same for the functions $a(t)$ and $b(t)$. Consider now the partition of $[0, 1]$ formed by the intervals $\{D_i : i = 1, \ldots, n^m\}$, where $s_1^{(n)}, \ldots, s_m^{(n)}$ are constants in each D_i. Now define $E_m = \{t \in A_m : \|a(t)\| \leq \|a(t) + b(t)\|\}$. With the help of the two statements below:

(a) If D is an interval of the family $\{D_i : i = 1, \ldots, n^m\}$ and $t \in D$, then there are $t_1, \ldots, t_n \in D$ such that $b(t_i) = \lambda_i b(t)$ for $i = 1, \ldots, n$,

(b) $\|x\| \leq \max\{\|x + \lambda_i y\| : i = 1, \ldots, n\}$ for every $x, y \in E$,

we are able to prove that $\mu(A_m) \leq n\mu(E_m)$.

Since $E_m \subset (A_m \cap B)$ it follows that

$$\begin{aligned}\mu(A) = \mu\bigg(\bigcup_{m=1}^{k} A_m\bigg) &= \sum_{m=1}^{k} \mu(A_m) \\ &\leq n \sum_{m=1}^{k} \mu(A_m \cap B) = n\mu(A \cap B) \leq n\mu(B).\end{aligned}$$

Analogously $\mu(C_m) \leq n\mu(B)$.

Define the sets $A_{m,1}, \ldots, A_{m,n}$ and $C_{m,1}, \ldots, C_{m,n}$ by:

$$A_{m,1} = A_m \cap \{t : s_m(t) = \lambda_1\}, \ldots, A_{m,n} = A_m \cap \{t : s_m(t) = \lambda_n\}$$

and

$$C_{m,1} = C_m \cap \{t : s_m(t) = \lambda_1\}, \ldots, C_{m,n} = C_m \cap \{t : s_m(t) = \lambda_n\}.$$

For $i = 1, \ldots, n$, the independence of the n-Rademacher functions shows that

$$\begin{aligned}\mu(A_{m,i} \cap C_{m,i}) &= n\mu(A_{m,i})\mu(C_{m,i}), \\ \mu(A_m) = n\mu(A_{m,i}) \quad &\text{and} \quad \mu(C_m) = n\mu(C_{m,i}).\end{aligned}$$

Now we have

$$\begin{aligned}\mu(A_m \cap C_m) &= \mu\bigg(\bigcup_{i=1}^{n}(A_{m,i} \cap C_{m,i})\bigg)\\ &= \sum_{i=1}^{n} n\mu(A_{m,i})\mu(C_{m,i})\\ &= n^2\mu(A_{m,1})\mu(C_{m,1})\\ &= n^2\frac{1}{n}\mu(A_m)\frac{1}{n}\mu(C_m),\end{aligned}$$

then

$$\mu(A_m \cap C_m) = \mu(A_m)\mu(C_m).$$

Since $C \subset B \subset A$ and $(A_m \cap C) \subset (A_m \cap C_m)$ we can finish the proof of the lemma

$$\begin{aligned}\mu(C) = \mu(A \cap C) &= \mu\bigg(\bigcup_{m=1}^{k}(A_m \cap C)\bigg)\\ &\leq \sum_{m=1}^{k}\mu(A_m \cap C_m) = \sum_{m=1}^{k}\mu(A_m)\mu(C_m)\\ &\leq \sup\{\mu(C_m) : m = 1, \dots, k\} \cdot \sum_{m=1}^{k}\mu(A_m)\\ &\leq n\mu(B)\mu(A) \leq n^2\mu(B)^2. \qquad \square\end{aligned}$$

Theorem 5.2. (Generalized Kahane's inequalities). *Let $n \in \mathbf{N}$, $n > 1$, and $0 < p < q < \infty$. There exists a constant $K(n,p,q)$ such that, for every complex Banach space E, the inequalities*

$$\begin{aligned}\bigg\|\sum_{j=1}^{k} s_j^{(n)} x_j\bigg\|_{L_p(E)} &\leq \bigg\|\sum_{j=1}^{k} s_j^{(n)} x_j\bigg\|_{L_q(E)}\\ &\leq K(n,p,q)\bigg\|\sum_{j=1}^{k} s_j^{(n)} x_j\bigg\|_{L_p(E)},\end{aligned}$$

hold for every finite sequence $x_1, \dots, x_k \in E$.

