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ON THE HILBERT FUNCTION OF CERTAIN NON COHEN-MACAULAY ONE-DIMENSIONAL RINGS

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ABSTRACT. Let k be a field, $n_0, \dots, n_r$ a sequence of positive integers, A the coordinate ring of the algebroid monomial curve in $\mathbf{A}_k^{r+1}$ defined parametrically by $X_0 = t^{n_0}, \dots, X_r = t^{n_r}$. Let $G = \mathrm{gr}_M(A)$, where $M = (t^{n_0}, \dots, t^{n_r})$ and $P_G(z) = (h_G(z)/(1-z))$, where $h_G(z) = \sum_{i=0,\dots,s} h_i z^i \in Q[z]$, be the Poincaré series of G. In this paper we study some of the coefficients of the polynomial $\sum_{i=0,\dots,s} h_i z^i$ when G is not Cohen-Macaulay. We show $h_s > 0$ and $h_2 \geq 0$, $h_3 \geq 0$ under suitable assumptions.

Introduction. Let A be the coordinate ring of an algebroid monomial curve in the affine space $\mathbf{A}_k^{r+1}$ defined parametrically by $X_0 = t^{n_0}, \dots, X_r = t^{n_r}$ where $0 < n_0 < \cdots < n_r$, $\gcd(n_0, \dots, n_r) = 1$ and $n_0, \dots, n_r$ minimally generate the semigroup $\Gamma = \langle n_0, \dots, n_r \rangle$. Let $M = (t^{n_0}, \dots, t^{n_r})$ and $G = \mathrm{gr}_M(A) = \oplus_{i \geq 0}(M^i/M^{i+1})$. If $H_G(i) = \dim_k(M^i/M^{i+1})$ is the Hilbert function of G and $P_G(z) = \sum_{i \geq 0} H_G(i) z^i$ is the Poincaré series, it is known that $P_G(z) = h_G(z)/(1-z)$ where $h_G(z) = \sum_{i=0,\dots,s} h_i z^i \in Q[z]$, $h_s \neq 0$. When G is Cohen-Macaulay, it is known that $h_i > 0$ for each $i \in \{0, \dots, s\}$. When G is not Cohen-Macaulay, one may have $h_i < 0$ for some i. In this paper we study the coefficients h_i of $h_G(z)$ when G is not Cohen-Macaulay. To this purpose we use a characterization of the Cohen-Macaulayness of G we gave in [**3**]. In Section 1 we show h_s is always positive, see Corollary 1.11. In Section 2 we study h_2. There are examples where G is not Cohen-Macaulay and $h_2 < 0$, e.g., in [**1**] and [**2**]. First, we show an example where $h_2 < 0$ and $A = k[[t^{n_0}, \dots, t^{n_r}]]$ has both multiplicity and embedding dimension less than the ones considered in the examples of [**1**] and [**2**]. Then we prove $h_2 \geq 0$ when Γ is generated by $m_0, \dots, m_p, n$ where $m_0, \dots, m_p$ are an arithmetic sequence, i.e., $m_i = m_0 + id$ for $i \in \{1, \dots, p\}$ and n is arbitrary, with

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gcd $(m_0, d, n) = 1$, see Proposition 2.11. In Section 3 we show that $h_3 \geq 0$ when Γ is generated by $m_0, \dots, m_p, n$ as above with $p = 2$, see Corollaries 3.7, 3.11, 3.17, 3.23, 3.26; in Section 4 we exhibit various examples.

1. Let $A = k[[t^{n_0}, \dots, t^{n_r}]] = k[[\Gamma]]$ where k is a field, Γ is a numerical semigroup minimally generated by $n_0, \dots, n_r$ with $0 < n_0 < \cdots < n_r$ and gcd $(n_0, \dots, n_r) = 1$. Let $\mathbf{N}$ be the set of nonnegative integers, $\xi = \mathbf{N}^{r+1}$, $e_i = (\delta_{ij})_{0 \leq j \leq r}$, $0 \leq i \leq r$, and, for each $\alpha = \sum_{i=0,\dots,r} a_i e_i \in \xi$, let $\partial(\alpha) = \sum a_i n_i$ and $\deg(\alpha) = \sum a_i$. Further, for each $g \in \Gamma$ we put $\xi(g) = \{\alpha \in \xi \mid \partial(\alpha) = g\}$. In $\xi(g)$ we write $\alpha \leq_{\deg} \beta$ if $\deg(\alpha) \leq \deg(\beta)$. Then "$\leq_{\deg}$" is an order on $\xi(g)$ and, since $\xi(g)$ is a finite set, $\xi(g)$ has maximal elements with respect to "$\leq_{\deg}$." All the maximal elements in $\xi(g)$ have the same degree, we shall note by $\max\deg(g)$, see also [**3**, Section 2].

Let $M = (t^{n_0}, \dots, t^{n_r})$, $G = \mathrm{gr}_M(A) = \oplus_{i \geq 0}(M^i/M^{i+1}) = \oplus G_i$. It is known that $G \approx k[\tau_0, \dots, \tau_r]$ where $\tau_i = t^{n_i} \bmod (M^2)$. In [**3**] we gave the following characterization of the Cohen-Macaulayness of G.

