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POLYNOMIAL COMPACTNESS IN BANACH SPACES

PETER BISTRÖM, JESÚS A. JARAMILLO AND MIKAEL LINDSTRÖM

ABSTRACT. We investigate infinite dimensional Banach spaces equipped with the initial topology with respect to the continuous polynomials. We show nonlinear properties for this topology in both the real and the complex case. A new property for Banach spaces, polynomial Dunford-Pettis property, is introduced. For spaces with this property the compact sets in the topology induced by the polynomials are shown to be invariant under the summation map. For most real Banach spaces we characterize the polynomially compact sets as the bounded sets that are separated from zero by the positive polynomials.

Denote a Banach space X equipped with the topology induced by its continuous polynomials by $X_{\mathcal{P}(X)}$. This article investigates the topological space $X_{\mathcal{P}(X)}$ with a focus on its compact sets. In [**3**] Aron et al. prove that $X_{\mathcal{P}(X)}$ has a nonlinear topology if X is an infinite dimensional complex Hilbert space. We show that there are also real as well as entirely other complex Banach spaces, e.g., ℓ^∞, with nonlinear polynomial topologies. Although $X_{\mathcal{P}(X)}$ is not linear in general, we show that the compact sets in $X_{\mathcal{P}(X)}$ form an invariant class under the sum operation for large classes of spaces X. This is shown to be the case when X has the property (P) studied in [**2**] by Aron et al., or when X is a $\mathcal{P}$-Dunford-Pettis space, a new class of spaces containing all the Dunford-Pettis spaces and all the Λ-spaces. We investigate this class of $\mathcal{P}$-Dunford-Pettis spaces with emphasis on its connections with the polynomial Dunford-Pettis properties studied by Farmer and Johnson [**16**] as well as the Dunford-Pettis-like properties of Castillo and Sánchez [**9**].

For real Banach spaces X, we give an almost covering characterization of the relatively compact sets in $X_{\mathcal{P}(X)}$ as those bounded sets that are separated from zero by all strictly positive polynomials in $\mathcal{P}(X)$.

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This characterization holds when $\mathcal{P}_f({}^N X)$ is dense in $\mathcal{P}({}^N X)$ for the compact-open topology, for all N (the case when X has the approximation property), when the polynomials on X are weakly sequentially continuous, when the dual X' doesn't contain ℓ_1 and every null sequence in X' has a subsequence with an upper p-estimate, or when the separable subspaces are contained in separable complemented ones. We do not know whether there is any real Banach space X that doesn't satisfy any of the listed conditions above.

It is known that relatively compact sets in initial topologies with respect to some function classes, e.g., the C^∞-function on real Banach spaces, are characterized as those bounded sets that are separated from zero by the strictly positive functions in the inducing class, see [**4**]. On the other hand, any bounded set in a real Banach space X is separated from zero by the strictly positive polynomials in $\mathcal{P}_f(X)$. The bounded sets are relatively weakly compact if and only if the space X is reflexive. Hence there seems to be a big difference between the algebras $\mathcal{P}_f(X)$ and $\mathcal{P}(X)$ with respect to their ability to measure relative compactness in their induced topologies with such an elementary method as the testing with strictly positive functions is.

Preliminaries. In the sequel X will always be an infinite dimensional Banach space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. For any $N \in \mathbf{N} := \{1, 2, \dots\}$ the set of all N-homogeneous continuous polynomials $P : X \to \mathbf{K}$ is denoted by $\mathcal{P}({}^N X)$. Given $P \in \mathcal{P}({}^N X)$, we denote by $\overline{P}$ the N-linear symmetric continuous map associated with P. The topology on $\mathcal{P}({}^N X)$ will be the usual normed one. The set $\mathcal{P}(X)$ stands for the union of all $\mathcal{P}({}^N X)$ where N runs through $\mathbf{N}$ together with the constant $\mathbf{K}$-valued functions on X. The finite type polynomials on X, i.e., the algebra generated by the dual X', will be denoted by $\mathcal{P}_f(X)$. Also, for any $N \in \mathbf{N}$, the set $\mathcal{P}_f({}^N X)$ stands for all the N-homogeneous polynomials in $\mathcal{P}_f(X)$. The set of all $P \in \mathcal{P}(X)$ taking weakly convergent sequences into convergent sequences in $\mathbf{K}$ is denoted by $\mathcal{P}_{wsc}(X)$. By an operator we mean a bounded linear map.

Let X_σ denote X endowed with the weak topology $\sigma(X, X')$, and let $X_{\mathcal{P}(X)}$, respectively $X_{\mathcal{P}(\leq^N X)}$, be the set X endowed with the weakest topology making all $P \in \mathcal{P}(X)$, respectively $P \in \cup_{m=1}^N \mathcal{P}({}^m X)$, continuous. Then the *polynomial topology* $X_{\mathcal{P}(X)}$ and also $X_{\mathcal{P}(\leq^N X)}$ are regular Hausdorff topologies such that $(X, \|\cdot\|) \geq X_{\mathcal{P}(X)} \geq X_{\mathcal{P}(\leq^N X)} \geq X_\sigma$. Further, since $X_{\mathcal{P}(X)}$ and $X_{\mathcal{P}(\leq^N X)}$ are regular and X_σ is angelic, it follows that both $X_{\mathcal{P}(X)}$ and $X_{\mathcal{P}(\leq^N X)}$ are angelic, see [**17**]. This means that the concepts (relatively) countably compact, (relatively) sequentially compact and (relatively) compact all agree in these spaces.

We will say that a sequence in X is *w-null*, respectively *pol-null*, if it is convergent to zero in X_σ, respectively in $X_{\mathcal{P}(X)}$. In the same way a sequence $(x_n) \subset X$ is *$\mathcal{P}^{\leq N}$-convergent* to $x \in X$ if $x_n \to x$ in $X_{\mathcal{P}(\leq^N X)}$. A space X is said to be *$\mathcal{P}^{\leq N}$-Schur* if $\mathrm{id} : X_{\mathcal{P}(\leq^N X)} \to X$ is sequentially continuous and, more generally, a *Λ-space* if the pol-null sequences all converge in norm. Also X is said to have the *Dunford-Pettis property*, X is D.P., if for w-null sequences (x_n) and (l_n) in X and X', respectively, it holds that $l_n(x_n) \to 0$. All superreflexive spaces and ℓ_1 are Λ-spaces [**24**]. In Λ-spaces the norm compact and the polynomially compact sets agree. Spaces with the Dunford-Pettis property are, e.g., $C(K)$, ℓ_1 and c_0. If X is D.P., then $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$, see [**7**] and [**32**], and therefore the weakly compact and the polynomially compact sets in X agree. Recall that $X_\sigma \neq X_{\mathcal{P}(X)}$ for any infinite dimensional Banach space X, see [**27**].

Since a central idea in the paper is to investigate the sum of two compact sets in the polynomial topology, we start in the first section by showing that at least for most Banach spaces the polynomial topology is nonlinear. In Section 2 we investigate the new class of polynomial Dunford-Pettis spaces. In the third section we show that, for these spaces, the class of polynomially compact sets is invariant under the sum map. In the last section we give, for the real case, an almost covering description of the compact sets in the polynomial topology, as those closed and bounded sets that are separated from zero by all strictly positive polynomials.

