ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1999

GROUPS OF ISOMETRIES OF A TREE AND THE CCR PROPERTY

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1. Introduction. Let X be a homogeneous tree of order $q+1 \geq 3$. Let Ω be the tree boundary. Let $\mathrm{Aut}\,(X)$ be the locally compact group of all isometries of X. The reader is referred to [**10**] or [**3**] for undefined notions and terminology. In [**5**] a locally compact group G is called a CCR-group if $\pi(f)$ is a compact operator for every $f \in L^1(G)$ and for every $\pi \in \hat{G}$ where $\hat{G}$ is the set of equivalence classes of all unitary continuous irreducible representations of G. Every CCR-group is a type I group [**2**]. $\mathrm{Aut}\,(X)$ is a CCR-group, see [**7**] or [**3**, p. 113]. Also, $PGL(2, \mathbf{Q}_p)$ where $\mathbf{Q}_p$ is the field of the p-adic numbers, is a CCR-group [**9**]. It is known that $PGL(2, \mathbf{Q}_p)$ may be realized as a closed subgroup of $\mathrm{Aut}\,(X)$, for some tree X,in such a way that $PGL(2, \mathbf{Q}_p)$ acts transitively on X and Ω. If G is a locally compact totally disconnected group, then the property CCR is equivalent to the fact that every unitary irreducible representation of G is admissible, see Section 2 below. On the other hand, in the present paper, we prove that if G is a closed unimodular CCR-subgroup of $\mathrm{Aut}\,(X)$ acting transitively on X, then G acts transitively on Ω. We conjecture that the converse is true. This conjecture is supported by the fact that all noncuspidal irreducible representations of a closed subgroup of $\mathrm{Aut}\,(X)$ acting transitively on X and on Ω are in fact admissible representations. This follows from the classification given in [**3**, p. 84]. It is also true that every irreducible subrepresentation of the regular representation is admissible [**4**, p. 6].

2. The result. There exists a K-invariant probability measure on the tree boundary, Ω, say ν. Let $P(g, \omega)$ be the Poisson kernel associated with ν, that is, $P(g, \omega) = (d\nu_g/d\nu)(\omega)$ for $g \in \mathrm{Aut}\,(X)$ and $\omega \in \Omega$ with $\nu_g(\omega) = \nu(g^{-1}\omega)$, see [**3**, pp. 34–35]. For every $t \in \mathbf{R}$, we

Received by the editors on October 12, 1996, and in revised form on December 6, 1996.

define the following representation of $\mathrm{Aut}\,(X)$ on $L^2(\Omega,\nu)$:

$$[\pi_{1/2+it}(g)f](\omega) = P^{1/2+it}(g,\omega)f(g^{-1}\omega).$$

For every $t \in \mathbf{R}$, $\pi_{1/2+it}$ is irreducible. The series of representations $\{\pi_{1/2+it} : t \in \mathbf{R}\}$ is called the unitary principal series of representations of $\mathrm{Aut}\,(X)$. In [**1**], Bouaziz-Kellil proved that the representations of the unitary principal series restrict irreducibly to any closed unimodular subgroup of $\mathrm{Aut}\,(X)$ acting transitively on X. A locally compact group G is called a CCR-group if, for every irreducible representation π and for every $f \in L^1(G)$, the operator $\pi(f) = \int_G f(x)\pi(x)\,dx$ is compact. As observed in the introduction, if G is a totally disconnected locally compact group, then G is a CCR-group if and only if every unitary irreducible representation is admissible. (Recall that a representation π is called admissible if, for every compact open subgroup H of G, the subspace of H-invariant vectors is finite-dimensional.) In fact, the space $\mathcal{S}$ of locally constant functions with compact support is dense in $L^1(G)$ and, for every $f \in \mathcal{S}$ there exists an open compact subgroup H of G such that $f = \sum_{i=1}^n c_i \chi_{x_i H}$ where $\chi_{x_i H}$ is the characteristic function of the coset $x_i H$. Therefore, $\pi(f)$ is compact for every π and f in $L^1(G)$ if and only if $\pi(\chi_H)$ is compact for every compact open subgroup H. This means that π is admissible because $(1/\lambda(H))\pi(\chi_H)$ (where λ is a fixed left Haar measure) is the orthogonal projection on the space of H-invariant vectors. The aim of this note is to prove that, if G is a closed unimodular subgroup of $\mathrm{Aut}\,(X)$ which acts transitively on X but does not act transitively on Ω, then G is not a CCR-group.

