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# QUALITATIVE PROPERTIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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ABSTRACT. In this paper some qualitative properties of the solutions for the functional differential equations with "maxima" of the form

$$[x(t) - p(t)x(t-\tau)]' + q(t) \max_{[t-\sigma,t]} x(s) = 0$$

are established.

**1. Introduction.** Consider the neutral differential equation

$$(1) \qquad [x(t) - p(t)x(t-\tau)]' + q(t) \max_{[t-\sigma,t]} x(s) = 0,$$

where $\tau > 0$, $\sigma \geq 0$ and $p, q \in C([t_0, \infty), R)$. The differential equations with "maxima" are often met in the applications, for instance, in the theory of automatic control [**8, 9**]. The qualitative theory of these equations has been developed relatively little. The existence of periodic solutions of the equations with "maxima" is considered in [**10**] and [**11**]. The oscillatory properties of Equation (1) are considered in [**1**–**3**]. The main goal of this paper is to discuss more comprehensively the oscillation and nonoscillation of Equation (1).

By a solution of (1) we mean a function $x$ which is defined for $t \geq -\max(\sigma, \tau)$ and which satisfies (1) for $t \geq 0$. By the method of steps, we know that, for a given initial function $\phi \in C([-\max(\sigma, \tau), 0], R)$, there exists a unique solution defined for $t \geq -\max(\sigma, \tau)$ and which satisfies the initial condition for $-\max(\sigma, \tau) \leq t \leq 0$.

A nontrivial solution of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, the solution

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is called nonoscillatory. A solution is called $Z$-type if it is either nonpositive or nonnegative.

Equation (1) is much different from the equation without "maxima" of the form

$$[x(t) - p(t)x(t-\tau)]' + q(t)x(t-\sigma) = 0. \tag{2}$$

For example, it is easy to see that (1) is nonlinear and (2) is linear. In particular, the following result is obvious.

**Property 1.** *The function $x(t)$ is an eventually negative solution of Equation* (1) *if and only if $y(t) = -x(t)$ is an eventually positive solution of the equation*

$$[y(t) - p(t)y(t-\tau)]' + q(t) \min_{[t-\sigma,t]} y(s) = 0. \tag{3}$$

From Property 1, we can see that the positive and negative solutions of Equation (1) need to be discussed separately.

In Section 2, we will discuss the delay differential equations with "maxima"

$$x'(t) + q(t) \max_{[t-\sigma,t]} x(s) = 0, \tag{4}$$

$$x'(t) + q_1(t) \max_{[t-\sigma,t]} x(s) + q_2(t)x(t-r) = 0, \tag{5}$$

and

$$x'(t) + q(t) \max_{[t-\sigma,t]} x(s) = r(t), \tag{6}$$

respectively.

In Section 3, we obtain some oscillation and nonoscillation results for Equation (1).

**2. Delay equations.** The following results show that the behavior of solutions of (4) is much different from the delay equation

$$x'(t) + q(t)x(t-\sigma) = 0 \tag{7}$$

where $q \in C(R^+, R)$ is one sign and $\sigma > 0$.

**Theorem 1.** *If $q(t)$ is of one sign, then all solutions of Equation (4) are nonoscsillatory.*

*Proof.* When $q(t) \equiv 0$ or $\sigma = 0$, Theorem 1 is obvious. Therefore, we assume that $q(t) \not\equiv 0$ and $\sigma > 0$.

If $q(t) \geq 0$, suppose that $x(t)$ is an oscillatory solution of Equation (4). Then $x(t)$ is not a $Z$-type solution and otherwise $x(t) \equiv 0$ eventually. Therefore, there exist $t_1, t_2$ and $t_3$ such that $x(t_1) = x(t_2) = x(t_3) = 0$ and $x(t) < 0$ for $t \in (t_1, t_2)$ and $x(t) > 0$ for $t \in (t_2, t_3)$. Thus, $x'(t_2) = -q(t_2) \max_{[t_2-\sigma, t_2]} x(s) \leq 0$, which is a contradiction. For $q(t) \leq 0$, Theorem 1 can be proved similarly. The proof is complete. □

*Remark* 1. If $q(t)$ has the same sign, by Theorem 1, solutions of (4) are more nonoscillatory in nature than those of (7).

When $q(t)$ is oscillatory, we see that Equation (4) may have oscillatory solutions. For example, consider the equation

$$x'(t) + \sin t \max_{[t-2\pi, t]} x(s) = 0, \tag{8}$$

which has an oscillatory solution $x = \cos t$. But it also has a nonoscillatory solution $x = -2 - \cos t$. This example shows that (4) is different from the ordinary differential equation without delay.