1. Nonlinear polynomial topologies. Given $P \in \mathcal{P}(^N X)$ and $x \in X$, the map $y \mapsto P(x+y)$ is a bounded polynomial on X of degree N and therefore the topologies of $X_{\mathcal{P}(\leq^N X)}$ and $X_{\mathcal{P}(X)}$ are semilinear. Also, for spaces X like c_0, the Tsirelson space T^* and $C(K)$ for scattered compacts K, the topologies of $X_{\mathcal{P}(X)}$ and X_σ agree on bounded sets, and thus $X_{\mathcal{P}(X)}$ has a linear topology on bounded sets for these spaces X. On the other hand, in [**3**] Aron et al. prove that, for any infinite dimensional complex Hilbert space X, the space $X_{\mathcal{P}(X)}$

does not have a linear topology even when restricted to the unit ball of X. If Y is a superspace of an infinite dimensional complex Hilbert space X, with $X_{\mathcal{P}(X)}$ a topological subspace of $Y_{\mathcal{P}(Y)}$, then of course also $Y_{\mathcal{P}(Y)}$ has a nonlinear topology. (Recall that if X and Y are two Banach spaces with $X \subset Y$ then $X_{\mathcal{P}(X)}$ is a topological subspace of $Y_{\mathcal{P}(Y)}$ if, for each $P \in \mathcal{P}(X)$ there is an extension $\tilde{P} \in \mathcal{P}(Y)$. This is the case if $Y = X''$ or when X is complemented in Y; in fact, it is enough that there is a linear Hahn-Banach extension operator $X' \to Y'$.)

The real case is different from the complex one; if X is a real Hilbert space, then obviously $X = X_{\mathcal{P}(X)}$. We now provide new examples of both real and complex Banach spaces X such that $X_{\mathcal{P}(X)}$ is not a topological vector space. Much inspired by [**25**], we obtain the following result.

Theorem 1.1. *Assume that X is not a Λ-space and that Y is a Banach space such that there is a bounded sequence $(y_n) \subset Y$ and a continuous linear operator $T : Y \to \ell_{2m}$ for some $m \in \mathbf{N}$ such that the sequence $\{T(y_n)\}$ is the unit vector basis of ℓ_{2m}. Then $Z_{\mathcal{P}(Z)}$, where $Z := X \times Y$, is not a topological vector space.*

Proof. Since X is not a Λ-space, there is a pol-null sequence in X not converging in norm. By the Bessaga-Pelczynski selection principle, there is a basic subsequence (x_n) which is associated with a bounded biorthogonal sequence (ψ_n) in X'. Let (ϕ_n) in ℓ'_{2m} be the biorthogonal sequence to the unit vector basis in ℓ_{2m}. Define

$$P(z) := \sum_{n=1}^{\infty} (\psi_n \circ pr_X)(z) \cdot (\phi_n^{2m} \circ T \circ pr_Y)(z),$$

where pr_X and pr_Y are the natural projections. Since $P(z)$ is well-defined for each $z \in Z$, the Banach-Steinhaus theorem assures that $P \in \mathcal{P}(^{2m+1}Z)$. Let $U := \{z \in Z : |P(z)| < 1\}$. Assume that $Z_{\mathcal{P}(Z)}$ has a linear topology. Since $U \ni 0$ is open, there is an open set $V \ni 0$ in $Z_{\mathcal{P}(Z)}$ with $V + V \subset U$. The sequence $(0, y_n)$ is bounded in Z and hence there is an $\varepsilon > 0$ such that $\varepsilon(0, y_n) \in V$ for all n. Take $k > 0$ with $k\varepsilon^{2m} > 1$. Since $(k(x_n, 0))$ is a pol-null sequence in Z, there is some n_0 such that $k(x_{n_0}, 0) \in V$. Therefore, $k(x_{n_0}, 0) + \varepsilon(0, y_{n_0}) \in V + V \subset U$.

However,

$$P(k(x_{n_0},0)+\varepsilon(0,y_{n_0}))=\varepsilon^{2m}\cdot\psi_{n_0}(k(x_{n_0}))=k\cdot\varepsilon^{2m}>1,$$

which is a contradiction. ◻

Remark. For spaces X and Y as in the above theorem, we have that $(X\times Y)_{\mathcal{P}(X\times Y)}\neq X_{\mathcal{P}(X)}\times Y_{\mathcal{P}(Y)}$. Indeed, otherwise the polynomial P would be continuous on $X_{\mathcal{P}(X)}\times Y_{\mathcal{P}(Y)}$ and $(W_1,W_2)\subset U$ for some open sets $W_1\ni 0$ and $W_2\ni 0$ in $X_{\mathcal{P}(X)}$ and $Y_{\mathcal{P}(Y)}$, respectively. But then $(kx_{n_0},\varepsilon y_{n_0})\in(W_1,W_2)\subset U$ gives a contradiction.

Corollary 1.2. *Assume that X is not a Λ-space and that Y is a Banach space containing a copy of ℓ_1. Then $Z_{\mathcal{P}(Z)}$, where $Z\simeq X\times Y'$, has a nonlinear topology. Especially, for any infinite set Γ the topology of $\ell^\infty(\Gamma)_{\mathcal{P}(\ell^\infty(\Gamma))}$ is nonlinear.*

Proof. Clearly there is a continuous linear map from Y' onto ℓ^∞. According to [**30**], there exists a continuous, linear map from ℓ^∞ onto $\ell_2([0,1])$, hence there is an onto operator $T:Y'\to\ell_2([0,1])$. Since $[0,1]$ is uncountable, we can find a bounded infinite sequence in Y' that is mapped by T into the set of unit vectors in $\ell_2([0,1])$. By Theorem 1.1, the topology of $Z_{\mathcal{P}(Z)}$ is nonlinear. The space $\ell^\infty=(\ell^1)'$ is not a Λ-space, so the last statement follows by considering $\ell^\infty\simeq\ell^\infty\times\ell^\infty$ and the fact that ℓ^∞ is complemented in any $\ell^\infty(\Gamma)$ if Γ is infinite. ◻

The space c_0 is not a Λ-space and therefore, by Theorem 1.1, the spaces $c_0\oplus\ell_p$, $1\le p<\infty$, have a nonlinear polynomial topology. For the case $1<p<\infty$, there is the generalization below. Recall that, by [**26**], a Banach space X is said to have *property S_p* (for some $1<p<\infty$) if every weakly null sequence (x_n) in X has a subsequence (y_n) with an upper p-estimate; that is, $(x'(y_n))\in\ell_{p^*}$ for every $x'\in X'$ where $1/p^*+1/p=1$.

Corollary 1.3. *Assume that X is not a Λ-space and that Y is a Banach space with an infinite dimensional quotient $\pi(Y)$ such that $\ell_1\not\subset\pi(Y)'$ and $\pi(Y)'$ has property S_p for $p^*\in\mathbf{N}$ with $1/p+1/p^*=1$. Then $Z_{\mathcal{P}(Z)}$, where $Z\simeq X\times Y$, has a nonlinear topology.*

Proof. The quotient $\pi(Y)$ is not a Schur space, since otherwise $\ell_1 \subset \pi(Y)$, contradicting the fact $\ell_1(2^{\mathbf{N}}) \not\subset \pi(Y)'$ [**12**, p. 211]. Again, using the Bessaga-Pelczynski selection principle, we find a normalized sequence (y_n) in $\pi(Y)$ with an associated bounded biorthogonal sequence (l_n) in $\pi(Y)'$. Since $\ell_1 \not\subset \pi(Y)'$, we have $l_{2m+1} - l_{2m} =: u_m \to 0$ in $\sigma(\pi(Y)', \pi(Y)'')$. Put $v_k := y_{2k+1}$. Then $u_m(v_k) = \delta_{mk}$. Now, by the S_p-property of $\pi(Y)'$, there is a subsequence (u_{m_n}) with $(u_{m_n}(x)) \in \ell_{p^*}$ for each $x \in \pi(Y)$. Thus we obtain a well-defined operator $T : \pi(Y) \to \ell_{p^*}$, $x \mapsto (u_{m_n}(x))$ with $T(v_k) = e_k$ for all k. The rest follows from Theorem 1.1. $\square$

Remark. If Y is any superreflexive space, the dual Y' has property S_p for some $1 < p < \infty$, see [**10**] or [**23**], and $\ell_1 \not\subset Y'$. Now if $Z \simeq X \times Y$, where X is not a Λ-space, we obtain that $Z_{\mathcal{P}(Z)}$ has a nonlinear topology.