Theorem. *Let G be a closed unimodular* CCR*-subgroup of* $\mathrm{Aut}\,(X)$ *acting transitively on X; then G acts transitively on Ω.*

Proof. We will prove that, if G is a closed unimodular subgroup of $\mathrm{Aut}\,(X)$ acting transitively on X but not on Ω, then G does not satisfy the CCR property. As observed in the previous remarks, the restriction to G of $\pi_{1/2+it}|_G$ is irreducible for every t [**1**]. Therefore it is enough to prove that $\pi_{1/2+it}|_G$ is not admissible. Let x_0 be a fixed vertex of X. Let K be the stability subgroup of x_0. The subgroup K is compact open in G; let

$$M_K = \{f \in L^2(\Omega) : \text{ for every } k \in K f(k\omega) = f(\omega) \text{ a.e.}\}.$$

Since $P(k,\omega) = 1$ for every $k \in K$, it follows that M_K is exactly the closed subspace of $L^2(\Omega)$ consisting of all K-invariant vectors of $\pi_{1/2+it}|_G$ for every $t \in \mathbf{R}$. Hence, it suffices to show that $\dim M_K = +\infty$. This is a consequence of the following two lemmas. $\square$

Lemma 1. *Let G be a closed unimodular subgroup of* Aut (X) *acting transitively on X. If G is not transitive on Ω, then no orbit of G on Ω is open.*

Proof. We recall that, if an orbit E of G on Ω has an interior point, then E is open. In [**6**] we prove that,if a closed subgroup of Aut (X) acts transitively on X and on an open subset of Ω, then either G fixes one end of X or G acts transitively on Ω. Also we can deduce from the proof of [**6**, Theorem 3, p. 377] that, if the action of G on Ω is not transitive, then G fixed an end $\omega \in \Omega$ and it acts transitively on $\Omega\backslash\{\omega\}$. Therefore, to prove Lemma 1, it suffices to show that, if G fixes ω and it acts transitively on $\Omega\backslash\{\omega\}$, then G is not unimodular. In fact, G contains a step one translation [**6**, Lemma 1, p. 378] because G acts transitively on X. For α and β in Ω with $\alpha \neq \beta$, let (α, β) be the unique infinite geodesic joining α to β. If G fixes ω and it acts transitively on $\Omega\backslash\{\omega\}$ then, for every $\omega_0 \neq \omega_1$ with $\omega_0 \neq \omega$ and $\omega_1 \neq \omega$, there exists $g \in G$ such that $g(\omega) = \omega$ and $g(\omega_0) = \omega_1$, that is, $g((\omega, \omega_0)) = (\omega, \omega_1)$. Therefore, if w is a step one translation along (ω, ω_0), then it is easy to see that gwg^{-1} is a step one translation along the geodesic (ω, ω_1). This means that, for every geodesic (ω, ω_1) with $\omega \neq \omega_1$, there exists a step one translation in G along (ω, ω_1). We now fix an infinite geodesic (ω, ω_0) with $\omega \neq \omega_0$ and a step one translation $\omega \in G$ along (ω, ω_0). Let $\{s_n\}$ be the sequence of distinct vertices of (ω, ω_0) for $n \in \mathbf{Z}$. Let K_n be the stability subgroup of s_n for $n \in \mathbf{Z}$. For every n, K_n is compact open in G. We can suppose that the sequence $\{s_0, s_1, s_2, \dots, s_n, \dots\}$ for $n \geq 0$ identifies ω_0 while the sequence $\{s_0, s_{-1}, s_{-2}, \dots, s_{-n}, \dots\}$ identifies ω. So $K_n \subseteq K_{n-1}$ for every n because G fixes ω. Moreover, we can suppose that $w(s_n) = s_{n+1}$. Therefore $wK_{n-1}w^{-1} \subseteq K_n$. If G is unimodular, then $\lambda(wK_{n-1}w^{-1}) = \lambda(K_{n-1}) \leq \lambda(K_n)$ (λ is a fixed left Haar measure). On the other hand, $K_n \subseteq K_{n-1}$ and $\lambda(K_{n-1}) = \lambda(K_n)$. Since the subgroup K_n is compact open, then $K_{n-1} = K_n$ for every n. The same argument applies to every geodesic (ω, ω_1) with $\omega \neq \omega_1$ by replacing w with gwg^{-1} as observed. We have that X is the union of different geodesics (ω, ω_0) where $\omega_0 \in \Omega \backslash \{\omega\}$, as is easily seen. Therefore, it is easy to see that, if $g(v) = v$ for a vertex v, then $g = e$, that is, $G \cap K_v = \{e\}$. This means that G acts faithfully and transitively on X. This is a contradiction because such a group is discrete and so it does not act transitively on $\Omega \backslash \{\omega\}$. In fact, a discrete subgroup of $\mathrm{Aut}\,(X)$ is countable; therefore, every orbit of a discrete subgroup is countable. □