Proposition 1.1. *Let G, Γ, be as above. The following conditions are equivalent*:

(1) *G is Cohen-Macaulay*;

(2) *for each $g \in \Gamma$ the following fact holds*:

$$\max\deg(g + cn_0) = \max\deg(g) + c \quad \text{for each } c \geq 1. \tag{1.1}$$

In this paper we show some properties of the Hilbert function $H_G(i) = \dim_k(M^i/M^{i+1})$ when G is *not* Cohen-Macaulay. We recall the Poincaré series $P_G(z) = \sum_{i \geq 0} H_G(i) z^i$ is equal to $h_G(z)/(1-z)$ where $h_G(z) = \sum_{i=0,\dots,s} h_i z^i \in Q[z]$ with $h_i = H_G(i) - H_G(i-1)$ for each $i \geq 1$. Let $i \geq 1$. The set G_i is generated over k by the elements of the set $M_i = \{N = \tau_0^{a_0} \cdots \tau_r^{a_r} \in M^i/M^{i+1} \mid N \neq O_G\}$. One has

Lemma 1.2. *For $i \in \mathbf{N}$, put*

$$Y_i = \{g \in \Gamma \mid \max\deg(g) = i\} \quad \textit{if } i \geq 1, \quad Y_0 = \{0\}.$$

Then for $i \geq 1$, one has $H_G(i) = \mathrm{card}(Y_i)$ and $h_i = \mathrm{card}(Y_i) - \mathrm{card}(Y_{i-1})$.

Proof. Since there is a one-to-one correspondence between Y_i and M_i the result follows from the above definitions. $\square$

From now on, we suppose G is not Cohen-Macaulay. Then, according to Proposition 1.1 and Lemma 1.2 there is some $g \in \Gamma$ that does not satisfy (1.1) and we can express h_i, $i \geq 2$, in terms of particular subsets of Y_i and Y_{i-1}. We give some definitions.

Definition 1.3. Let $g \in \Gamma$. Put

$$\sum = \{g \in \Gamma \mid g \text{ does not satisfy } (1.1)\}$$

and if $g \in \sum$ we put $c_g = \min\{c \geq 1 \mid \max\deg(g + cn_0) > \max\deg(g) + c\}$. Further, let S be the standard basis of Γ with respect to n_0, i.e.,

$$\begin{aligned} S &= \{g \in \Gamma \mid g - n_0 \notin \Gamma\} \text{ and put, for } i \geq 2: \\ S_i &= \{s \in S \mid \max\deg(s) = i\}, \\ A_{i-1} &= \{g \in Y_{i-1} \mid g \notin \Sigma \text{ or } g \in \Sigma \text{ and } c_g > 1\}, \\ B_{i-1} &= \{g \in Y_{i-1} \mid g \in \Sigma \text{ and } c_g = 1\}, \\ C_i &= \{g + n_0 \in Y_i \mid g \in B_j \text{ with } j < i-1\}. \end{aligned}$$

One can immediately see that $Y_{i-1} = A_{i-1} \cup B_{i-1}$, $Y_i = \{g + n_0 \mid g \in A_{i-1}\} \cup C_i \cup S_i$, where there is one-to-one correspondence between A_{i-1} and $\{g + n_0 \mid g \in A_{i-1}\}$) and, according to Lemma 1.2, one has for $i \geq 2 : h_i = \operatorname{card}(C_i) + \operatorname{card}(S_i) - \operatorname{card}(B_{i-1})$. In particular, one can note

Remark 1.4. (1) One has $C_2 = \varnothing$, so that $h_2 = (\operatorname{card} S_2) - (\operatorname{card} B_1)$.
(2) If $Y_1 \cap \sum = \varnothing$, then $C_3 = \varnothing$ and $h_3 = (\operatorname{card} S_3) - (\operatorname{card} B_2)$.

In this section we shall prove that the leading coefficient h_s of the polynomial $h_G(z)$ is positive. We need some results. One can easily prove the following.

Lemma 1.5. *Let $g \in \Gamma$. Then*

(1) *if* $c \in \mathbf{N}$, $c \geq 1$ *is such that* $g + cn_0 \in \sum$, *then* $g + c'n_0 \in \sum$ *for each* c' *such that* $0 \leq c' \leq c - 1$;

(2) *if* $g \in \sum$ *and* c_g *is as in Definition* 1.3, *then for each* i *such that* $1 \leq i \leq c_g$, *one has* $g_i = g + (c_g - i)n_0 \in \sum$ *and* $c_{g_i} = i$.

Further, one has

Lemma 1.6. *Let* $s_1 \in S \cap \sum$. *Then there exists* $c \in N$, $c \geq 1$, *such that* $s_1 + cn_0$ *does not belong to* $\sum$.