Proof. Of course, we only have to deal with the righthand side inequality. We can assume $\int_0^1 V(t)^p\,dt = 1$, so Chebyshev's inequality yields

$$\mu(\{t : V(t) > n^{3/p}\}) \leq \left(\frac{1}{n^{3/p}}\|V(t)\|_p\right)^p = \frac{1}{n^3}.$$

Applying Lemma 5.1 inductively for $r = 1, 2, \ldots$, we obtain

$$\begin{aligned}\mu(\{t : V(t) > 2^r n^{3/p}\}) &\leq n^2 \mu(\{t : V(t) > 2^{r-1} n^{3/p}\})^2 \\ &\leq (n^2)^4 \mu(\{t : V(t) > 2^{r-2} n^{3/p}\})^4 \\ &\leq (n^2)^{2^r} \mu(\{t : V(t) > n^{3/p}\})2^r \\ &\leq (n^2)^{2^r} (n^3)^{-2^r} = n^{-2^r}.\end{aligned}$$

But $[0,1] = \{t : V(t) \leq n^{3/p}\} \cup (\cup_{r=1}^{\infty} \{t : 2^{r-1} n^{3/p} < V(t) \leq 2^r n^{3/p}\})$, then

$$\begin{aligned}\int_0^1 V(t)^q\,dt &\leq (n^{3/p})^q \mu(\{t : V(t) \leq n^{3/p}\}) \\ &\qquad + \sum_{r=1}^{\infty} (2^{rq} n^{3q/p}) \mu(\{t : V(t) > 2^{r-1} n^{3/p}\}) \\ &\leq n^{3q/p} + n^{3q/p} \cdot \sum_{r=1}^{\infty} (2^{rq} n^{-2^{r-1}}) \\ &\leq n^{3q/p} \left[1 + \sum_{r=1}^{\infty} (2^{rq} \cdot 2^{-2^{r-1}})\right].\end{aligned}$$

Choose $K(n,p,q) = n^{3/p}[1 + \sum_{r=1}^{\infty} (2^{rq} \cdot 2^{-2^{r-1}})]^{1/q}$. $\square$

Remark 5.3. It should be noted that, contrary to the proof that [**8**] provides for the generalized Khintchine's inequalities, the proof above gives no hint whether the constants may be chosen independently from n. If $K_{p,q}$ denotes the constant that [**18**, Theorem III.A.18] results in the classical Kahane's inequality (the case $n = 2$) then all we have is that $K(n,p,q) \leq (n^3/8) K_{p,q}$.

6. Application. The aim of this section is to show that the previous results can be used to establish interesting results concerning tensor products of multilinear mappings and homogeneous polynomials between complex Banach spaces.

6.1 *Basic definitions and notations.* n will be an integer not smaller than 2 and $E_1, \dots, E_n$, E and F will stand for complex Banach spaces. The Banach space of all continuous n-linear mappings A from $E_1 \times \cdots \times E_n$ to F endowed with the norm

$$\|A\| = \sup\{\|A(x_1, \dots, x_n)\| : \|x_j\| \leq 1, j = 1, \dots, n\}$$

will be denoted by $L(E_1, \dots, E_n; F)$. If $E_1 = \cdots = E_n = E$, we write $L(^nE; F)$. $P(^nE; F)$ is the Banach space of all continuous n-homogeneous polynomials P from E to F with the norm

$$\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\} = \inf\{C : \|P(x)\| \leq C\|x\|^n, \forall\, x \in E\}.$$

Remember that, for each $P \in P(^nE; F)$, there is a unique symmetric n-linear continuous mapping $\check{P} \in L(^nE; F)$ such that $P(x) = \check{P}(x, \dots, x)$ for every $x \in E$ (further details can be found in Mujica [**15**]).

