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CYCLIC SUBGROUP SEPARABILITY OF CERTAIN HNN EXTENSIONS OF FINITELY GENERATED ABELIAN GROUPS

P.C. WONG AND C.K. TANG

ABSTRACT. In this note we give characterizations for certain HNN extensions of finitely generated abelian groups to be cyclic subgroup separable.

1. Introduction. The residual finiteness of the Baumslag-Solitar groups $G_{k,l} = \langle t, a; t^{-1}a^k t = a^l \rangle$ were exhaustively studied and completely characterized by Baumslag and Solitar [**5**], Meskin [**8**] and Collins and Levin [**7**]. Their results can be summarized as follows:

Theorem A. *Let* $G_{k,l} = \langle t, a; t^{-1}a^k t = a^l \rangle$. *Then* $G_{k,l}$ *is residually finite if and only if* $|k| = 1$ *or* $|l| = 1$ *or* $|k| = |l|$.

Observing that the Baumslag-Solitar groups are HNN extensions with base group an infinite cyclic group, Andreadakis, Raptis and Varsos in a series of papers [**2, 3, 4, 9**] gave characterizations for the HNN extensions $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ where the base group K is a finitely generated abelian group, to be residually finite. Motivated by the above results, Wong in [**11**] and [**12**] gave characterizations for the Baumslag-Solitar groups and some of the above HNN extensions to be subgroup separable.

In this note we shall extend the results of Andreadakis, Raptis and Varsos by giving characterizations for those HNN extensions to be cyclic subgroup separable.

Our main results are contained in Theorems 1–5. In addition, we shall give some applications.

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The layout of this paper is as follows. We give the definitions and essential lemmas in Section 2, prove our main results in Sections 3 and 4, and give the applications in Section 5.

The notation used here is standard. In addition, the following will be used for any group G:

$N <_f G$, respectively $N \triangleleft_f G$, means N is a subgroup, respectively normal subgroup, of finite index in G.

$G = \langle t, K; t^{-1}At = B, \varphi\rangle$ denotes an HNN extension where K is the base group, A, B are the associated subgroups and φ is the associated isomorphism $\varphi : A \to B$.

2. Preliminaries. We begin with the definition of cyclic subgroup separable groups. Following Stebe [**10**], we will use the term π_c instead of cyclic subgroup separable.

Definition 1. A group G is called *H-separable* for the subgroup H if for each $x \in G\backslash H$, there exists $N \triangleleft_f G$ such that $x \notin HN$.

G is termed subgroup separable, respectively π_c, if G is H-separable for every finitely generated subgroup H, respectively cyclic subgroup H.

Lemma 1. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension where K is a finite group. Then G is subgroup separable and hence π_c.*

Proof. Lemma of Wong [**11**]. □

Definition 2. Let K be a group, $A, B < K$ and $\varphi : A \to B$ be an isomorphism. A family $\{N_i\}$, $i \in I$, of normal subgroups of K is called an (A, B, φ) *filtration* of K if the following hold:

(i) $(A \cap N_i)\varphi = B \cap N_i$, for all $i \in I$;

(ii) $\cap_{i\in I} N_i = 1$;

(iii) $\cap_{i\in I} N_i A = A$, $\cap_{i\in I} N_i B = B$.

Furthermore the $\{N_i\}$, $i \in I$, will be called an *extended*-(A, B, φ)

filtration of K if in addition the following hold:

(iv) $\cap_{i\in I} N_i\langle x\rangle = \langle x\rangle$, for all $x \in K$.

Let $\{N_i\}$, $i \in I$, be an extended-(A, B, φ) filtration of K. We can form the HNN extension $G_i = \langle t, K/N_i; t^{-1}(AN_i/N_i)t = BN_i/N_i, \varphi_i\rangle$ where φ_i is the isomorphism from AN_i/N_i onto BN_i/N_i induced by φ. Clearly, each G_i is a homomorphic image of G.