Proof. Let $s_1 \in S \cap \sum$. Put $s_2 = s_1 + c_{s_1} n_0$. If $s_2 \notin \sum$, then $c = c_{s_1}$. Otherwise, we repeat the procedure, and we obtain a sequence of elements in $\sum : s_1, \dots, s_i$ such that

$$s_{i+1} = s_i + c_{s_i} n_0 \tag{1.2}$$

and

$$\max\deg(s_{i+1}) \geq \max\deg(s_1) + c_{s_1} + \cdots + c_{s_i} + i.$$

Now we show

$$i < \sum_{j=1\cdots r} (n_j - n_0).$$

Let $s_1 = \sum_{j=1\cdots r} a_j n_j$ be such that $\sum a_j = \max\deg(s_1)$. Further, let $s_{i+1} = \sum_{j=0\cdots r} b_j n_j$ be such that $\sum b_j = \max\deg(s_{i+1})$. Putting $\gamma = c_{s_1} + \cdots + c_{s_i}$, one has

$$b_0 n_0 + \sum_{j=1\cdots r} b_j n_j = s_1 + \gamma n_0 = \sum_{j=1\cdots r} a_j n_j + \gamma n_0,$$

where $b_0 \leq \gamma$ since $s_1 \in S$, so

$$\sum_{j=1\cdots r} b_j \geq \sum_{j=1\cdots r} a_j + i,$$

according to (1.2). Now, by adding $\sum_{j=1\cdots r}(n_0 - a_j)n_j$, one obtains

$$\sum_{j=1\cdots r} (b_j + n_0 - a_j)n_j + b_0 n_0 = \left(\sum_{j=1\cdots r} n_j + \gamma \right) n_0,$$

so

$$\Big(\sum_{j=1\cdots r} n_j + \gamma\Big) n_0 > \Big(\sum_{j=1\cdots r} (b_j + n_0 - a_j) + b_0\Big) n_0,$$

then

$$\begin{aligned}\sum_{j=1\cdots r} n_j + \gamma &> \sum_{j=1\cdots r} (b_j + n_0 - a_j) + b_0 \\ &= \sum_{j=0\cdots r} b_j + r n_0 - \sum_{j=1\cdots r} a_j \\ &= \max\deg(s_{i+1}) + r n_0 - \max\deg(s_1) \\ &\geq \gamma + i + r n_0,\end{aligned}$$

according to (1.2). Then $i < \sum_{j=1\cdots r}(n_j - n_0)$.

It follows that if $k \geq \sum_{j=1\cdots r}(n_j - n_0)$, one has $s_{k+1} \notin \sum$. □

Remark 1.7. Let $g \in \sum$. According to the properties of S and Lemma 1.5, one has $g = s + kn_0$ with $s \in S \cap \sum$, $k \geq 0$. According to Lemma 1.6, $\sum$ is a finite set and, putting $c'_s = \min\{c \mid s + cn_0 \notin \sum\}$ for each $s \in S \cap \sum$, one has $g = s + kn_0$ with $k < c'_s$. Further, $\sum = \{s, s + n_0, \ldots, s + (c'_s - 1)n_0 \mid s \in S \cap \sum\}$.

Now one has

Lemma 1.8. *Let*

$$n = \max\{\max\deg(g + c_g n_0) \mid g \in \Sigma\}$$

and

$$m = \max\{\max\deg(s) \mid s \in S\}.$$

The following facts hold:

(1) *if* $i \geq n - 2$, *then* $\{g \in Y_i \cap \sum \mid c_g > 1\} = \varnothing$;

(2) *if* $j \geq n - 1$, *then* $Y_i \cap \sum = \varnothing$;

(3) $\{g \in \sum \mid c_g = 1$ *and* $g + n_0 \in Y_n\} \neq \varnothing$;

(4) *if* $p > n$, *then* $C_p = \varnothing$ *and* $A_{p-1} = \{g \in Y_{p-1} \mid g \notin \sum\}$.

Proof. (1) Let $i \geq n-2$. If there is a $g \in Y_i \cap \sum$ such that $c_g > 1$, then $\max\deg(g+c_g n_0) > i+c_g \geq i+2$, so that $\max\deg(g+c_g n_0) > n$. This is a contradiction.

(2) Let $j \geq n-1$. If there is a $g \in Y_j \cap \sum$, then $\max\deg(g+c_g n_0) > j+c_g \geq j+1 \geq n$. This is a contradiction.

(3) According to the definition of n, there exists $g' \in \sum$ such that $\max\deg(g'+c_{g'}n_0) = n$ (and $\max\deg\ (g'+c_{g'}n_0) > \max\deg(g')+c_{g'}$). Then, putting $g = g' + (c_{g'}-1)n_0$, one has (Lemma 1.5): $g \in \sum$ and $c_g = 1$. Further, $\max\deg(g+n_0) = \max\deg(g'+c_{g'}n_0) = n$, so that $g + n_0 \in Y_n$.

(4) Let $p > n$; then $\{g \in \sum \mid c_g = 1 \text{ and } g+n_0 \in Y_p\} = \varnothing$, so that $C_p = \varnothing$.

Further, according to (1), one has $\{g \in Y_{p-1} \cap \sum \mid c_g > 1\} = \varnothing$, so $A_{p-1} = \{g \in Y_{p-1} \mid g \notin \sum\}$. □

Proposition 1.9. *Let n be as in Lemma* 1.8*. Then $H_G(n-1) < H_G(n)$.*

Proof. According to Lemma 1.8, one has $Y_{n-1} \cap \sum = \varnothing$. Then one can define $\varphi : Y_{n-1} \to Y_n$ by $\varphi(g) = g + n_0$ (that is injective). Such φ is not surjective by Lemma 1.8. So $\operatorname{card}(Y_{n-1}) < \operatorname{card}(Y_n)$. □

Proposition 1.10. *Let n, m be as in Lemma* 1.8*. Then*

(a) *if $m \geq n+1$, one has*

$$H_G(q-1) = H_G(q)$$

for each $q \geq m+1$ and $H_G(m-1) < H_G(m)$;

(b) *if $m \leq n$, one has*

$$H_G(q-1) = H_G(q)$$

for each $q \geq n+1$.