Let $L_a(E; F)$ be the vector space of all (not necessarily continuous) linear operators from E to F. Given $T \in L_a(E_1; F_1)$ and $U \in L_a(E_2; F_2)$, from the universal property of tensor products we know that there is a (unique) linear operator

$$T \otimes U : E_1 \otimes E_2 \to F_1 \otimes F_2$$

such that $T \otimes U(x \otimes y) = T(x) \otimes U(y)$ for all $x \in E_1$ and $y \in E_2$. Let us see that this notion can be naturally extended to multilinear and polynomial cases.

Definition 6.2. Let $L_a(E_1, \dots, E_n; F)$ be the vector space of all (not necessarily continuous) n-linear mappings from $E_1 \times \cdots \times E_n$ to F. $P_a(^nE; F)$ is defined analogously for polynomials. Given $A \in L_a(E_1, \dots, E_n; F)$ and $B \in L_a(G_1, \dots, G_n; H)$, let A_L and B_L be their linearizations, i.e.,

$$A_L \in L_a(E_1 \otimes \cdots \otimes E_n; F) \quad \text{and} \quad A_L(x_1 \otimes \cdots \otimes x_n) = A(x_1, \dots, x_n),$$
$$B_L \in L_a(G_1 \otimes \cdots \otimes G_n; H) \quad \text{and} \quad B_L(y_1 \otimes \cdots \otimes y_n) = B(y_1, \dots, y_n).$$

Consider now the tensor product operator

$$\begin{gathered} A_L \otimes B_L : (E_1 \otimes \cdots \otimes E_n) \otimes (G_1 \otimes \cdots \otimes G_n) \to F \otimes H, \\ A_L \otimes B_L((x_1 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes \cdots \otimes y_n)) \\ = A(x_1, \dots , x_n) \otimes B(y_1, \dots , y_n). \end{gathered}$$

Since $[(E_1 \otimes \cdots \otimes E_n) \otimes (G_1 \otimes \cdots \otimes G_n)]$ and $[(E_1 \otimes G_1) \otimes \cdots \otimes (E_n \otimes G_n)]$ are isomorphic as linear spaces, it can be rewritten as

$$\begin{gathered} A_L \otimes B_L \in L_a((E_1 \otimes G_1) \otimes \cdots \otimes (E_n \otimes G_n); F \otimes H), \\ A_L \otimes B_L((x_1 \otimes y_1) \otimes \cdots \otimes (x_n \otimes y_n)) \\ = A(x_1, \dots , x_n) \otimes B(y_1, \dots , y_n). \end{gathered}$$

The n-linear mapping associated with $A_L \otimes B_L$ is called $\mathbf{A} \otimes \mathbf{B}$. Then $A \otimes B \in L_a(E_1 \otimes G_1, \dots , E_n \otimes G_n; F \otimes H)$ and for every $x_j \in E_j$ and $y_j \in G_j$,

$$(*) \quad A \otimes B(x_1 \otimes y_1, \dots , x_n \otimes y_n) = A(x_1, \dots , x_n) \otimes B(y_1, \dots , y_n).$$

With the help of [**16**, Proposition 1.1] it is easy to see that $A \otimes B$ is the unique n-linear mapping from $E_1 \otimes G_1 \times \cdots \times E_n \otimes G_n$ to $F \otimes H$ satisfying $(*)$.

Given $P \in P_a(^nE; F)$ and $Q \in P_a(^nG; H)$ the polynomial associated with the symmetric n-linear mapping $\check{P} \otimes \check{Q}$ is called $\mathbf{P} \otimes \mathbf{Q}$. Then $P \otimes Q \in P_a(^nE \otimes G; F \otimes H)$ and

$$P \otimes Q(x \otimes y) = P(x) \otimes Q(y) \quad \text{for every } x \in E \text{ and } y \in G.$$

$P \otimes Q$ is unique in the following sense: if R is an n-homogeneous polynomial from $E \otimes G$ to $F \otimes H$ such that

$$(**) \qquad \check{R}(x_1 \otimes y_1, \dots , x_n \otimes y_n) = \check{P}(x_1, \dots , x_n) \otimes \check{Q}(y_1, \dots , y_n),$$

for every $x_j \in E$ and $y_j \in G$, then $R = P \otimes Q$.