Open problem. Is the polynomial topology of X nonlinear if $X \neq X_{\mathcal{P}(X)}$?

2. Polynomial Dunford-Pettis property. In order to study sequences and compactness in the polynomial topology, we introduce the $\mathcal{P}$-*Dunford Pettis* property. This can be described as an analogue of the classical Dunford-Pettis property, where the weak topology on the space is replaced by the polynomial topology. This property is obtained as a weakening of the polynomial Dunford-Pettis properties studied by Farmer and Johnson [**16**], where polynomials of a fixed degree are considered. We also study the connections between all these properties.

Definition. Let $P : X \to Y$ be an m-homogeneous polynomial. We say that P is *pol-compact*, respectively $\mathcal{P}^{\leq N}$-compact if, for each bounded sequence (x_n) in X, there exists a subsequence (x_{n_j}) so that $(P(x_{n_j}))$ converges in $Y_{\mathcal{P}(Y)}$, respectively in $Y_{\mathcal{P}(\leq^N Y)}$.

An operator (or a polynomial) is of course $\mathcal{P}^{\leq 1}$-compact if and only if it is weakly compact.

Theorem 2.1. *For a Banach space X, the following are equivalent.*

(1) *For every* Y, *each pol-compact operator* $T : X \to Y$ *maps convergent sequences in* $X_{\mathcal{P}(X)}$ *into norm convergent sequences in* Y.

(1′) *For every* Y, *each pol-compact polynomial* $P : X \to Y$ *maps convergent sequences in* $X_{\mathcal{P}(X)}$ *into norm convergent sequences in* Y.

(2) *For every* Y, *each weakly compact operator* $T : X \to Y$ *maps convergent sequences in* $X_{\mathcal{P}(X)}$ *into norm convergent sequences in* Y.

(2′) *For every* Y, *each weakly compact polynomial* $P : X \to Y$ *maps convergent sequences in* $X_{\mathcal{P}(X)}$ *into norm convergent sequences in* Y.

(3) *For every* w*-null sequence* (l_n) *in* X' *and every pol-null sequence* (x_n) *in* X, *we have* $l_n(x_n) \to 0$.

(3′) *For each* m, *for every* w*-null sequence of* m*-homogeneous polynomials* $(P_n) \subset \mathcal{P}(^mX)$, *and every pol-null sequence* (x_n) *in* X, *we have* $P_n(x_n) \to 0$.

Proof. (2′) $\Rightarrow$ (1′) is trivial since pol-compactness implies weak compactness.

(1′) $\Rightarrow$ (3′). Let (x_n) and (P_n) be as in (3′). Since $P_n \overset{w}{\to} 0$ we can define $P : X \to c_0$ by $P(x) := (P_n(x))$. In order to show that P is weakly compact, it is sufficient by [**31**] to check that $P^t : \ell_1 \to \mathcal{P}(^mX)$ is weakly compact, and this follows from the fact that $P^t(e_n) = P_n \overset{w}{\to} 0$, see, e.g., [**12**, p. 114]. Now $P : X \to c_0$ is pol-compact since c_0 is D.P. Thus $\|P(x_n)\|_\infty \to 0$ by (1′) and then $|P_n(x_n)| \le \|P(x_n)\|_\infty \to 0$.

(3′) $\Rightarrow$ (2′). Let P be an m-homogeneous weakly compact polynomial, let $(x_n) \subset X$ be pol-null, and suppose that $\|P(x_n)\| \ge \varepsilon > 0$ for all n. For each n, choose $y_n^* \in Y'$ with $\|y_n^*(P(x_n))\| = \|P(x_n)\|$; and define $P_n := y_n^* \circ P \in \mathcal{P}(^mX)$. By [**31**] we have that $P^t : Y' \to \mathcal{P}(^mX)$ is weakly compact, so there exists a subsequence $(y_{n_j}^*)$ such that $(P_{n_j}) = (P^t(y_{n_j}^*))$ is weakly convergent in $\mathcal{P}(^mX)$ to some $Q \in \mathcal{P}(^mX)$. Then by (3′), $(P_{n_j} - Q)(x_{n_j}) \to 0$, but $Q(x_{n_j}) \to 0$ since $x_{n_j} \to 0$ in $X_{\mathcal{P}(X)}$, and we obtain that $P_{n_j}(x_{n_j}) \to 0$. This is a contradiction since $P_{n_j}(x_{n_j}) = y_{n_j}^*(P(x_{n_j})) = \|P(x_{n_j})\| \ge \varepsilon$.

Hence (1′) $\Leftrightarrow$ (2′) $\Leftrightarrow$ (3′). In the same way it can be shown that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). On the other hand, it is clear that (1′) $\Rightarrow$ (1). So the proof is complete if we show that (2) yields (3′).

(2) $\Rightarrow$ (3′). We use induction on m. For $m = 1$, we have (2) $\Rightarrow$ (3),

which already has been established. Now suppose that the result is true for polynomials of degree m, and we are going to prove it for $m+1$.

So let $(x_n) \subset X$ be pol-null, and let $(P_n) \subset \mathcal{P}(^{m+1}X)$ be w-null. For each n, define $Q_n := \overline{P}_n(x_n; \cdot, \dots, \cdot) \in \mathcal{P}(^mX)$. We show that $(Q_n) \subset \mathcal{P}(^mX)$ is w-null. Define $P : X \to c_0$ as before by $P(x) := (P_n(x))$ and consider $P^t : \ell_1 \to \mathcal{P}(^{m+1}X)$; since $P^t(e_n) = P_n \xrightarrow{w} 0$ we have that P^t is weakly compact and, therefore, the bitranspose P^{tt} is also weakly compact. It is not difficult to check that $P^{tt} : \mathcal{P}(^{m+1}X)' \to c_0$ is given by $P^{tt}(\phi) = (\phi(P_n))$ for every $\phi \in \mathcal{P}(^{m+1}X)'$. Now, given $\xi \in \mathcal{P}(^mX)'$, we define $\tilde{\xi} : X \to \mathcal{P}(^{m+1}X)'$ by $\tilde{\xi}(x)(R) := \xi(\overline{R}(x; \cdot, \dots, \cdot))$. Then $P^{tt} \circ \tilde{\xi} : X \to c_0$ is a weakly compact operator, since P^{tt} is, and then, by (2), $\|P^{tt}(\tilde{\xi}(x_n))\|_\infty \to 0$. Since $P^{tt}(\tilde{\xi}(x_n)) = (\tilde{\xi}(x_n)(P_j))_{j \in \mathbf{N}}$, we obtain that

$$|\xi(Q_j)| = |\tilde{\xi}(x_j)(P_j)| \le \|P^{tt}(\tilde{\xi}(x_j))\|_\infty \to 0.$$

This shows that $(Q_n) \subset \mathcal{P}(^mX)$ is w-null and then, by the induction hypothesis, $P_n(x_n) = \overline{P}_n(x_n, \dots, x_n) = Q_n(x_n) \to 0$. $\square$

Definition. A Banach space X is said to have the $\mathcal{P}$-*Dunford-Pettis* property, X is $\mathcal{P}$-D.P., if X satisfies the equivalent conditions of Theorem 2.1.

It is clear that the class of $\mathcal{P}$-D.P. spaces contains all the D.P. spaces and the Λ-spaces. Conversely, by using (2) and (3) in Theorem 2.1, we get

Corollary 2.2. *If X is a $\mathcal{P}$-D.P. space and X is reflexive, then X is a Λ-space. If X is a $\mathcal{P}$-D.P. space and $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$, then X is a D.P. space.*

Remark. No reflexive space is D.P., and hence there are spaces such as T^*, the Tsirelson space, failing to be $\mathcal{P}$-D.P., note that $\mathcal{P}(T^*) = \mathcal{P}_{wsc}(T^*)$, according to [**1**]. To be more specific, X is polynomially reflexive, see [**15**], if it is reflexive ad $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$; the converse holds if X, in addition, has the approximation property. So it follows from Corollary 2.2 that an infinite dimensional polynomially reflexive space with the approximation property is not $\mathcal{P}$-D.P.