Proof. Let $q \geq \max(m+1, n+1)$. Then $Y_{q-1} \cap \sum = \varnothing$ (Lemma 1.8) and $Y_q = \{g + n_0 \mid g \in A_{q-1}\}$, since $C_q = \varnothing$ according to Lemma 1.8

and $S_q = \varnothing$ since $q > m$, so one can define $\varphi : Y_{q-1} \to Y_q$ as $\varphi(g) = g + n_0$ that is one-to-one and onto. Then, according to Lemma 1.2, one has (b) if $m \leq n$, and the first statement of (a) for $q \geq m+1$ when $m \geq n+1$. Further, when $m \geq n+1$, one has $Y_{m-1} \cap \sum = \varnothing$, $A_{m-1} = \{g \in Y_{m-1} \mid g \notin \sum\}$ (Lemma 1.8) so $A_{m-1} = Y_{m-1}$; besides $Y_m = \{g + n_0 \mid g \in A_{m-1}\} \cup S_m$ (Lemma 1.8) where $S_m \neq \varnothing$ according to the definition of m. Then $\operatorname{card}(Y_m) > \operatorname{card}(Y_{m-1})$, so $H_G(m) > H_G(m-1)$, see Lemma 1.2. □

From Propositions 1.9 and 1.10, one has

Corollary 1.11. *Let m, n be as in Lemma* 1.8. *The polynomial $\sum_{i=0,\dots,s} h_i z^i$ has degree $s = \max(m,n)$ and $h_s > 0$.*

2. In this section we consider the coefficient h_2 of the polynomial $\sum_{i=0,\dots,s} h_i z^i$, and we show $h_2 \geq 0$ when Γ is minimally generated by an "almost arithmetic sequence." First we recall some known examples where $h_2 < 0$, e.g.,

$$A = [[t^{30}, t^{35}, t^{42}, t^{47}, t^{148}, t^{153}, t^{157}, t^{169}, t^{181}, t^{193}]]$$

defined by Herzog and Waldi in [**2**] and

$$A = [[t^{15}, t^{21}, t^{23}, t^{47}, t^{48}, t^{49}, t^{50}, t^{52}, t^{54}, t^{56}, t^{58}]]$$

defined by Eakin and Sathaye in [**1**]. By Remark 1.4, the following Lemma 2.1 and technical calculations, we have seen that the minimum value of n_0 such that $h_2 = H_G(2) - H_G(1)$ may be negative is $n_0 = 13$. We have found the following example with $h_2 < 0$, where embedding dimension or multiplicity are less than the ones in the previous examples:

$$A = [[t^{13}, t^{19}, t^{24}, t^{44}, t^{49}, t^{54}, t^{55}, t^{59}, t^{60}, t^{66}]]$$

where

$$S_2 = \{2n_1 = 38, n_1 + n_2 = 43, 2n_2 = 48\},$$
$$B_1 = \{n_3, n_4, n_5, n_7\} = \{44, 49, 54, 59\},$$

so one has $h_2 = -1$ according to Remark 1.4. Further, one can see that

$$n_3 + n_0 = 3n_1, \quad n_4 + n_0 = 2n_1 + n_2,$$
$$n_5 + n_0 = n_1 + 2n_2, \quad n_7 + n_0 = 3n_2.$$

The following Lemma 2.1 and Corollary 2.2 show that certain elements of Γ are in S under suitable assumptions.

Lemma 2.1. *Let $i \geq 1$. Let $g = \sum a_k n_k$ be such that $g \in B_i$ with $i = \sum a_k$, and $g + n_0 = \sum b_k n_k$ with $j = \sum b_k > i + 1$. Then the elements of the type $g' = \sum b'_k n_k$ with $b'_k \leq b_k$ for each k and $i + 1 \leq \sum b'_k \leq j$ belong to $S_a \cup C_a$ where $a = \max deg\,(g')$.*

Proof. Let g' be as above. If $\sum b'_k = j$, i.e., if $b'_k = b_k$ for each k, then obviously $g' \in C_a$ where $a = \max\deg\,(g')$. Now we suppose $\sum b'_k \leq j - 1$. If $g' \notin S_a$, then $g' = n_0 + h$ with $h \in \Gamma$, then $g + n_0 = g' + \sum(b_k - b'_k)n_k = h + n_0 + \sum(b_k - b'_k)n_k$, so $g = h + \sum(b_k - b'_k)n_k$.