If E and F are normed spaces, $E \otimes F$ can be normed in many different ways. Once tensor norms are fixed in $E_1 \otimes E_2$ and $F_1 \otimes F_2$, given continuous linear operators $T : E_1 \to F_1$ and $U : E_2 \to F_2$ it is natural to ask whether $T \otimes U$ is continuous or not. Except for special norms, including the projective norm, the tensor product operator is not always continuous. In Defant and Floret [**6**] the reader can find numerous situations where the tensor product operator is continuous and several examples where it is not. An important case arises when we are working with tensor products of the form $L_p(\mu) \otimes F$: if μ is an arbitrary measure and $1 \leq p \leq \infty$, let Δ_p be the natural norm induced on $L_p(\mu) \otimes F$ from $L_p(\mu, F)$ and $\Delta_{p,t}$ its natural transposition, i.e., for $f_j \in L_p(\mu)$ and $y_j \in F$,

$$\Delta_{p,t}\bigg(\sum_j y_j \otimes f_j\bigg) = \Delta_p\bigg(\sum_j f_j \otimes y_j\bigg) = \bigg\|\sum_j f_j y_j\bigg\|_{L_p(\mu,F)}.$$

It is not difficult to see that $L_p(\mu) \otimes_{\Delta_p} F$ and $F \otimes_{\Delta_{p,t}} L_p(\mu)$, respectively $l_p \otimes_{\Delta_p} F$ and $F \otimes_{\Delta_{p,t}} l_p$, can be viewed as dense subspaces of $L_p(\mu, F)$, respectively $l_p(F)$, see [**6**, Sections 7 and 8]. The paper [**5**] is entirely devoted to the study of the continuity of tensor product operators with respect to Δ_p (many important concepts in Banach space theory can be defined by means of the continuity of certain operators with respect to Δ_p, e.g., absolutely summing operators, K-convexity and even type and cotype). If a_p is the constant from Khintchine's inequality and $K_{p,q}$ is the constant from Kahane's inequality, it is not difficult to prove the following particular case where we have continuity with respect to Δ_p.

Proposition 6.2. *If E has type r, T is a bounded linear operator from E to $L_p(\mu)$ and I denotes the canonical embedding $l_r \hookrightarrow l_2$, then the tensor product operator*

$$I \otimes T : l_r \otimes_{\Delta_r} E \longrightarrow l_2 \otimes_{\Delta_{p,t}} L_p(\mu)$$

is continuous. Moreover, $\|I \otimes T\| \leq a_p \cdot K \cdot T_r(E) \cdot \|T\|$, where $K = 1$ if $p \leq 2$ and $K = K_{p,2}$ if $p < 2$.

Next we prove multilinear and polynomial versions of Proposition 6.2. At the heart of our argument are the following inequalities.

Lemma 6.3. *Let n be an integer not smaller than 2 and $1 \leq p < \infty$.*

(i) *If E_1 has type $r_1, \ldots, E_n$ has type r_n and $A \in L(E_1, \ldots, E_n; L_p(\mu))$*

then there exists a constant K_A *such that the inequality*

$$\bigg(\int_\Omega\bigg(\sum_{j=1}^k |A(x_j^1,\dots,x_j^n)(w)|^2\bigg)^{p/2} d\mu(w)\bigg)^{1/p} \\ \leq K_A\cdot\prod_{m=1}^n\bigg(\sum_{j=1}^k \|x_j^m\|^{r_m}\bigg)^{1/r_m}$$

holds for every finite sequence $x_1^1,\dots,x_k^1\in E_1,\dots,x_1^n,\dots,x_k^n\in E_n$.

(ii) *If* E *has type* r *and* $P\in P(^nE;L_p(\mu))$, *then there exists* K_p *such that the inequality*

$$\bigg(\int_\Omega\bigg(\sum_{j=1}^k |P(x_j)(w)|^2\bigg)^{p/2} d\mu(w)\bigg)^{1/p}\leq K_p\cdot\bigg(\sum_{j=1}^k\|x_j\|^r\bigg)^{n/r}$$

holds for every finite sequence $x_1,\dots,x_k\in E$.