Clearly the $\mathcal{P}$-Dunford-Pettis property cannot be closed under formation of subspaces, although the property is hereditary with respect to complemented subspaces.

Definition. Following [**16**], X is said to be $\mathcal{P}^{\leq N}$-*Dunford Pettis* if, for each $\mathcal{P}^{\leq N}$-null sequence $(x_n) \subset X$ and every w-null sequence $(P_n) \subset \mathcal{P}(^N X)$, it holds that $P_n(x_n) \to 0$.

Theorem 2.3. *For a Banach space X and $N \in \mathbf{N}$ fixed, the following are equivalent*:

(1) *For every Y, each $\mathcal{P}^{\leq N}$-compact operator $T : X \to Y$ maps $\mathcal{P}^{\leq N}$-convergent sequences in X into norm convergent sequences in Y.*

(1′) *For every Y, each $\mathcal{P}^{\leq N}$ compact polynomial $P : X \to Y$ maps $\mathcal{P}^{\leq N}$-convergent sequences in X into norm convergent sequences in Y.*

(2) *For every Y, each weakly compact operator $T : X \to Y$ maps $\mathcal{P}^{\leq N}$-convergent sequences in X into norm convergent sequences in Y.*

(2′) *For every Y, each weakly compact polynomial $P : X \to Y$ maps $\mathcal{P}^{\leq N}$-convergent sequences in X into norm convergent sequences in Y.*

(3) *For every w-null sequence (l_n) in X' and every $\mathcal{P}^{\leq N}$-null sequence (x_n) in X, we have $l_n(x_n) \to 0$.*

(3′) *For each m, for every w-null sequence of m-homogeneous polynomials $(P_n) \subset \mathcal{P}(^m X)$, and every $P^{\leq N}$-null sequence (x_n) we have $P_n(x_n) \to 0$.*

(4) *X is $\mathcal{P}^{\leq N}$-D.P.*

Proof. The first six cases can be treated as in Theorem 2.1, only the step (3′) to (2′) needs some light. Assume that (3′) holds. We only need to show that $P(x_n) \to P(x)$ for every $P \in \mathcal{P}(^m X)$ and every sequence (x_n) that is $\mathcal{P}^{\leq N}$-convergent to x. If $m \leq N$ this is true by definition. We proceed by induction. Assume that it holds for m. Take $Q \in \mathcal{P}(^{m+1} X)$ and a $\mathcal{P}^{\leq N}$-null sequence (x_n). Consider the linear operator $T : X \to \mathcal{P}(^m X)$ defined by $T(x) := \overline{Q}(x; \cdot, \overset{m}{\ldots}, \cdot)$. Now $T(x_n)$ is w-null and by (3′) we have $Q(x_n) = T(x_n)(x_n) \to 0$.

(3′) ⇒ (4). We choose $m = N$ and we obtain that X is $\mathcal{P}^{\leq N}$-D.P.

(4) ⇒ (3). Let $(x_n) \subset X$ be $\mathcal{P}^{\leq N}$-null, and let $(l_n) \subset X'$ be w-null. Define $P_n := l_n^N \in \mathcal{P}_f(^N X) \subset \mathcal{P}(^N X)$, and it is enough to prove that (P_n) is w-null. Consider $\phi \in \mathcal{P}(^N X)'$. By [**13**], $\mathcal{P}_f(^N X)'$ is isomorphic to the space of integral polynomials $P_I(^N X')$ on X'. In particular, there exists a regular, countably additive Borel measure of bounded variation μ on the compact set $(B_{X''}, w^*)$ such that

$$\phi(l^N) = \int_{B_{X''}} \langle l, z\rangle^N \, d\mu(z), \quad \forall\, l \in X'.$$

Now (P_n) is a sequence of continuous functions on $(B_{X''}, w^*)$ which is uniformly bounded and pointwise null on $B_{X''}$. Therefore

$$\phi(P_n) = \int_{B_{X''}} P_n(z) \, d\mu(z) \longrightarrow 0. \qquad \square$$

By definition, X satisfies the $\mathcal{P}^{\leq 1}$-Dunford-Pettis condition precisely when it is a Dunford-Pettis space.

Corollary 2.4. *D.P.* $\Rightarrow \cdots \Rightarrow \mathcal{P}^{\leq N}$*-D.P.* $\Rightarrow \mathcal{P}^{\leq N+1}$*-D.P.* $\Rightarrow \cdots \Rightarrow$ $\mathcal{P}$*-D.P.*

Next we give some stability properties of $\mathcal{P}$-D.P. and $\mathcal{P}^{\leq N}$-D.P. spaces.

Proposition 2.5. *Let* (X_n) *be a sequence of Banach spaces with the* $\mathcal{P}$*-Dunford-Pettis property, respectively* $\mathcal{P}^{\leq N}$*-D.P. Then the spaces* $(\oplus_n X_n)_{c_0}$ *and* $(\oplus_n X_n)_{\ell_p}$ *for* $1 \leq p < \infty$*, respectively for* $1 \leq p \leq N$*, have the* $\mathcal{P}$*-Dunford-Pettis property, respectively* $\mathcal{P}^{\leq N}$*-D.P.*

Proof. The proof can be carried out as in results by [**5**] and [**8**]. We restrict to the case when the spaces X_n are $\mathcal{P}$-D.P., the other case being similar. Let X either be the ℓ_p-sum or the c_0-sum, and take a weakly compact operator $T : X \to Y$. Further, let $T_n : X_n \to Y$ be the weakly compact operators determined by T so that

$$T(x) = \sum_{n=1}^{\infty} T_n(x^n), \quad \text{if } x = (x^n) \in X.$$

Take a pol-null sequence (x_m) in X. If X is the c_0-sum, then Theorem 1.7 in [**5**] yields that

$$(\dagger) \qquad \sup_{m\in\mathbf{N}} \left\| \sum_{n=1}^{N} T_n(x_m^n) - T(x_m) \right\| \xrightarrow{N} 0.$$

Since the spaces X_n are $\mathcal{P}$-D.P., the sequences $\{T_n(x_m^n)\}_{m\in\mathbf{N}}$ are norm-null in Y by Theorem 2.1 (2). Therefore the sequence $\{T(x_m)\}$ is norm-null in Y, and the statement concerning c_0-sums is proved by Theorem 2.1 (2). We are finished if we show that ($\dagger$) holds also for the case $X = (\oplus_n X_n)_{\ell_p}$. According to Lemma 1.3 in [**5**], this is the case if the following claim holds.