It follows that $i = \max\deg\,(g) \geq \max\deg\,(h) + \sum(b_k - b'_k) = \max\deg\,(h) + j - \sum b'_k$, then $\max\deg\,(h) \leq i - 1$. On the other hand, $h + n_0 = g' = \sum b'_k n_k$ so that $\max\deg\,(h + n_0) \geq \sum b'_k \geq i + 1$. Putting $a = \max\deg\,(g')$, it follows that $h \in \sum \cap Y_q$ with $q < a - 1$. Then $g' \in C_a$. $\square$

Corollary 2.2. *Let $i \geq 1$, and let g, g', a be as in Lemma 2.1. If $\sum \cap Y_q = \varnothing$ for $q < a - 1$, then $g' \in S_a$.*

Proof. If $\sum \cap Y_q = \varnothing$ for $q < a - 1$, then $C_a = \varnothing$, so the result follows from Lemma 2.1. $\square$

When, in particular, Γ is minimally generated by an "almost arithmetic sequence," one can prove other properties of the elements of $\sum$ and S. So, let $\Gamma = \langle m_0, \dots, m_p, n\rangle$ where $m_0, \dots, m_p$ are an arithmetic sequence, i.e., $m_i = m_0 + id$ for each $i \in \{1, \dots, p\}$ and n is arbitrary; suppose $\gcd\,(m_0, d, n) = 1$. First we show some properties of the elements of $\sum$ and S. We consider the two cases $n > m_0$ and $n < m_0$.

Case $n > m_0$. We need the following lemma.

Lemma 2.3. *Let* $\Gamma = \langle m_0, \dots, m_p, n\rangle$ *where* $m_i = m_0 + id$ *for* $1 \le i \le p$ *and* $n > m_0$.

Let $g = \sum_{k=0,\dots,p} a_k m_k \in \sum$ *with* $\sum a_k = \max\deg\ (g)$, *and suppose* $g + c_g m_0 = \sum_{k=0,\dots,p} b_k m_k + bn$, *where* $\sum b_k + b \ge \sum a_k + c_g + 1$. *If* $\sum b_k - \sum a_k - c_g \ge 0$, *one has* $\sum k b_k \ge p(\sum a_k - 1)$.

Proof. Put $a' = \sum a_k$, $b' = \sum b_k$. According to the assumptions, one has

$$(a' + c_g)m_0 + \left(\Sigma k a_k\right)d = b' m_0 + \left(\Sigma k b_k\right)d + bn.$$

We want to prove $\sum k b_k \ge p(a' - 1)$. Otherwise, $\sum k b_k = cp + r$ with $c \le a' - 2$, $0 \le r < p$, so

$$\left(\Sigma k a_k\right)d = (b' - a' - c_g)m_0 + cpd + rd + bn,$$

where c, r are as above and $b' - a' - c_g \ge 0$ by the assumption. By adding $a' m_0$, one gets

$$g = (b' - a' - c_g - c - 1 + a')m_0 + cm_p + m_r + bn,$$

then $\max\deg\,(g) \ge b' - a' - c_g + a' + b$. So, if $b' - a' - c_g \ge 1$, one has $\max\deg\,(g) \ge 1 + a' + b' \ge a' + 1$; if $b' - a' - c_g = 0$, one has $b \ge 1$, so $\max\deg\,(g) \ge a' + 1$. In any case, there is a contradiction. □

Through Lemma 2.3, one can prove

Corollary 2.4. *Let* Γ *be as in Lemma* 2.3. *Let* $g = \sum a_k m_k \in \sum$ *with* $\sum a_k = \max\deg\,(g)$, *and let* $g + c_g m_0 = \sum b_k m_k + bn$ *with* $\sum b_k + b \ge \sum a_k + c_g + 1$. *Then* $\sum b_k \le \sum a_k + c_g - 1$ *and* $b \ge 2$.

Proof. Put $a = \sum a_k$ and $b' = \sum b_k$. According to the assumption

$$b' + b \ge a + c_g + 1 \tag{2.1}$$

it is enough to show that $b' \leq a + c_g - 1$.

If $b' \geq a + c_g$, one has

$$\left(\Sigma k a_k\right) d = (b' - a - c_g) m_0 + \left(\Sigma k b_k\right) d + bn. \tag{2.2}$$

Now, if $b' - a - c_g = 0$, one has $b \geq 1$ according to (2.1), otherwise $b' - a - c_g \geq 1$. In both cases, one has $\sum k a_k > \sum k b_k$ by (2.2). This inequality and Lemma 2.3 imply $p(a-1) \leq \sum k b_k < \sum k a_k \leq pa$, so $0 < \sum k a_k - \sum k b_k \leq p$. Putting $q = \sum k a_k - \sum k b_k$ one has, according to the assumption, $m_q = (b' - a - c_g + 1) m_0 + bn$, a contradiction. □

Corollary 2.5. *Let Γ be as above, and let $g = \sum a_k m_k \in B_a$ with $a = \sum a_k$, $g + m_0 = \sum b_k m_k + bn$ with $\sum b_k + b \geq a + 2$. Then $\sum b_k \leq a - 1$ and $b \geq 3$.*

Proof. Let $b' = \sum b_k$. According to Corollary 2.4, one has $b' \leq a$, $b \geq 2$. So it is enough to show that $b' \neq a$. Suppose $b' = a$. Then, by the assumption, one has

$$m_0 + \left(\Sigma k a_k\right) d = \left(\Sigma k b_k\right) d + bn, \tag{2.3}$$

where $n > m_0$, $b \geq 2$, so that $\sum k a_k - \sum k b_k > 0$.