Proof. Since the proofs of (i) and (ii) are quite similar, we shall prove only (i). a_p will denote the constant from Khintchine's inequality and $K(n,2,np)$ is the constant from the generalized Kahane's inequality, Theorem 5.2, (remember that $np\geq 2$). From the multiorthogonality of the generalized Rademacher functions we have that

$$\sum_{j=1}^k A(x_j^1,\dots,x_j^n)=\int_0^1 A\bigg(\sum_{j=1}^k s_j^{(n)}(t)x_j^1,\dots,\sum_{j=1}^k s_j^{(n)}(t)x_j^n\bigg)\,dt.$$

For every $t\in[0,1]$ and $j=1,\dots,k$, the symbol $r_j(t)^{1/n}$ will be used to denote a fixed nth root of the number $r_j(t)$; for example, we can fix $r_j(t)^{1/n}$ as the nth root of $r_j(t)$ of the smallest argument. So $|r_j(t)^{1/n}|=1$. At this point Hölder's inequality and Corollary 4.2 can be applied to show that

$$\bigg(\int_\Omega\bigg(\sum_{j=1}^k |A(x_j^1,\dots,x_j^n)(w)|^2\bigg)^{p/2} d\mu(w)\bigg)^{1/p} \\ \leq a_p\bigg(\int_\Omega\int_0^1\bigg|\sum_{j=1}^k r_j(t)A(x_j^1,\dots,x_j^n)(w)\bigg|^p dt\,d\mu(w)\bigg)^{1/p}$$

$$
\begin{aligned}
&= a_p\bigg(\int_\Omega \int_0^1 \bigg|\bigg[\int_0^1 A\bigg(\sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^1,\\
&\qquad\qquad \ldots, \sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^n\bigg)\, dz\bigg](w)\bigg|^p \,dt\, d\mu(w)\bigg)^{1/p}\\
&\le a_p\bigg(\int_\Omega \int_0^1 \int_0^1 \bigg|A\bigg(\sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^1,\\
&\qquad\qquad \ldots, \sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^n\bigg)(w)\bigg|^p \,dz\,dt\, d\mu(w)\bigg)^{1/p}\\
&= a_p\bigg(\int_0^1 \int_0^1 \bigg\|A\bigg(\sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^1,\\
&\qquad\qquad \ldots, \sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^n\bigg)(\cdot)\bigg\|_{L_p(\mu)}^p \,dz\,dt\bigg)^{1/p}\\
&\le a_p\|A\|\bigg(\int_0^1 \int_0^1 \prod_{m=1}^n \bigg(\bigg\|\sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^m\bigg\|^p\bigg)\,dz\,dt\bigg)^{1/p}\\
&\le a_p\|A\|\bigg(\int_0^1 \prod_{m=1}^n \bigg(\int_0^1 \bigg\|\sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(z) x_j^m\bigg\|^{np} dz\bigg)^{1/n} dt\bigg)^{1/p}\\
&\le a_p\|A\|\bigg(\int_0^1 \prod_{m=1}^n K(n,2,np)^p \cdot \bigg\|\sum_{j=1}^k r_j(t)^{1/n} s_j^{(n)}(\cdot) x_j^m\bigg\|_{L_2(E_m)}^p dt\bigg)^{1/p}\\
&\le a_p\|A\| K(n,2,np)^n \bigg[\prod_{m=1}^n T_{(r_m,n)}(E_m)\bigg] \cdot \prod_{m=1}^n \bigg(\sum_{j=1}^k \|x_j^m\|^{r_m}\bigg)^{1/r_m}.
\end{aligned}
$$