Claim. $s_N = \sup_{m\in\mathbf{N}} \sum_{n>N} \|x_m^n\|^p \xrightarrow{N} 0.$

Indeed, otherwise there is a strictly increasing unbounded sequence (N_i) so that $s_{N_i} > \varepsilon > 0$. Thus, there exists a sequence (m_i) such that

$$\sum_{n=N_i+1}^{N_{i+1}} \|x_{m_i}^n\|^p > \varepsilon \quad \text{for each } i \in \mathbf{N}.$$

For each i, $n \in \mathbf{N}$, choose $y_{i,n}^* \in X_n'$ with $\|y_{i,n}^*\| = 1$ and $y_{i,n}^*(x_{m_i}^n) = \|x_{m_i}^n\|$. Let $M_i := N_{i+1} - N_i$ and consider the operator

$$R : X \longrightarrow \left(\bigoplus_n \ell_p^{M_n} \right) \simeq \ell_p,$$
$$R(x) := ((y_{i,n}^*(x^n))_{N_i<n\le N_{i+1}})_{i\in\mathbf{N}}.$$

Now $R(x_m)$ is a pol-null sequence in $(\oplus_n \ell_p^{M_n})_{\ell_p} \simeq \ell_p$ and therefore $\|R(x_m)\| \to 0$ since ℓ_p is a Λ-space. Nevertheless, this is a contradiction establishing the claim since $\|R(x_{m_i})\|^p > \varepsilon$ for every $i \in \mathbf{N}$. ◻

The spaces $\ell_p, \ell_p \oplus c_0$ and $\ell_p \oplus \ell^\infty$ are $\mathcal{P}^{\le N}$-D.P. spaces for $N \ge p$. Also the James space J is $\mathcal{P}^{\le 2}$-D.P. according to [**21**]. If $\ell_1 \not\subset X'$ and X' has property S_{p^*}, where $1/p + 1/p^* = 1$, then X is $\mathcal{P}^{\le N}$-D.P. for $N \ge p$, see Proposition 3.5 below. On the other hand, it follows from [**16**] that every space with nontrivial type is $\mathcal{P}$-D.P. Some more examples of $\mathcal{P}$-D.P. and $\mathcal{P}^{\leq N}$-D.P. spaces are provided in the following.

Proposition 2.6. *If X is a Λ-space, respectively $\mathcal{P}^{\leq N}$-Schur, then for each compact set K we have that $C(K,X)$ is $\mathcal{P}$-D.P., respectively $\mathcal{P}^{\leq N}$-D.P.*

Proof. The proof is as in Proposition 3.4 in [**9**]. Assume that X is a Λ-space, the other case is analogous. Let $T : C(K,X) \to Y$ be a weakly compact operator, and let (f_n) be a pol-null sequence in $C(K,X)$. Then, for each $t \in K$, $(f_n(t))$ is a pol-null sequence in X, and thus $\|f_n(t)\| \to 0$. Now by Theorem 2.1 in [**6**], we conclude that $\|T(f_n)\| \to 0$ in Y. □

According to Castillo and Sánchez in [**9**], a Banach space X is said to have the Dunford-Pettis property of order p, in short $(\text{D.P.})_p$, if, for every w-null sequence (l_n) in X' and every sequence (x_n) in X such that $(x'(x_n)) \in \ell_p$ for every $x' \in X'$, we have that $l_n(x_n) \to 0$. This property is related to the $\mathcal{P}^{\leq N}$-D.P. property as follows. *If X is $\mathcal{P}^{\leq N}$-D.P., then X is also $(D.P.)_p$ for every p with $1 \leq p < N^*$, where $1/N + 1/N^* = 1$.* Indeed, let $(l_n) \subset X'$ be a w-null sequence, and $(x_n) \subset X$ be a sequence with an upper-p^*-estimate, where $1/p + 1/p^* = 1$. If $1 \leq p < N^*$, then $p^* > N$. Now if P is a polynomial on X of degree $\leq N$, by [**20**] we have that $P(x_n) \to 0$. That is, (x_n) is $\mathcal{P}^{\leq N}$-null. Since X has the $\mathcal{P}^{\leq N}$-D.P. property, we obtain that $l_n(x_n) \to 0$.

Talagrand has given examples of spaces $C(K,X)$ that are not D.P. Using Example 3.7 in [**9**] we find a compact space K and a sequence (X_N) of D.P. spaces such that each $C(K,X_N)$ fails the $\mathcal{P}^{\leq N}$-D.P. property.

3. The sum of polynomially compact sets. In [**19**], González and Gutiérrez ask the question if the sum operation $+ : X_{\mathcal{P}(X)} \times X_{\mathcal{P}(X)} \to X_{\mathcal{P}(X)}$ is sequentially continuous. By means of the linearity for the weak topology and the angelic property for the polynomial topology, the question actually asks whether the sum of two compact sets in $X_{\mathcal{P}(X)}$ is again a compact set in $X_{\mathcal{P}(X)}$. If one of the sets is norm compact, there is always the following affirmative answer.

Proposition 3.1. *Let X be a Banach space, $K \subset X$ norm-compact and $B \subset X$ compact in $X_{\mathcal{P}(X)}$. Then $K + B$ is compact in $X_{\mathcal{P}(X)}$.*

Proof. Since $X_{\mathcal{P}(X)}$ is angelic we have to show that $(x_n + y_n)$ converges to $x + y$ in $X_{\mathcal{P}(X)}$ whenever (x_n) is a sequence in K that converges to x in norm and (y_n) is a sequence in B that converges to $y \in B$ in $X_{\mathcal{P}(X)}$. Set $z_n := x - x_n$ and $z'_n := y - y_n$. Let $P \in \mathcal{P}(^N X)$. Then

$$P(z_n + z'_n) = P(z_n) + P(z'_n) + \sum_{j=1}^{N-1} \binom{N}{j} \overline{P}(z_n, \overset{j}{\ldots}, z_n, z'_n, \overset{N-j}{\ldots}, z'_n).$$

Clearly the first two terms tend to zero. Also, since $\|z_n\| \to 0$, the last term converges to zero. $\square$

When $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$ or when X is a Λ-space, the compact sets in $X_{\mathcal{P}(X)}$ are compact either in the weak or in the norm topologies. Hence, by linearity of these topologies, the sum of compact sets in $X_{\mathcal{P}(X)}$ is a compact set in $X_{\mathcal{P}(X)}$ under these conditions. However, for some spaces X not satisfying these conditions such as $c_0 \oplus \ell_2$, we still know that the sum of compact sets in $X_{\mathcal{P}(X)}$ is again compact in $X_{\mathcal{P}(X)}$, because of the following result.

Theorem 3.2. *Suppose that X has the $\mathcal{P}$-Dunford-Pettis property. Then the sum of two polynomially compact sets in X is again polynomially compact.*

Proof. Without loss of generality, we need only to prove that if (x_n) and (y_n) are pol-null, then $P(x_n + y_n) \to 0$ for an N-homogeneous polynomial P on X. For a fixed $j \leq N$, consider the j-homogeneous polynomials $P_n := \overline{P}(\cdot, \overset{j}{\ldots}, \cdot, y_n, \overset{N-j}{\ldots}, y_n)$. In order to see that (P_n) is w-null, consider a functional $\phi \in \mathcal{P}(^j X)'$. Since the map

$$Q : X \to \mathcal{P}(^j X), \qquad Q(y) := \overline{P}(\cdot, \overset{j}{\ldots}, \cdot, y, \overset{N-j}{\ldots}, y)$$

is an $(N - j)$-homogeneous polynomial, we have $\phi \circ Q \in \mathcal{P}(^{N-j} X)$. Thus, $(\phi \circ Q)(y_n) = \phi(P_n) \to 0$, and therefore $P_n(x_n) = \overline{P}(x_n, \overset{j}{\ldots}, x_n, y_n, \overset{N-j}{\ldots}, y_n) \to 0$ by the $\mathcal{P}$-Dunford-Pettis property of X. Hence, $P(x_n + y_n) = P(x_n) + P(y_n) + \sum_{j=1}^{N-1} \binom{N}{j} P_n(x_n)$ converges to zero. $\square$

Remark. In [**19**] González and Gutiérrez also ask if $x_n \otimes y_n$ is w-null in $X\hat{\otimes}_\pi X$, whenever (x_n) and (y_n) are pol-null in X. If X or Y has the $\mathcal{P}$-D.P. property, then we can even show that $x_n \otimes y_n$ is pol-null in $X\hat{\otimes}_\pi Y$, whenever (x_n) and (y_n) are pol-null sequences in X and Y, respectively. Indeed, let $B : X \times Y \to Z$ be a continuous bilinear map. Take $P \in \mathcal{P}(^N X)$ and let $Q : Y \to \mathcal{P}(^N X)$ be the polynomial defined by $Q(y)(x) := P(B(x, y))$. Then $Q(y_n)$ is w-null in $\mathcal{P}(^N X)$ and hence, if X is $\mathcal{P}$-D.P., $Q(y_n)(x_n) \to 0$ by $(3')$ in Theorem 2.1. The statement follows if we consider $Z := X\hat{\otimes}_\pi Y$ and $B(x, y) = x \otimes y$.