Lemma 2. *Let G be a closed unimodular subgroup of* $\mathrm{Aut}\,(X)$ *which acts transitively on X but does not act transitively on Ω. Then* $\dim M_K = +\infty$.

Proof. Lemma 1 implies that no orbit of G on Ω is open. Obviously, this property is true also for the subgroup K. We recall that an orbit is open if and only if it contains an interior point. Let x_0 be the fixed vertex of X such that $K = K_{x_0}$. For $x \in X$, $x \neq x_0$, let $C(x)$ be the subset of Ω consisting of all ends $\omega \in \Omega$ such that the infinite geodesic $[x_0, \omega)$ contains x. $C(x)$ is open in Ω, therefore K is not transitive on $C(x)$ for every x. Let $S_n = \{y \in X : d(x_0, y) = n\}$ for $n = 1, 2, 3, \dots$. Obviously S_n is K-invariant for every n. Since K is compact, it is easy to see that K acts transitively on Ω if and only if K acts transitively on S_n for every n. Also, for a fixed vertex $x \neq x_0$, K acts transitively on $C(x)$ if and only if K acts transitively on $E(x, n)$ for every $n > d(x_0, x)$ where

$$E(x, n) = \{y \in S_n : \exists \omega \in C(x) \text{ such that } y \in [x_0, \omega)\}.$$

Hence, by Lemma 1, it follows that, for every $x \neq x_0$, there exists $n > d(x_0, x)$ such that K is not transitive on $E(x, m)$ for every $m \geq n$. As observed, S_n is K-invariant and so S_n is a disjoint union of different orbits of K on X. Let $\{S_n^1, S_n^2, \dots, S_n^{i_n}\}$ be the partition of S_n into the orbits of K. Since K is not transitive on Ω,then $i_n > 1$ for n sufficiently large. Let S_n^j be a fixed orbit of K contained in S_n; then the following subset of Ω,

$$F_n^j = \bigcup_{y \in S_n^j} C(y)$$

is K-invariant, and so χ_{jn}, the characteristic function of the set F_n^j, is a K-invariant continuous function with compact support, that is,

$\chi_{jn} \in M_K$. On the other hand, if $F_n^j \cap F_m^h = \varnothing$, then the functions χ_{jn} and χ_{hm} are linearly independent in $L^2(\Omega, \nu)$. Therefore, Lemma 2 is a consequence of the following claim. For every integer p there exist p sets $A_1, A_2, \dots, A_p$ of type F_n^j such that $A_s \cap A_t = \varnothing$ for every $s \neq t$, $s, t = 1, 2, \dots, p$. The claim follows easily from the first part of this proof. Indeed, there exists n such that $i_n > 1$; let S_n^1, S_n^2 be two distinct orbits of S_n. We define $A_1 = F_n^1$. There exists $m > n$ such that K is not transitive on the union of the sets $E(x, m)$ with $x \in S_n^2$; therefore, there exist two distinct orbits, say S_m^1 and S_m^2, in this set union of the sets $E(x, m)$. We define $A_2 = F_m^1$; obviously $F_n^1 \cap F_m^1 = \varnothing$ because $S_n^1 \cap S_n^2 = \varnothing$. Similarly, there exists $h > m$ such that the action of K on the union of the sets $E(x, h)$ with $x \in S_m^2$ is not transitive, and the lemma follows. $\square$