**Theorem 2.** *If $q(t)$ is oscillatory, then (4) has at least one nonoscillatory solution.*

*Proof.* Assume that $q(t_n) = 0$ for $\{t_n\}_{n=1}^{\infty}$ and $\lim_{n\to\infty} t_n = \infty$ and $q(t) \geq 0$ for $t \in (t_1, t_2)$, $q(t) \leq 0$ for $t \in (t_2, t_3)$, $q(t) \geq 0$ for $t \in (t_3, t_4), \ldots$ . We define a function $\phi(t)$ for $t \in [t_1 - \sigma, t_1]$, which is nondecreasing and negative; then (4) has a solution $y(t) = \phi(t_1) \exp(-\int_{t_1-\sigma}^{t} q(s)\, ds)$ for $t \in [t_1 - \sigma, t_2]$. It is obvious that $y(t) < 0$ for $t \in [t_1 - \sigma, t_2]$ and $\max_{[t-\sigma, t]} y(s) = y(t)$. By the method of steps, we can obtain $y(t)$ for $t \geq t_1 - \sigma$. In view of $q(t) \leq 0$ for

$t \in (t_2, t_3)$, we know that $y(t) < 0$ for $t \in [t_1 - \sigma, t_3]$ and $y(t) \le y(t_2)$ for $t \in [t_2, t_3]$. We note that $q(t) \ge 0$ for $t \in [t_3, t_4]$. If there exists $\xi \in (t_3, t_4)$ such that $\max_{[\xi-\sigma,\xi]} y(s) = y(t_2)$, then we have $y'(t) = -q(t)\max_{[t-\sigma,t]} y(s) = -q(t)y(t)$ for $\xi \le t \le t_4$. By induction we know that $y(t) < 0$ for $t \ge t_1 - \sigma$. The proof is complete. □

Now we consider Equation (5), where $q_1, q_2 \in C([t_0, \infty), R^+)$, $\sigma_1 \ge 0$, $\sigma_2 \ge 0$. It is obvious that, if $x(t)$ is an eventually positive solution of (5), then it satisfies

$$x'(t) + q_1(t)x(t - \sigma_1) + q_2(t)x(t - \sigma_2) = 0 \tag{9}$$

and if $x(t)$ is an eventually negative solution of (5), then it satisfies the equation

$$x'(t) + q_1(t)x(t) + q_2(t)x(t - \sigma_2) = 0. \tag{10}$$

By the comparison result, we have

**Theorem 3.** *If Equation* (10) *is oscillatory, then so is Equation* (5).

By comparing (9) and (10), we know that the solutions of (5) are more nonoscillatory in nature than those of Equation (10). For example, it is well known that the equation

$$x'(t) + q_1(t)x(t - \sigma_1) + q_2(t)x(t) = 0, \tag{11}$$

may have oscillatory solutions, see [**7**]. But the equation

$$x'(t) + (q_1(t) + q_2(t))x(t) = 0, \tag{12}$$

is nonoscillatory. By Theorem 3, the equation

$$x'(t) + q_1(t)\max_{[t-\sigma_1,t]} x(s) + q_2(t)x(t) = 0, \tag{13}$$

has nonoscillatory solutions.

We now consider the forced equation (6), where $r \in C([t_0, \infty), R)$ and $q, \sigma$ are the same as in (1). It is different from the equation

$$x'(t) + q(t)x(t - \sigma) = r(t). \tag{14}$$

**Theorem 4.** *Assume that* $q(t) \geq 0$ *and that there exists* $R(t)$ *such that* $R'(t) = r(t)$. *Let* $R_+(t) = (|R(t)| + R(t))/2$ *and* $R_-(t) = -(|R(t)| - R(t))/2$ *such that*

$$\begin{aligned}\int_T^\infty q(t) \max_{[t-\sigma,t]} R_+(s)\,dt = \infty,\\ \int_T^\infty q(t) \max_{[t-\sigma,t]} R_-(s)\,dt = -\infty.\end{aligned} \tag{15}$$

*Then all solutions of* (6) *oscillate.*

The proof is similar to (14); for example, see [**6**]. It is omitted.

Consider the equation

$$x'(t) + \max_{[t-\pi,t]} x(s) = \cos t. \tag{16}$$

Theorem 4 does not hold for (16) because $\max_{[t-\pi,t]} R_-(s) \equiv 0$. In fact, $x = \sin t - t$ is a nonoscillatory solution of (16). But, by the known result [**6**], all solutions of the equation

$$x'(t) + x(t-\pi) = \cos t, \tag{17}$$

oscillate.