The sum of two compact sets in $X_{\mathcal{P}(X)}$ is of course compact in X_σ since that topology is a linear one. For some spaces, although not necessarily $\mathcal{P}$-D.P. spaces, the sum is also compact in the finer topology $X_{\mathcal{P}(\leq^2 X)}$.

Proposition 3.3. *Let X be a Banach space such that every symmetric operator $T : X \to X'$ factors through a $\mathcal{P}$-Dunford-Pettis space. Suppose that (x_n) and (y_n) are pol-null sequences in X. Then $P(x_n + y_n) \to 0$ for all $P \in \mathcal{P}(^2 X)$.*

Proof. Take $P \in \mathcal{P}(^2 X)$. Then there is a symmetric operator $T : X \to X'$ such that $P(x) = \langle x, Tx\rangle$ for all $x \in X$. Thus, we have

$$|\langle x_n + y_n, T(x_n + y_n)\rangle| \leq |P(x_n)| + |P(y_n)| + 2|\langle x_n, Ty_n\rangle|.$$

The first two terms converge to zero. Now $\langle x_n, T(y_n)\rangle$ converges to zero since T factors through a $\mathcal{P}$-Dunford-Pettis space. Hence the statement is proved. □

Corollary 3.4. *The sum of two polynomially compact sets in a C^*-algebra X is compact in $X_{\mathcal{P}(\leq^2 X)}$.*

Proof. The dual of any C^*-algebra is of cotype 2, see [**29**]. Since any operator $T : X \to F$, where F has cotype 2, factors through a Hilbert space [**29**], the statement follows from Proposition 3.3. □

According to [**2**], a Banach space X has *property* (P) if $P(u_n - v_n) \to 0$ for all $P \in \mathcal{P}(X)$ whenever (u_n) and (v_n) are bounded sequences in X such that $P(u_n) - P(v_n) \to 0$ for all $P \in \mathcal{P}(X)$. Clearly each X with $\mathcal{P}(X) = \mathcal{P}_{wsc}(X)$ has property (P). On the other hand, if a Banach space X has property (P), then the sum of two polynomially compact sets in X is polynomially compact; indeed, if (x_n) and (y_n) are pol-null sequences in X, then $(-y_n)$ is also pol-null; therefore, $P(x_n) - P(-y_n) \to 0$ for all $P \in \mathcal{P}(X)$, and $(x_n + y_n)$ is then pol-null.

Our next result is similar to Theorem 1.7 in [**2**].

Proposition 3.5. *Assume that $\ell_1 \not\subset X'$ and that X' has property S_p and $p^* \in \mathbf{N}$ with $1/p + 1/p^* = 1$. Then X is a $\mathcal{P}^{\leq p^*}$-Schur space with property (P).*

Proof. We only show that X has property (P); the other statement can be proved in the same way. Suppose that (x_i) and (y_i) are bounded sequences in X such that $P(x_i) - P(y_i) \to 0$ for all $P \in \mathcal{P}(^N X)$, $N \geq 1$. We claim that $\|x_i - y_i\| \to 0$. If not, then there is an $\varepsilon > 0$ and a w-null sequence (z_k) of form $z_k := x_{i_k} - y_{i_k}$ such that $\|z_k\| \geq \varepsilon$ for all k. Proceeding as in Corollary 1.3, we find a subsequence (v_k) of (z_k) and an operator $T : X \to \ell_{p^*}$, $x \mapsto (u_{m_n}(x))$, with $T(v_k) = e_k$ for all k. Take $P \in \mathcal{P}(^N \ell_{p^*})$. Then $P \circ T \in \mathcal{P}(^N X)$ and thus $P(T(x_i)) - P(T(y_i)) \to 0$. Now we conclude from the proof of Theorem 1.7 in [**2**] about ℓ_q that $\|T(x_i) - T(y_i)\| \to 0$. Hence, $e_k \to 0$ in norm, giving a contradiction. $\square$

Proposition 3.6. *If X and Y have property (P) and X, in addition, is a $\mathcal{P}$-D.P. space, then $X \times Y$ has property (P).*

Proof. Let (x_n, y_n) and (x'_n, y'_n) be bounded sequences in the space $X \times Y$ such that $P(x_n, y_n) - P(x'_n, y'_n) \to 0$ for all $P \in \mathcal{P}(X \times Y)$. Since X and Y have property (P), we have that $(x_n - x'_n)$ and $(y_n - y'_n)$ are pol-null sequences in X and Y, respectively. For any $P \in \mathcal{P}(^N X \times Y)$, it follows that

$$\begin{aligned}P((x_n, y_n) - (x'_n, y'_n)) &= P(x_n - x'_n, 0) + P(0, y_n - y'_n) \\ &\quad + \sum_{j=1}^{N-1} \binom{N}{j} \overline{P}((x_n - x'_n, 0), \overset{j}{\ldots}, (x_n - x'_n, 0), \\ &\qquad (0, y_n - y'_n), \overset{N-j}{\ldots}, (0, y_n - y'_n)).\end{aligned}$$

The first two terms clearly tend to zero. Fix j and consider the j-homogeneous polynomials $P_n = \overline{P}((\cdot, 0), \overset{j}{\ldots}, (\cdot, 0), (0, y_n - y'_n), \overset{N-j}{\ldots}, (0, y_n - y'_n))$. Take $\phi \in \mathcal{P}(^j X)'$. Since the map

$$Q : Y \longrightarrow \mathcal{P}(^j X), \qquad Q(y) := \overline{P}((\cdot, 0), \overset{j}{\ldots}, (\cdot, 0), (0, y), \overset{N-j}{\ldots}, (0, y))$$

is an $(N - j)$ homogeneous polynomial, we have $\phi \circ Q \in \mathcal{P}(^{N-j} Y)$. Thus, $(\phi \circ Q)(y_n - y'_n) = \phi(P_n) \to 0$ and therefore $P_n(x_n - x'_n) \to 0$ by the $\mathcal{P}$-Dunford-Pettis property of X. Hence, $P((x_n, y_n) - (x'_n, y'_n))$ converges to zero. □

Now $\mathcal{P}(T^*) = \mathcal{P}_{wsc}(T^*)$ for the Tsirelson space, and hence $T^* \oplus \ell_2$ is a Banach space with property (P) that fails the $\mathcal{P}$-Dunford-Pettis property.

Although we have not been able to show that the sum operation for every separable or every reflexive Banach space would be sequentially continuous for the polynomial topology, we know at least the following. *If the sum of two polynomially compact sets in X is polynomially compact in X for every separable Banach space X, then the same holds for all* WCG *Banach spaces X.* Indeed, let X be WCG and take two pol-null sequences (x_n) and (y_n) in X. Let $S_0 \subset X$ be the space spanned by these sequences. Since X is WCG, the separable space S_0 is contained into a complemented separable subspace $S \subset X$, see [**12**], and thus we have that (x_n) and (y_n) are pol-null sequences in $S_{\mathcal{P}(S)}$ as well. Then also $(x_n + y_n)$ is pol-null in S by the assumption and hence also in X.