The above computations can be easily adapted for polynomials, for this reason we did not write $A(r_j(t)x_j^1, x_j^2, \ldots, x_j^n)$ instead of $A(r_j(t)^{1/n}x_j^1, \ldots, r_j(t)^{1/n}x_j^n)$. If all the details are filled in, we get

$$K_p = a_p\|P\|K(n,2,np)^n (T_{(r,n)}(E))^n. \qquad \square$$

Theorem 6.4. *Let n be an integer not smaller than 2 and $1 \le p < \infty$.*

(i) *Given* $1 \leq r_1, \dots, r_n \leq 2$, *consider the continuous* n*-linear mapping*

$$J_n : l_{r_1} \times \cdots \times l_{r_n} \longrightarrow l_2;$$
$$J_n((\xi_j^1)_{j=1}^\infty, \dots, (\xi_j^n)_{j=1}^\infty) = (\xi_j^1 \cdots \xi_j^n)_{j=1}^\infty.$$

If E_1 *has type* $r_1, \dots, E_n$ *has type* r_n, *then for every continuous* n*-linear mapping* $A \in L(E_1, \dots, E_n; L_p(\mu))$ *the* n*-linear mapping*

$$J_n \otimes A : l_{r_1} \otimes_{\Delta_{r_1}} E_1 \times \cdots \times l_{r_n} \otimes_{\Delta_{r_n}} E_n \longrightarrow l_2 \otimes_{\Delta_{p,t}} L_p(\mu)$$

is also continuous. Moreover, $\|J_n \otimes A\| \leq a_p K(n, 2, np)^n [\prod_{m=1}^n T_{(r_m, n)}(E_m)] \|A\|$.

(ii) *Given* $r \in [1, 2]$ *consider the continuous* n*-homogeneous polynomial*

$$I_n : l_r \to l_2; I_n((\xi_j)_{j \in \mathbf{N}}) = ((\xi_j)^n)_{j \in \mathbf{N}}.$$

If E *has type* r *then, for every continuous* n*-homogeneous polynomial* $P \in P(^n E; L_p(\mu))$ *the* n*-homogeneous polynomial*

$$I_n \otimes P : l_r \otimes_{\Delta_r} E \longrightarrow l_2 \otimes_{\Delta_{p,t}} L_p(\mu)$$

is also continuous. Moreover, $\|I_n \otimes P\| \leq a_p K(n, 2, np)^n (T_{(r,n)}(E))^n \|P\|$.

Proof. Again (i) and (ii) have similar proofs. Now let us prove (ii). Let $l_r^f(E)$ be the space of all finite sequences of elements in E endowed with $\| \cdot \|_r$. It is clear from Lemma 6.3 that the following map

$$\overline{P} : l_r^f(E) \longrightarrow L_p(\mu, l_2) : \overline{P}((x_j)_{j=1}^m)(w) = (P(x_j)(w))_{j=1}^m, \quad \forall\, w \in \Omega,$$

is a continuous n-homogeneous polynomial; of course, we are identifying a finite sequence $\{\alpha_1, \dots, \alpha_m\}$ with the infinite sequence $\{\alpha_1, \dots, \alpha_m, 0, 0, \dots\}$). Since $l_r^f(E)$ is dense in $l_r(E)$ we can consider the extension

$$\overline{P} : l_r(E) \longrightarrow L_p(\mu, l_2) : \overline{P}((x_j)_{j=1}^\infty)(w) = (P(x_j)(w))_{j=1}^\infty, \quad \forall\, w \in \Omega.$$

Then $\overline{P} \in P(^n l_r(E); L_p(\mu, l_2))$ and $\|\overline{P}\| \leq K_p$. Now all we have to do is to prove that the restriction of $\overline{P}$ to $l_r \otimes \Delta_r E$ coincides with $I_n \otimes P$, because in this case $I_n \otimes P$ will inherit from $\overline{P}$ the continuity and the estimate for the norm. To prove this consider $\sum_j \alpha_j \otimes x_j \in l_r \otimes_{\Delta_r} E$ with $\alpha_j = (\xi_j^i)_{i=1}^\infty \in l_r$ and $x_j \in E$:

$$\begin{aligned}
I_n \otimes P\bigg(\sum_j \alpha_j \otimes x_j\bigg) &= \breve{I}_n \otimes \breve{P}\bigg(\sum_j \alpha_j \otimes x_j, \cdots, \sum_j \alpha_j \otimes x_j\bigg) \\
&= \sum_{j_1} \cdots \sum_{j_n} \breve{I}_n \otimes \breve{P}(\alpha_{j_1} \otimes x_{j_1}, \dots, \alpha_{j_n} \otimes x_{j_n}) \\
&= \sum_{j_1} \cdots \sum_{j_n} \breve{I}_n(\alpha_{j_1}, \dots, \alpha_{j_n}) \otimes \breve{P}(x_{j_1}, \dots, x_{j_n}) \\
&= \sum_{j_1} \cdots \sum_{j_n} (\xi_{j_1}^i \cdots \xi_{j_n}^i)_{i=1}^\infty \otimes \breve{P}(x_{j_1}, \dots, x_{j_n}) \\
&= \sum_{j_1} \cdots \sum_{j_n} (\xi_{j_1}^i \cdots \xi_{j_n}^i \cdot \breve{P}(x_{j_1}, \dots, x_{j_n}))_{i=1}^\infty \\
&= \bigg(\sum_{j_1} \cdots \sum_{j_n} \breve{P}(\xi_{j_1}^i x_{j_1}, \dots, \xi_{j_n}^i x_{j_n})\bigg)_{i=1}^\infty \\
&= \bigg(\breve{P}\bigg(\sum_j \xi_j^i x_j, \dots, \sum_j \xi_j^i x_j\bigg)\bigg)_{i=1}^\infty \\
&= \bigg(P\bigg(\sum_j \xi_j^i x_j\bigg)\bigg)_{i=1}^\infty = \overline{P}\bigg(\bigg(\sum_j \xi_j^i x_j\bigg)_{i=1}^\infty\bigg) \\
&= \overline{P}\bigg(\sum_j \alpha_j x_j\bigg) = \overline{P}\bigg(\sum_j \alpha_j \otimes x_j\bigg). \qquad \square
\end{aligned}$$

Remark 6.5. Thinking of E as a (complemented) subspace of $l_r(E)$ through the "inclusion" $x \in E \to (x, 0, 0, \dots) \in l_r(E)$, and of $L_p(\mu)$ as a (complemented) subspace of $L_p(\mu, l_2)$ through the "inclusion" $f \in L_p(\mu) \to (f, 0, 0, \dots) \in L_p(\mu, l_2)$, Theorem 6.4 can be viewed as an "extension theorem" in the following sense:

Theorem. *Let n be an integer not smaller than 2 and $1 \le p < \infty$.*

(i) *If E_1 has type $r_1, \dots, E_n$ has type r_n, then every n-linear mapping $A \in L(E_1, \dots, E_n; L_p(\mu))$ induces an n-linear mapping $\overline{A} \in L(l_{r_1}(E_1), \dots, l_{r_n}(E_n); L_p(\mu, l_2))$ which extends A and satisfies*

$$\overline{A}((\xi_j^1, x_1)_{j=1}^\infty, \dots, (\xi_j^n x_n)_{j=1}^\infty) = (\xi_j^1 \cdots \xi_j^n)_{j=1}^\infty \cdot A(x_1, \dots, x_n),$$

for every $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$ *and* $(\xi_j^m)_{j=1}^{\infty} \in l_{r_m}(E_m)$, $m = 1, \ldots, n$. *Moreover,*

$$\|A\| \leq \|\overline{A}\| \leq a_p K(n, 2, np)^n \left[\prod_{m=1}^{n} T_{(r_m, n)}(E_m) \right] \|A\|.$$

(ii) *If* E *has type* r, *then every* n*-homogeneous polynomial* $P \in P(^n E; L_p(\mu))$ *induces an* n*-homogeneous polynomial* $\overline{P} \in P(^n l_r(E); L_p(\mu, l_2))$ *which extends* P *and satisfies*

$$\overline{P}((\xi_j x)_{j=1}^{\infty}) = ((\xi_j)^n)_{j=1}^{\infty} \cdot P(x),$$

for every $x \in E$ *and* $(\xi_j)_{j=1}^{\infty} \in l_r$. *Moreover,*

$$\|P\| \leq \|\overline{P}\| \leq a_p K(n, 2, np)^n (T_{(r,n)}(E))^n \|P\|.$$

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