Lemma 2. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an* HNN *extension. Let $\{N_i\}$, $i \in I$, be an extended-(A, B, φ) filtration of K and G_i as defined above. If each G_i is π_c, then G is π_c.*

Proof. Straightforward. □

3. The main theorems.

Theorem 1. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an* HNN *extension where K is a finitely generated abelian group, $K \neq A$, $K \neq B$ and A, B of finite index in K. Then G is π_c if and only if there exists a subgroup H of finite index in K and H is normal in G.*

Proof. Suppose there exists a subgroup H of finite index in K and H is normal in G. Let $x, y \in G$ with $x \notin \langle y\rangle$.

Suppose that $x \notin \langle y\rangle H$. Since $H \triangleleft G$, we have $t^{-1}Ht = H$ and so $H\varphi = H$. This implies $H \subseteq A \cap B$ since φ is the associated isomorphism from A to B. Thus we can form the HNN extension $\overline{G} = \langle t, K/H; t^{-1}(A/H)t = B/H, \bar{\varphi}\rangle$ where $\bar{\varphi}$ is the isomorphism from A/H onto B/H induced by φ. Clearly $\overline{G}$ is a homomorphic image of G. Let $\bar{g}$ denote the image of any element g of G in $\overline{G}$. Then $\bar{x} \notin \langle\bar{y}\rangle$. Since K/H is finite, $\overline{G}$ is π_c by Lemma 1. Thus, there exists $\overline{N} \triangleleft_f \overline{G}$ such that $\bar{x} \notin \langle\bar{y}\rangle\overline{N}$, namely, there exists $N \triangleleft_f G$ such that $x \notin \langle y\rangle N$.

Suppose that $x \in \langle y\rangle H$. Then $x = y^k h$, $h \in H$, but $h \notin H \cap \langle y\rangle$, since $x \notin \langle y\rangle$. Now H is finitely generated abelian and, hence, π_c. Since $H \cap \langle y\rangle$ is cyclic, there exists $S \triangleleft_f H$, with index s say, such that $h \notin (H \cap \langle y\rangle)S$. Since H is finitely generated, there exists only a finite number of subgroups of index s in H. Let R be the intersection of all these subgroups of index s in H. Then R is characteristic in H and so $R \lhd G$. Furthermore, $R \lhd_f H$ and $h \notin (H \cap \langle y \rangle)R$. If $x \in \langle y \rangle R$, then $x = y^k h = y^n r$, $r \in R$. Hence, $hr^{-1} = y^{n-k} \in H \cap \langle y \rangle$, since $R < H$, and thus $h \in (H \cap \langle y \rangle)R$, a contradiction. So $x \notin \langle y \rangle R$. So we can argue, as before, with R in place of H and find $N \lhd_f G$ such that $x \notin \langle y \rangle N$.

On the other hand, if G is π_c, then G is residually finite and the result follows from Theorem 1 of [**2**]. □

We next prove Theorem 2 which is a refinement of Lemma 1.

Theorem 2. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ be an* HNN *extension where K is a finitely generated abelian group. If $A \cap B$ is finite, then G is π_c.*

Proof. We can assume that A, B are infinite since, if A, B are finite, then G is π_c by Theorem 2 of [**13**]. Furthermore, we can assume that A, B are of infinite index in K since $A \cap B$ is finite.

Let $A = A_1 \times A_2$, $B = B_1 \times B_2$, where A_1, $B_1 = A_1\varphi$ are the torsion parts of A and B, respectively, and $A_2, B_2 = A_2\varphi$ are torsion free. So $A \cap B = A_1 \cap B_1$ and $A_2 \cap B_2 = 1$. Similarly, let $K = K_1 \times K_2$, where K_1 is the torsion part of K and K_2 is chosen to be torsion free and such that $\langle A_2, B_2 \rangle = A_2 \times B_2 < K_2$. Hence, we can find $C_2 < K_2$ such that $(A_2 \times B_2) \cap C_2 = 1$ and $|K_2 : A_2B_2C_2|$ is finite.

Let $N_r = (A_2B_2C_2)^r$, $r \in Z^+$. Then N_r is a normal subgroup of finite index in K since $|K_2 : A_2B_2C_2|$ is finite. It is easy to see that $AN_r = AB_2^rC_2^r$, $BN_r = A_2^rBC_2^r$ and $(A \cap N_r)\varphi = (A_2^r)\varphi = B_2^r = B \cap N_r$. Furthermore, $\cap A_2^rB_2^rC_2^r = 1$.