When is the topology of $X_{\mathcal{P}(X)}$ metrizable? At least it is not metrizable (even on bounded sets) for Λ-spaces X with $X \neq X_{\mathcal{P}(X)}$, e.g., if X is any infinite dimensional complex Λ-space, see Theorem 4.3 in [**3**]. On the other hand, for spaces like c_0 where the weak and the polynomial topologies agree on the unit ball, the polynomial topology is metrizable on bounded sets. When the summation operation is sequentially continuous in $X_{\mathcal{P}(X)}$ and $X_{\mathcal{P}(X)}$ has a nonlinear topology, then the polynomial topology is not metrizable. An example of this situation is $X = \ell^\infty$, and some other examples are given in the following result.

Corollary 3.7. *Assume that X is a Banach space with property (P), but not a Λ-space, and that Y' does not contain a copy of ℓ_1 but has property S_p for some $1 < p < \infty$. Then $Z_{\mathcal{P}(Z)}$ is not metrizable, where $Z = X \times Y$.*

Proof. By Proposition 3.5, Y is a $\mathcal{P}$-D.P. space with property (P). Then Proposition 3.6 gives that $Z = X \times Y$ has property (P) and therefore the summation operation is sequentially continuous in $Z_{\mathcal{P}(Z)}$. On the other hand, $Z_{\mathcal{P}(Z)}$ has a nonlinear topology by Corollary 1.3. □

4. Polynomially compact sets in real Banach spaces. In [**14**], the case $X = X_{\mathcal{P}(X)}$ was studied in terms of the existence of a polynomial in $\mathcal{P}(X)$ separating $0 \in X$ from the unit sphere $\{x : \|x\| = 1\}$. Our objective in this section is to characterize the compact sets in $X_{\mathcal{P}(X)}$ in terms of another separating condition for the polynomials on X. Now, no complex Banach space X can have a polynomial $P \in \mathcal{P}(X)$ with $0 \notin P(X)$. However, if X is real there exist a lot of polynomials $P \in \mathcal{P}(X)$ with $P > 0$. It is trivial that any relatively compact set in $X_{\mathcal{P}(X)}$ is bounded and separated from zero by any such strictly positive $P \in \mathcal{P}(X)$. How about the converse? Are the relatively compact sets in $X_{\mathcal{P}(X)}$ precisely those bounded sets B that can be separated from zero by all strictly positive polynomials on X; that is, those bounded sets that satisfy

$$(*) \qquad \inf_{x \in B} P(x) > 0 \quad \text{for all strictly positive } P \in \mathcal{P}(X)?$$

It is clear that we cannot leave out the assumption on B to be bounded since the set $B := \{nx : n \in \mathbf{N}\}$ satisfies the condition $(*)$ above for any $x \neq 0$ in X.

Obviously a bounded set satisfies $(*)$ if and only if every rational form of polynomials on X is bounded on the set. We can then give an affirmative answer to our question if we know that the set Hom $\mathcal{R}(X)$ of all nonzero real valued homomorphisms on the algebra $\mathcal{R}(X) := \{P/Q : P, Q \in \mathcal{P}(X), 0 \notin Q(X)\}$ only consists of the point evaluations δ_x, $x \in X$. Indeed,

$$X_{\mathcal{P}(X)} \hookrightarrow \operatorname{Hom} \mathcal{R}(X) \hookrightarrow \mathbf{R}^{\mathcal{R}(X)}$$

and $\operatorname{Hom} \mathcal{R}(X)$ is closed in $\mathbf{R}^{\mathcal{R}(X)}$. Hence, any bounded set satisfying $(*)$ is relatively compact in $\mathbf{R}^{\mathcal{R}(X)}$ by the Tychonoff theorem and hence also in $\operatorname{Hom} \mathcal{R}(X)$. If $X = \operatorname{Hom} \mathcal{R}(X)$, a bounded set with $(*)$ has to be relatively compact in $X_{\mathcal{P}(X)}$.

Proposition 4.1. *Let X be a real Banach space such that either X' is $\sigma(X', X)$-separable or each closed separable subspace of X is contained into a closed complemented separable subspace of X, e.g., X is a* WCG *Banach space. Then a bounded subset of X is relatively compact in $X_{\mathcal{P}(X)}$ if and only if it is separated from zero by all strictly positive polynomials on X.*

Proof. According to [**18**], $X = \operatorname{Hom} \mathcal{R}(X)$ if X' is $\sigma(X', X)$-separable, and hence the first statement follows from the discussion above. Now each separable space has a weak* separable dual. So let X be such that its separable subspaces are contained into separable and complemented ones. Take a set $B \subset X$ satisfying $(*)$, and suppose that it is not relatively compact in $X_{\mathcal{P}(X)}$. Then there is a sequence (x_n) in X such that $F := \{x_n : n \in \mathbf{N}\}$ satisfies $(*)$ and the set F is not relatively compact in $X_{\mathcal{P}(X)}$. By assumption, there is a separable and complemented space S in X that contains F. Then $\inf_{x \in F} P(x) > 0$ for all $P > 0$ in $\mathcal{P}(S)$. Since S is separable, the set F is relatively compact in $S_{\mathcal{P}(S)}$ and hence also in $X_{\mathcal{P}(X)}$, a contradiction. □

There is, in fact, a large class of Banach spaces that satisfy the assumptions in Proposition 4.1. Recall that a projectional resolution of identity (PRI) on a Banach space X is a collection $P_\alpha : \omega_0 \le \alpha \le \mu$, where μ is the smallest ordinal such that its cardinality $|\mu| = \operatorname{dens}(E)$, of projections of X into X that satisfy, for every α, $\omega_0 \le \alpha \le \mu$, the following five conditions:

(i) $\|P_\alpha\| = 1$,

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\omega_0 \leq \alpha \leq \beta \leq \mu$,

(iii) $\text{dens}\,(P_\alpha(X)) \leq |\alpha|$,

(iv) $\cup P_{\beta+1}(x)$; $\beta < \alpha$ is norm-dense in $P_\alpha(X)$,

(v) $P_\mu = \text{Id}_X$.

If a Banach space X has a PRI, then not every separable subspace need be contained in a complemented separable subspace, see [**22**, p. 154]. However, if X has a PRI and, in addition, each $P_\alpha(X)$ has a PRI whenever α is a limit ordinal, $\omega_0 \leq \alpha \leq \mu$, then each separable subspace of X is contained in a complemented separable subspace. Indeed, proceeding by transfinite induction over the density number $|\mu|$ of X, if $\text{dens}\,(X) = \aleph_0$, the statement is obviously true. Suppose the statement holds for all X in the assumption with a density number smaller than μ. Take a separable space S in X and let (x_n) be a dense sequence in S. Let α_n be the smallest ordinal with $x_n \in P_{\alpha_n}(X)$. Set $\alpha = \sup_n \alpha_n$. Then α is a limit ordinal strictly smaller than μ. Hence S is a separable subspace of $P_\alpha(X)$. By the induction hypothesis, there is a complemented separable subspace S_α of $P_\alpha(X)$ containing S. But then S_α is a separable complemented subspace of X as well.

Examples of Banach spaces with such a strong PRI are all $C(K)$-spaces with K Valdivia compact, WCD-spaces and all duals of Asplund spaces, see [**11**].

Remark. Also following the proof of Proposition 3.5, one can show that if the dual X' of a real Banach space X has property S_p for some $p > 1$ and $\ell_1 \not\subset X'$, then any bounded set satisfying $(*)$ is in fact relatively compact in the norm topology of X.

The key in showing that bounded sets with $(*)$ are relatively compact in $X_{\mathcal{P}(X)}$ has been the study of $\text{Hom}\,\mathcal{R}(X)$. In order to obtain our main result in this section we use the properties of $\text{Hom}\,\mathcal{R}(X)$ for showing that the bounded set in X with $(*)$ have the interchangeable double limit property (IDLP) with the equicontinuous sets in all $\mathcal{P}(^N X)$. Recall that, if Z is a topological space, X is a set and $M \subset Z^X$, then X and M have the IDLP (in Z) if, for every sequence (x_k) in X and every sequence (f_m) in M, we have that $\lim_m \lim_k f_m(x_k) = \lim_k \lim_m f_m(x_k)$ whenever all involved limits exist. We need the following result, see [**17**].