Further, we note that if $b' = a$, one has $\sum k b_k \neq 0$, otherwise $\sum b_k m_k = b' m_0 = a m_0$, then $g = (a-1) m_0 + bn$, that implies $g \notin Y_a$ since $b \geq 2$. So

$$0 < \Sigma k a_k - \Sigma k b_k < \Sigma k a_k \leq ap,$$

then

$$\Sigma k a_k - \Sigma k b_k = cp + r \quad \text{with } c \leq a - 1, \quad 0 \leq r < p.$$

If $c = 0$, one has $0 < \sum k a_k - \sum k b_k < p$; putting $q = \sum k a_k - \sum k b_k$, according to (2.3), one has $m_q = bn$, a contradiction. Now we have a contradiction also when $c \geq 1$. By the previous facts, one has

$$ap \geq \Sigma k a_k = \Sigma k b_k + cp + r \geq \Sigma k b_k + cp,$$

then

$$\Sigma k b_k \leq (a-c)p,$$

so

$$\Sigma k b_k = \delta p + r' \quad \text{with} \quad \delta < a-c, \quad 1 \leq r' \leq p.$$

Then $g = (a-c-\delta-1)m_0 + \delta m_p + m_{r'} + (c-1)m_0 + bn$ with $b \geq 2$, so $\max\deg(g) \geq a+1$, a contradiction. □

Corollary 2.6. *Let Γ be as in Lemma 2.3. If $\{m_1, \dots, m_p\} \cap B_1 \neq \varnothing$, then there is only one $i' \in \{1, \dots, p\}$ such that $m_{i'} \in B_1$ and one has $m_{i'} + m_0 = bn$ with $b = \max\deg(bn) \geq 3$.*

Proof. If $m_i \in B_1$, according to Corollary 2.5 with $a = 1$, one has necessarily $m_i + m_0 = bn$ with $b \geq 3$ and there is only one i, say i', satisfying the previous equation since Γ is minimally generated; further, according to the above uniqueness, one has $b = \max\deg(bn)$. □

Now we show some properties of S.

Lemma 2.7. *Let Γ be as in Lemma 2.3. Let $g = \sum a_k m_k \in B_a$ with $a = \sum a_k$. Then $g + m_0 = \sum b_k m_k + bn$ with $\sum b_k + b \geq a+2$, $b \geq 3$, and, further, if*

(i) *$b \geq a$ and*

(ii) *$\{h \in \sum \mid \max\deg(h) \leq a-2\} = \varnothing$, one has*

$$\Sigma b_k m_k + (b-1)n \in S.$$

Proof. Let g be as above. Since $c_g = 1$, one has

$$g + m_0 = \Sigma b_k m_k + bn \quad \text{with} \sum b_k + b \geq a+2 \quad \text{and } b \geq 3 \tag{2.4}$$

according to Corollary 2.5. Now put $a = \sum a_k$ and suppose that (i) and (ii) hold.

If $\sum b_k m_k + (b-1)n \notin S$, then $\sum b_k m_k + (b-1)n = m_0 + h$ with $h \in \Gamma$, so that $g = h + n$, that implies $\max\deg(h) \leq a-1$. Now let

$h = \sum c_k m_k + c'n$ be such that $\sum c_k + c' = \max\deg(h)$; then putting $c = \sum c_k$, one has $c + c' \le a - 1$.

Step 1. First we show that $c' = 0$. One has, according to (2.4):

$$\Sigma c_k m_k + (c'+1)n + m_0 = \Sigma b_k m_k + bn$$

where $\sum b_k + b \ge a + 2$. Now, if $c' + 1 \le b$, one has $\sum c_k m_k + m_0 = \sum b_k m_k + (b - c' - 1)n$ where $c = \max\deg(\sum c_k m_k)$ (since $c + c' = \max\deg(h)$) and

$$\Sigma b_k + b - c' - 1 \ge a + 2 - c' - 1 = a - c' + 1 \ge c + 2,$$

so $\sum c_k m_k \in \sum \cap Y_c$ where $c \le a - c' - 1$; this implies $c' = 0$ according to assumption (ii).

On the other hand, $c' + 1$ cannot be greater than b; otherwise, $\sum c_k m_k + (c' + 1 - b)n + m_0 = \sum b_k m_k$, where $b \ge 2$, $c + c' + 1 - b = \max\deg(\sum c_k m_k + (c' + 1 - b)n)$, since $\max\deg\ (h) = c + c'$, and

$$\Sigma b_k \ge a + 2 - b \ge c + c' + 1 + 2 - b = (c + c' + 1 - b) + 2,$$

then

$$\Sigma c_k m_k + (c' + 1 - b)n \in \Sigma \cap Y_{c+c'+1-b},$$

where $c + c' + 1 - b \le a - b \le a - 3$, since $b \ge 3$, that contradicts (ii).