Suppose $x \notin \langle y \rangle$ where $x, y \in K$. Since K is π_c, there exists $M \lhd_f K$ such that $x \notin \langle y \rangle M$. Let $M = M_1 \times M_2$, where M_1 is the torsion part of M and M_2 is torsion free. Then $M_2 \lhd_f K_2$ and hence there is a positive integer s such that K_2^s is a subgroup of M_2 since K is finitely generated. Thus $N_s = (A_2B_2C_2)^s$ is a subgroup of K_2^s in which $x \notin \langle y \rangle N_s$. Hence, $\{N_r\}$, $r \in Z^+$, is an extended-(A, B, φ) filtration of K. Let $G_r = \langle t, K/N_r; t^{-1}(AN_r/N_r)t = BN_r/N_r, \varphi_r \rangle$ where φ_r is the isomorphism from AN_r/N_r onto BN_r/N_r induced by φ. Then, by

Lemma 1, G_r is π_c since K/N_r is finite. Thus G is π_c by Lemma 2. □

Theorem 3. *Let* $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ *be an* HNN *extension where* K *is a finitely generated abelian group,* A, B*, of infinite index in* K*. Then* G *is* π_c *if and only if there exists a subgroup* H *of finite index in* $A \cap B$ *and* H *is normal in* G*.*

Proof. The proof of Theorem 3 is the same as the proof of Theorem 1 except that we use Theorem 2 above and Proposition 3 of [**9**] instead of Lemma 1 above and Theorem 1 of [**2**], respectively. □

Next we define a subgroup D which is needed in Theorem 4, see Andreadakis, Raptis and Varsos [**3**]. Let $M_0 = A \cap B$. Define $M_1 = M_0\varphi^{-1} \cap M_0 \cap M_0\varphi$ and inductively $M_{i+1} = M_i\varphi^{-1} \cap M_i \cap M_i\varphi$. Then $M_{i+1} \leq M_i$ for every i. If $r(M_i)$ denotes the free rank of M_i, then there must be an integer λ such that $r(M_\lambda) = r(M_{\lambda+1})$ since K is finitely generated. Set D to be the isolated closure of the subgroup M_λ in K (see Definition 3 below). In the HNN notation, D consists of all elements a with the following property: for each $\nu \in Z$ there exists a $\lambda \in Z^+$ such that $t^{-\nu}a^\lambda t^\nu \in K$.

Theorem 4. *Let* $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ *be an* HNN *extension where* K *is a finitely generated abelian group and* $K \neq A$*,* $K \neq B$*. Then* G *is* π_c *if and only if there exists a subgroup* H *of finite index in* D *with* $H\varphi = H$*.*

To prove Theorem 4 we shall need the following definitions and lemmas from Andreadakis, Raptis and Varsos [**3**].

Definition 3. Let G be a group and $H < G$. Then

(i) H is called an *isolated subgroup* of G if, whenever $g^n \in H$ for some $n \in Z^+$ then $g \in H$;

(ii) $i_G(H)$ shall denote the *isolated closure of* H *in* G, that is, $i_G(H)$ is the intersection of all isolated subgroups of G containing H. We shall write $i(H)$ for $i_G(H)$ when the group G is clear.

Lemma 3. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an* HNN *extension where K is a finitely generated abelian group and $H < K$ such that $H\varphi = H$. Then $\{H^r\}$, $r \in Z^+$, is an (A, B, φ) filtration of K.*

Proof. Lemma 2 in Andreadakis, Raptis and Varsos [**3**]. □

Lemma 4. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an* HNN *extension where K is a finitely generated abelian group and D be the subgroup as in Theorem* 4. *Let $i(A \cap B)$ be the isolated closure of $A \cap B$ in K. Then*

(i) *$D \leq i(A \cap B)$ and $i(A \cap B)/D$ is a free abelian group*;

(ii) *If D is a proper subgroup of $i(A \cap B)$, then there exists a nontrivial torsion-free subgroup L of $i(A \cap B)$ such that $L \cap D = 1$ and $\{L^r\}$, $r \in Z^+$, is an (A, B, φ) filtration of K.*

Proof. Proposition 2 and Lemma 3 in Andreadakis, Raptis and Varsos [**3**]. □

We now prove Theorem 4.