Lemma 4.2. *Let X be a countably compact space, Z a compact metric space and $M \subset C(X,Z)$. Then X and M have the* IDLP *if and only if the pointwise limit of functions in M is continuous.*

We now state our main result in this section. In what follows, we denote by τ_s and τ_{co} the pointwise and the compact-open topologies, respectively.

Theorem 4.3. *Let X be a real Banach space such that $\mathcal{P}_f(^N X)$ is τ_{co}-dense in $\mathcal{P}(^N X)$ for every $N \in \mathbf{N}$. Then a bounded subset of X is relatively compact in $X_{\mathcal{P}(X)}$ if and only if it is separated from zero by all $P > 0$ in $\mathcal{P}(X)$.*

Proof. Let $B \subset X$ be bounded and separated from zero by all $P > 0$ in $\mathcal{P}(X)$.

Step 1. We first show that B and the set $E_N := \{P \in \mathcal{P}(^N X) : \|P\| \leq 1\}$ have the IDLP for each $N \in \mathbf{N}$. The set E_N is equicontinuous and hence, by the Ascoli theorem, compact in $(\mathcal{P}(^N X), \tau_s)$. Fix $N \in \mathbf{N}$. Take sequences (x_k) in B and (P_m) in E_N. By Tychonoff's theorem, and the fact that $\operatorname{Hom}\mathcal{R}(X)$ is closed in $\mathbf{R}^{\mathcal{R}(X)}$, the set B is also relatively compact in the induced topology on $\operatorname{Hom}\mathcal{R}(X)$. Let $\phi \in \operatorname{Hom}\mathcal{R}(X)$ and $P_0 \in \mathcal{P}(^N X)$ be cluster points to the sequences (x_k) and (P_m), respectively. Choose a sequence $(\alpha_m) \in \mathbf{R}_+$ such that the sums $f := \sum_{m=0}^{\infty} \alpha_m (P_m - \phi(P_m))^2$ and $g := \sum_{m=0}^{\infty} (\alpha_m/m)(P_m - \phi(P_m))^2$ are pointwise convergent and therefore belong to $\mathcal{P}(X)$ by the Banach-Steinhaus theorem. Since the maps in Hom $\mathcal{R}(X)$ are strictly monotone, for each $n \in \mathbf{N}$ we have $0 \leq n \cdot \phi(g) \leq \phi(f)$, by which $\phi(g) = 0$. Hence, there is some point $a \in X$ with $\phi(P_m) = P_m(a)$ for all $m \in \{0, 1, 2, \dots\}$. If all limits involved exist, then

$$\begin{aligned}
\lim_m \lim_k P_m(x_k) &= \lim_m \phi(P_m) \\
&= \lim_m P_m(a) = P_0(a) = \phi(P_0) \\
&= \lim_k P_0(x_k) = \lim_k \lim_m P_m(x_k).
\end{aligned}$$

Step 2. Since B is bounded, there is a $\lambda > 0$ such that $B \subset \lambda B_X$. Fix again $N \in \mathbf{N}$. Hence $|P(B)| \leq \lambda^N$ for all $P \in E_N$. As we noticed before, E_N is compact for the pointwise topology. Now consider the evaluation mapping $ev : B \to C(E_N, [-\lambda^N, \lambda^N])$ defined by $ev(x)(P) = P(x)$. Since B and E_N have the IDLP by Step 1, also $ev(B)$ and E_N have the IDLP. Now Lemma 4.2 gives that the pointwise limit on E_N of functions $ev(x)|_{E_N}$, $x \in B$, is pointwise-continuous for each $N \in \mathbf{N}$.

Step 3. By Tychonoff's theorem B is relatively compact in the induced topology on $\operatorname{Hom} \mathcal{R}(X)$. In order to prove that B is relatively compact in $X_{\mathcal{P}(X)}$, we must show that $\overline{B}^{\operatorname{Hom} \mathcal{R}(X)} \subset X$. Therefore, let $\psi \in \overline{B}^{\operatorname{Hom} \mathcal{R}(X)}$. Hence there is a net $(x_\alpha) \subset B$ such that $P(x_\alpha) \to \psi(P)$ for all $P \in \mathcal{P}(X)$. For every $N \in \mathbf{N}$ we have that $P(x_\alpha) \to \psi_{|E_N}(P)$ for all $P \in E_N \subset \mathcal{P}(^N X)$. This means by Step 2 that, for every $N \in \mathbf{N}$ the restriction map $\psi_{|E_N}$ is pointwise-continuous. Now $\mathcal{P}(^N X) = \cup_{\rho > 0} \rho E_N$ and the compact-open topology τ_{co} is the finest topology on $\mathcal{P}(^N X)$ which coincide with τ_s on each equicontinuous subset of $\mathcal{P}(^N X)$ by Theorem 2.1 in [**28**]. Hence, for all $N \in \mathbf{N}$, the restriction map $\psi_{|\mathcal{P}(^N X)} : (\mathcal{P}(^N X), \tau_{co}) \to \mathbf{R}$ is continuous. Since $\psi_{|X'} : (X', \tau_{co}) \to \mathbf{R}$ is continuous, there is $a \in X$ such that $\psi(l) = l(a)$ for all $l \in X'$. Thus, for all $N \in \mathbf{N}$, $\psi(P) = P(a)$ for all $P \in \mathcal{P}_f(^N X)$. By assumption and continuity of the restriction maps $\psi_{|\mathcal{P}(^N X)}$ we conclude that $\psi(P) = P(a)$ for all $P \in \mathcal{P}(^N X)$ and all $N \in \mathbf{N}$. Hence, there exists a unique point $a \in X$ such that $\psi(P) = P(a)$ for all $P \in \mathcal{P}(X)$. This means that ψ is represented by a point in X, and the proof is complete. □

It is well-known, see [**28**], that if X is a Banach space with the approximation property, then $\mathcal{P}_f(^N X)$ is τ_{co}-dense in $\mathcal{P}(^N X)$ for every $N \in \mathbf{N}$. Does there exist a Banach space X such that $\mathcal{P}_f(^N X)$ is not τ_{co}-dense in $\mathcal{P}(^N X)$ for some $N \in \mathbf{N}$?

We now obtain the following result from [**4**] as a consequence of Theorem 4.3.

Corollary 4.4. *In real Banach spaces X, every bounded set $B \subset X$*

that is separated from zero by all $P > 0$ in $\mathcal{P}(X)$ is relatively weakly compact.

Proof. Since X is isomorphic to a subspace of $C(B_{X'}, \text{weak}^*)$ and also since $Y := C(B_{X'}, \text{weak}^*)$ has the approximation property, every bounded set B in X satisfying $(*)$ is relatively compact in $Y_{\mathcal{P}(Y)}$. Hence B is a relatively weakly compact set in X. $\square$

It should be pointed out that it is of no interest to study $(*)$ if $\mathcal{P}(X)$ is replaced by $\mathcal{P}_f(X)$. Indeed, any $P \in \mathcal{P}_f(X)$ with $P > 0$ is of the form $P = \hat{P} \circ (l_1, \dots, l_n)$, where $l_i \in X'$, $i = 1, \dots, n$, are linearly independent and $\hat{P}$ is a polynomial on $\mathbf{R}^n$ with $\hat{P} > 0$. Hence it is obvious that $\inf_{x \in kB_X} P(x) > 0$ for all k.

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Department of Mathematics, Åbo Akademi University, FIN-20500 Åbo, Finland

Dpto de Análisis Matemático, Universidad Complutense, 28040 Madrid, Spain

Department of Mathematics, Åbo Akademi University, FIN-20500 Åbo, Finland