Remarks. (1) Using an argument similar to that for Lemma 2, we can also prove that $\dim M_H = +\infty$ for every compact open subgroup H of G.

(2) Finally, we provide examples of closed unimodular nondiscrete subgroups of $\mathrm{Aut}\,(X)$ as in Lemma 2. Let r be an integer such that $1 \leq r \leq q+1$ where $q+1$ is the order of the tree X. For $i = 1, 2, \dots, r$, let E_i be a set of indices such that

$$|E_1| + |E_2| + \cdots + |E_r| = q + 1.$$

We suppose that $E_i \cap E_j = \varnothing$ for every $i \neq j$. If $r = 1$, then $|E_1| = q+1$, if $r = q + 1$ then $|E_1| = |E_2| = \cdots = |E_r| = 1$. We may label the nonoriented edges of X in such a way that, for every vertex v of X, there is a bijection of the set of indices $E_1 \cup E_2 \cup \cdots \cup E_r$ onto the $q+1$ edges starting from the vertex v. We will only consider nonoriented edges. This means that, if x and y are adjacent vertices, then the edge $[x, y] = [y, x]$ is labeled in the same way from x's point of view or y's point of view. In this way, for every v, there is a partition $F_1^v, F_2^v, \dots, F_r^v$ into disjoint subsets of the set of edges starting from v such that, for every $i = 1, 2, \dots, r$, F_i^v is in one-to-one correspondence with E_i. Therefore, $X_{\mathcal{L}}$ becomes a labeled tree; we consider the set $G_{\mathcal{L}}$ of all isometries g of X such that $g(F_i^v) = F_i^{g(v)}$ for every $i = 1, 2, \dots, r$ and for every $v \in X$, that is, the set of all automorphisms of the labeled tree $X_{\mathcal{L}}$. It is easy to see that $G_{\mathcal{L}}$ is a closed subgroup of $\mathrm{Aut}\,(X)$ acting transitively on X. If $r = 1$, then $G_{\mathcal{L}} = \mathrm{Aut}\,(X)$ while $r = q + 1$ implies that $G_{\mathcal{L}} = \Gamma$ where Γ is the simply transitive subgroup of $\mathrm{Aut}\,(X)$ isomorphic with $\mathbf{Z}_2^*\mathbf{Z}_2^*\cdots^*\mathbf{Z}_2$ $q+1$-times [**3**, p. 16]. Moreover, it is easy to see that, for $1 < r < q+1$, $G_{\mathcal{L}}$ is a closed nondiscrete subgroup of $\mathrm{Aut}\,(X)$ which acts transitively on X but does not act transitively on Ω. We prove now that $G_{\mathcal{L}}$ is unimodular. It follows directly that, for every edge $[x, y]$ there exists an inversion of order 2 in $G_{\mathcal{L}}$ on the edge $[x, y]$. This implies that $G_{\mathcal{L}}$ contains a discrete simply transitive subgroup Γ isomorphic to $\mathbf{Z}_2^*\mathbf{Z}_2^*\cdots^*\mathbf{Z}_2$ $q+1$-times [**3**, pp. 14–15]. Let K be the stability subgroup of a fixed vertex v of X; K is compact open in $G_{\mathcal{L}}$ and $G_{\mathcal{L}} = \Gamma K$ with $\Gamma \cap K = \{e\}$. Since Γ is discrete and K is compact, it follows that $G_{\mathcal{L}}$ is unimodular.

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