**3. Neutral equations.** In this section we first obtain a lemma for Equation (1) which is useful for the proof of the main theorems.

**Lemma 1.** *Assume that*

(i) $p(t) \geq 0$ *for* $t \geq t_0$ *and there exists a* $T \geq t_0$ *such that*

$$p(T + j\tau) \leq 1, \quad j = 0, 1, 2, \dots, \tag{18}$$

(ii) $q(t) \geq 0 (\not\equiv 0)$ *for* $t \geq t_0$.

(iii) $x(t)$ *is an eventually positive solution of* (1) (*or* (3)). *Set*

$$y(t) = x(t) - p(t)x(t-\tau). \tag{19}$$

*Then $y(t) > 0$ eventually.*

The proof is similar to the proof for (2) in [**5**].

**Theorem 5.** *Assume that the assumptions of Lemma 1 hold and either $p(t) > 0$ or $\sigma > 0$ and $q(t) \geq 0 (\not\equiv 0)$ for $t \in [u-\sigma, u]$ for all large $u$. Then Equation (1) has eventually positive solutions if and only if*

$$[x(t) - p(t)x(t-\tau)]' + q(t) \max_{[t-\sigma,t]} x(s) \leq 0 \tag{20}$$

*has eventually positive solutions and Equation (3) has eventually positive solutions if and only if*

$$[x(t) - p(t)x(t-\tau)]' + q(t) \min_{[t-\sigma,t]} x(s) \leq 0 \tag{21}$$

*also has eventually positive solutions.*

The proof of Theorem 5 is similar to Theorem 1 in [**13**].

**Theorem 6.** *Assume that (i) and (ii) of Lemma 1 hold and that there exists some integer $N$ such that*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} q(s) \max_{[s-\sigma,s]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(u-i\tau)\, ds > \frac{1}{e}. \tag{22}$$

*Then each solution of Equation (1) oscillates.*

*Proof.* If $x(t)$ is an eventually positive solution of Equation (1), then $y'(t) \leq 0$ and $y(t) = x(t) - p(t)x(t-\tau) > 0$ eventually. Then

$$\begin{aligned} x &= y(t) + p(t)x(t-\tau) \\ &= y(t) + p(t)y(t-\tau) + p(t)p(t-\tau)x(t-2\tau) \\ &= \cdots \\ &\geq y(t) + p(t)y(t-\tau) + \cdots + \prod_{i=0}^{N-1} p(t-i\tau)y(t-(i+1)\tau) \\ &\geq \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(t-i\tau)y(t-\tau). \end{aligned}$$

Hence,

$$\begin{aligned}\max_{[t-\sigma,t]} x(s) &\geq \max_{[t-\sigma,t]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(s-i\tau)y(s-\tau)\\ &= \max_{[t-\sigma,t]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(s-i\tau)y(t-\sigma-\tau)\\ &\geq \max_{[t-\sigma,t]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(s-i\tau)y(t-\tau).\end{aligned}$$

Substituting the last inequality into (1), we have

$$y'(t) + q(t) \max_{[t-\sigma,t]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(s-i\tau)y(t-\tau) \leq 0, \tag{23}$$

which contradicts the fact that, under condition (22), the inequality (23) has no eventually positive solution [**7**].

If $Z(t)$ is an eventually negative solution of (1), then $x(t) = -Z(t)$ is an eventually positive solution of (3). Similarly, we have

$$y'(t) + q(t) \min_{[t-\sigma,t]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(s-i\tau)y(t-\tau) \leq 0.$$

That is,

$$[-y(t)]' + q(t) \max_{[t-\sigma,t]} \sum_{j=0}^{N-1} \prod_{i=0}^{j} p(s-i\tau)[-y(t-\tau)] \geq 0.$$

This is also a contradiction by the same reason to the positive solution. The proof is complete. □

For the equation

$$[x(t) - x(t-\tau)]' + q(t) \max_{[t-\sigma,t]} x(s) = 0, \tag{24}$$

we have the following result.