Proof of Theorem 4. Suppose there exists a subgroup H of finite index in D with $H\varphi = H$. We shall use induction on the rank of the free abelian group $i(A \cap B)/D$, see Lemma 4.

If $r(i(A\cap B)/D) = 0$, then $i(A \cap B) = D$ and $H \triangleleft_f A \cap B$. Hence the theorem follows from Theorem 3 above.

Suppose the theorem holds for all groups G for which there exists $H \triangleleft_f D$ with $H\varphi = H$ and $r(i(A\cap B)/D) < n$. Now let G be a group for which there exists $H \triangleleft_f D$ with $H\varphi = H$ and $r(i(A\cap B)/D) = n$. Since D is a proper subgroup of $i(A \cap B)$, there exists an (A, B, φ) filtration $\{L^r\}$, $r \in Z^+$, of K by Lemma 4. Furthermore, $\{H^r\}$, $r \in Z^+$, is also an (A, B, φ) filtration of K by Lemma 3. Hence $N_r = H^r L^r = (HL)^r$ is a normal subgroup of K for each $r \in Z^+$ and $\{N_r\}$, $r \in Z^+$, is also an (A, B, φ) filtration of K.

Suppose $x \notin \langle y\rangle$ where $x, y \in K$. Since K is π_c, there exists $M \triangleleft_f K$ such that $x \notin \langle y\rangle M$. Since K is finitely generated, there exists a positive integer s such that $K^s \subseteq M$. Now $N_s = (HL)^s$ is a subgroup of K^s in which $x \notin \langle y \rangle N_s$. Hence $\{N_r\}$, $r \in Z^+$, is an extended-(A, B, φ) filtration of K.

Let $G_r = \langle t, K/N_r; t^{-1}(AN_r/N_r)t = BN_r/N_r, \varphi_r \rangle$ where φ_r is the isomorphism from AN_r/N_r onto BN_r/N_r induced by φ. Let D_r be the subgroup in G_r which correspond to D in G. Then $D_r = i(DN_r)/N_r$ by Lemma 4 of [**3**]. Let $H_r = HN_r/N_r$. Then $H_r\varphi_r = H_r$ and $H_r \triangleleft_f D_r$. Let $S_r = i((AN_r/N_r) \cap (BN_r/N_r))$. Then $r(S_r/D_r) < r(i(A \cap B)/D)$ by Lemma 5 of [**3**]. Hence G_r is π_c by the induction hypothesis. Thus G is π_c by Lemma 2.

On the other hand, if G is π_c, then the result follows from the main theorem of [**3**]. □

4. Automorphisms. In [**4**], Andreadakis, Raptis and Varsos gave another characterization for $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ to be residually finite by embedding the base group K in a supergroup X with an automorphism $\bar{\varphi} \in \operatorname{Aut} X$ such that $\bar{\varphi}|_A = \varphi$. We shall prove the following:

Theorem 5. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ be an* HNN *extension where K is a finitely generated abelian group and $K \neq A$, $K \neq B$. Then G is π_c if and only if there exists a finitely generated abelian group X and an automorphism $\bar{\varphi} \in Aut\, X$ such that $K \subseteq X$ and $\bar{\varphi}|_A = \varphi$.*

To prove Theorem 5, we need the next result.

Lemma 5. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ be an* HNN *extension where K is a finitely generated abelian group. If φ comes from an automorphism of K, then G is π_c.*

Proof. By Lemma 4.4 of [**6**], G is residually finite. Let $\mathcal{N}$ be the set $\mathcal{N} = \{N \mid N \triangleleft_f K \text{ and } (A \cap N)\varphi = B \cap N\}$. Then, by Proposition 2 of [**2**], the set $\mathcal{N}$ forms an (A, B, φ) filtration of K.

Suppose $x \notin \langle y \rangle$ where $x, y \in K$. Since K is π_c, there exists $M \triangleleft_f K$ such that $x \notin \langle y \rangle M$. Let M have index s in K. Let R be the intersection of all these subgroups of index s in K. Then R is characteristic in K, and hence $(A \cap R)\varphi = B \cap R$ since φ is an automorphism of K. Clearly $R \triangleleft_f K$ and $x \notin \langle y \rangle R$. Therefore, $R \in \mathcal{N}$ and $\mathcal{N}$ is an extended-(A, B, φ) filtration of K.