According to Step 1, let $h = \sum c_k m_k$ with $c = \sum c_k = \max\deg(h)$. Then $c \le a - 1$ and $\sum a_k m_k = g = h + n = \sum c_k m_k + n$, so

$$(a - c)m_0 + \left(\Sigma k a_k\right)d = \left(\Sigma k c_k\right)d + n, \quad \text{where } a - c \ge 1. \tag{2.5}$$

Step 2. In (2.5) one has necessarily that $\sum k a_k < \sum k c_k$. In fact, if $\sum k a_k \ge \sum k c_k$, putting $r = \sum k a_k - \sum k c_k \ge 0$, one has, by (2.5),

$$(a - c)m_0 + rd = n.$$

So, if $r \le (a - c)p$, one has $n \in \langle m_0, \dots, m_p\rangle$, a contradiction.

If $r > (a-c)p$, then $n > (a-c)m_0 + (a-c)pd = (a-c)m_p \geq m_p$ so that $n > m_p$. So, if $\sum b_k = 0$, one has $g + m_0 = bn \geq (a+2)n > am_p + m_0 \geq g + m_0$; if $\sum b_k \geq 1$, one has $g + m_0 = \sum b_k m_k + bn \geq m_0 + an > m_0 + am_p \geq m_0 + g$, contradictions.

Step 3. Finally, we get a contradiction.

In (2.5), one has $(a-c)m_0 = (\sum kc_k - \sum ka_k)d + n$, where $a - c \geq 1$, according to Step 2. Now, if $a - c \geq 2$, one has $c = \max\deg(h) \leq a-2$, then $h \in \sum \cap Y_c$ with $c \leq a-2$ (in fact, as we have supposed, $h + m_0 = \sum b_k m_k + (b-1)n$ with $\sum b_k + b - 1 \geq a + 2 - 1 = a + 1 \geq 2 + c + 1 = c + 3$ that contradicts (ii); finally, if $a - c = 1$, one has $m_0 > n$, a contradiction. $\square$

From Lemma (2.7), one has immediately

Corollary 2.8. *Let* Γ *be as in Lemma* (2.3). *If* $g = \sum a_k m_k \in B_2$ *and* $g + m_0 = \sum b_k m_k + bn$ *with* $\sum b_k + b \geq 4$, *then* $b \geq 3$ *and* $\sum b_k m_k + (b-1)n \in S$.

Case $n < m_0$.

Lemma 2.9. *Let* $\Gamma = \langle m_0, \dots, m_p, n\rangle$ *where* $m_i = m_0 + id$ *for* $1 \leq i \leq p$, *and let* $n < m_0$. *Let* $g = \sum a_k m_k \in \sum$ *with* $\sum a_k = \max deg(g)$. *Then there exist* $b, c, r \in \mathbf{N}$ *such that*

$$g + c_g n = (b - c - 1)m_0 + cm_p + m_r$$

with $b - c - 1 > 0$, $b \geq \sum a_k + c_g + 1$,

$$c \leq \min\left\{\Sigma a_k - 1, \left[\frac{\Sigma k a_k}{p}\right]\right\}$$

and $0 \leq r < p$.

Proof. Let g be as above. Then $g + c_g n = \sum b_k m_k$ with $\sum b_k \geq \sum a_k + c_g + 1$, and one can choose $\sum b_k m_k$ such that $\sum b_k = \max\deg(g + c_g n)$. Put $a = \sum a_k$, $b = \sum b_k$. Then

$$am_0 + \left(\Sigma k a_k\right)d + c_g n = bm_0 + \left(\Sigma k b_k\right)d \tag{2.6}$$

where $b - a - c_g \geq 1$ so, since $n < m_0$, $(b - a - c_g)m_0 + (\sum kb_k)d < (\sum ka_k)d$, which implies $\sum ka_k > \sum kb_k$. So, if $\sum kb_k = cp + r$, $\sum ka_k = c'p + r'$ (where $0 \leq r < p$, $0 \leq r' < p$), one has

$$c \leq c' = \left[\frac{\sum ka_k}{p}\right]$$

and $c \leq a - 1$, since $cp \leq \sum kb_k < \sum ka_k \leq pa$. Then, according to (2.6),

$$g + c_g n = bm_0 + cpd + rd = (b - c - 1)m_0 + cm_p + m_r,$$

where

$$\begin{aligned} &b \geq a + c_g + 1 > a + 1 > c + 1, \\ &c \leq \min\left\{a - 1, \left[\frac{\Sigma ka_k}{p}\right]\right\}, \quad 0 \leq r < p. \qquad \square \end{aligned}$$

Corollary 2.10. *Let Γ be as in Lemma 2.9. If $B_1 \neq \varnothing$, let $g = m_i \in B_1$, $i \in \{1, \dots, p\}$. Then*

(a) *if $i = \min\{j \in \{1, \dots, p\} \mid m_j \in B_1\}$, one has $m_i + n = bm_0$ with $b \geq 3$ and $b = \max\deg(bm_0)$;*

(b) *if $i > \min\{j \in \{1, \dots, p\} \mid m_j \in B_1\}$, then $m_i + n = m_r + b'm_0$ with $0 < r < p$, $r \neq i$, $b' \geq 2$, $1 + b' = \max\deg(m_r + b'm_0)$.*

Proof. Let $m_i \in B_1$. Then, according to Lemma 2.9,

$$m_i + n = (b - 1)m_0 + m_r \quad \text{with } b \geq 3, \quad 0 \leq r < p, \tag{2.7}$$

and one can show that $b = \max\deg((b-1)m_0 + m_r)$. Then $m_0 + id + n = bm_0 + rd$, i.e.,

$$id + n = (b - 1)m_0 + rd, \tag{2.8}$$

so $(b - 1)m_0 + rd < id + m_0$, since $n < m_0$, where $b - 1 \geq 2$.