**Theorem 7.** *Assume that $q(t) \geq 0$. Then (24) has nonoscillatory solutions if and only if*

$$Z''(t) + \frac{1}{\tau} q(t) Z(t) = 0 \tag{25}$$

*also has nonoscillatory solutions.*

*Proof.* Assume that $x(t)$ is an eventually positive solution of (24). Let $y(t) = x(t) - x(t-\tau)$; then $y(t) > 0$ and $y'(t) \leq 0$ eventually. Let $T$ be a large number so that $x(t) > 0$, $y(t) > 0$ and $y'(t) \leq 0$ for $t \geq T - \tau$. Set $m = \min\{x(t) : -\tau \leq t \leq T\}$. When $N \leq t \leq N + \tau$, we have

$$x(t) = y(t) + x(t-\tau) \geq \frac{1}{\tau} \int_t^{t+\tau} y(s)\, ds + m.$$

By induction, for $T + k\tau \leq t \leq T + (k+1)\tau$,

$$x(t) \geq \frac{1}{\tau} \int_{t-k\tau}^{t+\tau} y(s)\, ds + m.$$

Hence,

$$x(t) \geq \frac{1}{\tau} \int_{T^*}^{t+\tau} y(s)\, ds + m, \quad t \geq T^* \geq T + \tau,$$

and

$$x(t) \geq \frac{1}{\tau} \int_{T^*+\tau}^{t} y(s)\, ds + m, \quad t \geq T^* + \tau.$$

Set

$$Z(t) = \frac{1}{\tau} \int_{T^*+\tau}^{t} y(s)\, ds + m.$$

Thus, we have

$$Z''(t) + \frac{1}{\tau} q(t) Z(t) \leq 0, \tag{26}$$

which implies that (25) has an eventually positive solution.

If $x(t)$ is an eventually positive solution of the equation

$$[x(t) - x(t-\tau)]' + q(t) \min_{[t-\sigma,t]} x(s) = 0, \tag{27}$$

we can also prove that (25) has an eventually positive solution by the above method.

If (25) has an eventually positive solution $Z(t)$, then $Z''(t) \leq 0$ and $Z'(t) > 0$ eventually. Therefore, there exist $T$ and $M > 0$ such that $Z(t) > M$ and $Z'(t) < M$ eventually. Set

$$H(t) = \begin{cases} \tau Z'(t) & t \geq T \\ (t - T + \tau)Z'(T) & T - \tau \leq t < T, \\ 0 & t < T - \tau. \end{cases}$$

Then $H(t) \geq 0$. Define

$$y(t) = \sum_{i=0}^{\infty} H(t - i\tau) > 0$$

and

$$y(t) - y(t - \tau) = H(t) \quad \text{for } t \geq T.$$

That is,

$$y(t) - y(t - \tau) = \tau Z'(t).$$

Setting

$$\mu = \max\{y(t), T - \tau \leq t \leq T\},$$

we have

$$\begin{aligned} y(t) &= \tau Z'(t) + y(t - \tau) \\ &\leq \int_{t-\tau}^{t} Z'(s)\, ds + y(t - \tau) \\ &\leq \int_{t-2\tau}^{t} Z'(s)\, ds + y(t - 2\tau) \\ &\leq \int_{t-n\tau}^{t} Z'(s)\, ds + y(t - n\tau). \end{aligned}$$

Therefore, we have

$$y(t) \leq \int_{T}^{t} Z'(s)\, ds + \mu \leq Z(t) \quad \text{for} \quad t \geq T.$$

Thus,

$$\max_{[t-\sigma,t]} y(s) \leq Z(t) \quad \text{and} \quad \min_{[t-\sigma,t]} y(s) \leq Z(t) \quad \text{for } t \geq T.$$

Therefore, we have

$$[y(t) - y(t-\tau)]' + q(t) \max_{[t-\sigma,t]} y(s) \le 0$$

and

$$[y(t) - y(t-\tau)]' + q(t) \min_{[t-\sigma,t]} y(s) \le 0.$$

By Theorem 5, (24) has nonoscillatory solutions. The proof is complete. □

**Theorem 8.** *Assume that* $p(t) \equiv p \neq -1$, $q(t) \ge 0$, *and*

$$\int_{t_0}^{\infty} q(s)\,ds = \infty. \tag{28}$$

*Then any nonoscillatory solution* $x(t)$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

**Theorem 9.** *Assume that* $p(t) \equiv -1$, $q(t) \ge 0$, $Q(t) = \min\{q(t), q(t-\tau)\}$ *and*

$$\int_{t_0}^{\infty} Q(t)\,dt = \infty.$$

*Then any eventually positive solution* $x(t)$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

The proofs of Theorems 8 and 9 are similar to Theorems 2 and 1 in [**12**].

*Remark* 2. Theorem 6 improves Theorem 3 in [**1**].

*Remark* 3. By Theorem 7 and the known results of Equation (25), we can obtain some improved results for Equation (24). For example, consider the equation

$$[x(t) - x(t-\tau)]' + \frac{k}{t^2} \max_{[t-\sigma,t]} x(s) = 0. \tag{30}$$

We may show that every solution of Equation (30) is oscillatory if and only if $k > 1/(4\tau)$. Therefore, we see that Theorem 7 improves Theorem 1 in [**1**].

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