For each $N \in \mathcal{N}$, we let $G_N = \langle t, K/N; t^{-1}(AN/N)t = BN/N, \varphi_N \rangle$ where φ_N is the isomorphism from AN/N onto BN/N induced by φ. Then, by Lemma 1, G_N is π_c since K/N is finite and so, by Lemma 2, G is π_c. □

Proof of Theorem 5. Theorem 5 follows from Theorem 1 of [**4**] and Lemma 5 since a subgroup of a π_c group is again a π_c group. □

5. Applications. In this section we give some applications of our theorems.

Corollary 1. *Let* $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ *be an* HNN *extension where* K *is a finitely generated abelian group and* $K \neq A$, $K \neq B$. *If there exists a subgroup* H *of* K *where* H *has finite index in both* A *and* B *and* H *is normal in* G, *then* G *is* π_c.

Proof. Since H has finite index in $A \cap B$, the result now follows easily from Theorems 1 and 3. □

Corollary 2. *Let* $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ *be an* HNN *extension where* K *is a finitely generated abelian group and* $A = B$. *Then* G *is* π_c.

Proof. If $K = A = B$, then $G = \langle t, K; t^{-1}Kt = K, \varphi \rangle$ is a split extension of K by $\langle t \rangle$ and hence is π_c by Theorem 4 of Allenby and Gregorac [**1**]. If $K \neq A = B$, then G is π_c by Corollary 1. □

Finally we show the following:

Corollary 3. *Let*

$$G = \langle t, a_1, a_2, \ldots, a_n; t^{-1}a_i^{h_i}t = a_i^{k_i}, \quad i = 1, 2, \ldots, r, r \leq n, [a_i, a_j] = 1 \rangle$$

and $K = \langle a_1, a_2, \ldots, a_n; [a_i, a_j] = 1 \rangle$. *Then the following are equiva-*

lent:

(i) *G is* π_c;

(ii) $|h_i| = |k_i|$, $i = 1, 2, \dots, r$;

(iii) *the map* φ *which sends* $a_i^{h_i}$ *to* $a_i^{k_i}$, $i = 1, 2, \dots, r$ *comes from an automorphism of* K.

Proof. Let $K = \langle a_1, a_2, \dots, a_n; [a_i, a_j] = 1\rangle$ be the free abelian group of rank n. Let $A = \langle a_1^{h_1}, a_2^{h_2}, \dots, a_r^{h_r}\rangle$, $B = \langle a_1^{k_1}, a_2^{k_2}, \dots, a_r^{k_r}\rangle$ be subgroups of K and $\varphi : A \to B$ be the map defined by $a_i^{h_i}\varphi = a_i^{k_i}$, $i = 1, 2, \dots, r$. Then G can be considered as the HNN extension $G = \langle t, K; t^{-1}At = B, \varphi\rangle$.

We show (i) $\Rightarrow$ (ii). Since G is π_c, by Theorems 1 and 3 there exists a subgroup H of finite index in $A \cap B$ and H is normal in G. Since $H < A \cap B$, we can write $H = \langle a_1^{c_1}, a_2^{c_2}, \dots, a_r^{c_r}\rangle$ with $h_i|c_i$, $k_i|c_i$. But

$$(a_i^{c_i})\varphi = (a_i^{c_i h_i/h_i})\varphi = (a_i^{h_i}\varphi)^{c_i/h_i} = a_i^{k_i c_i/h_i} = (a_i^{c_i})^{k_i/h_i}.$$

Since $H\varphi = H$, we have $|h_i| = |k_i|$, $i = 1, 2, \dots, r$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) follows from Lemma 5. □

From Corollary 3 we obtain the fact that the Baumslag-Solitar groups $G = \langle t, a; t^{-1}a^h t = a^k\rangle$ is π_c if and only if $|h| = |k|$.

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Department of Mathematics, University of Malaya, 59100 Kuala Lumpur, Malaysia
E-mail address: wpc@mnt.math.um.edu.my