Then $(b - 2)m_0 + rd < id$ that implies $r < i$, so, according to (2.8), $(i - r)d + n = (b - 1)m_0$ where $0 < i - r \leq p$. Then $m_{i-r} + n = bm_0$ with $b \geq 3$, so $m_{i-r} \in B_1$.

Now, if $i = \min\{j \in \{1, \dots, p\} \mid m_j \in B_1\}$ necessarily one has $r = 0$, so, according to (2.7), $m_i + n = bm_0$ with $b \geq 3$. On the other hand, if $i > \min\{j \in \{1, \dots, p\} \mid m_j \in B_1\}$, one has in (2.7): $r > 0$ and $r \neq i$. Then $m_i + n = b'm_0 + m_r$ with $b' \geq 2$, $0 < r < p$, $r \neq i$. □

Now one can show

Proposition 2.11. *Let $\Gamma = \langle m_0, \dots, m_p, n\rangle$ where $m_i = m_0 + id$, $1 \leq i \leq p$. Then $h_2 \geq 0$.*

Proof. According to Remark 1.4, it is enough to show that $\operatorname{card}(S_2) \geq \operatorname{card}(B_1)$, where

$$S_2 = \{s \in S \mid \max\deg(s) = 2\},$$
$$B_1 = \{g \in Y_1 \cap \Sigma \mid c_g = 1\}.$$

We consider the two cases $n > m_0$ and $n < m_0$.

Case 1. $n > m_0$. In view of Corollary 2.6, B_1 can be one of the following sets: $\{m_{i'}\}$ or $\{m_{i'}, n\}$ or $\{n\}$ or $\varnothing$ where $m_{i'}$ is as in Corollary 2.6. If $m_{i'} \in B_1$, then $m_{i'} + m_0 = bn$ with $b \geq 3$, $b = \max\deg\,(bn)$, see Corollary 2.6. So the element $2n \in Y_2$ and further $2n$ belongs to S_2 according to Corollary 2.2. If $n \in B_1$, one has $n + m_0 = \sum b_k m_k$ with $\sum b_k \geq 3$. So, by Corollary 2.2, each element of the type $\sum b'_k m_k$ with $b'_k \leq b_k$ and $\sum b'_k = 2$ belongs to S_2. So, in any case one has $\operatorname{card}(S_2) \geq \operatorname{card}(B_1)$.

Case 2. $n < m_0$. In this case n, m_0 cannot be in B_1, so $B_1 \subset \{m_1, \dots, m_p\}$. Let $B_1 = \{m_{i_0}, \dots, m_{i_k}\}$ with $i_0 < i_1 < \cdots < i_k$. Then, according to Corollary 2.10,

$$\begin{aligned}
&m_{i_0} + n = b_0 m_0,\\
&m_{i_1} + n = m_{j_1} + b_1 m_0,\\
&\vdots\\
&m_{i_k} + n = m_{j_k} + b_k m_0
\end{aligned}$$

with

$$b_0 = \max\deg(b_0 m_0) \geq 3$$

and for each $q \in \{1, \ldots, k\}$, one has

$$1 \leq j_q < p, \quad j_q \neq i_q, \quad 1 + b_q = \max\deg\ (m_{j_q} + b_q m_0) \geq 3.$$

So, $2m_0, m_{j_1} + m_0, \ldots, m_{j_k} + m_0$ belong to Y_2. Further, one can easily note that $m_{j_q} \neq m_{j_{q'}}$ when $m_{i_q} < m_{i_{q'}}$; otherwise, $b_q < b_{q'}$ and $m_{i_{q'}} = m_{i_q} + (b_{q'} - b_q)m_0)$. So the elements $m_{i_1}, \ldots, m_{j_k}$ are all distinct. Then, according to Corollary (2.2), the set S_2 contains $2m_0, m_{j_1} + m_0, \ldots, m_{j_k} + m_0$. Then $\operatorname{card}(S_2) \geq \operatorname{card}(B_1)$. □

3. In this section we show $h_3 \geq 0$ when $\Gamma = \langle m_0, m_1, m_2, n\rangle$ where m_0, m_1, m_2 are an arithmetic sequence. We recall that, according to Section 1, one has

$$h_3 = \operatorname{card}(S_3) + \operatorname{card}(C_3) - \operatorname{card}(B_2)$$

and, in particular,

$$h_3 = \operatorname{card}(S_3) - \operatorname{card}(B_2)$$

when $B_1 = \varnothing$, see Remark 1.4. Throughout this section Γ will be minimally generated by m_0, m_1, m_2, n, with $m_i = m_0 + id$ for $i \in \{1, 2\}$.

We start with the case $m_0 < n$. In this situation $m_0 \notin \sum$; further, we can show

Lemma 3.1. *Let $m_0 < n$. Only one of the following situations can happen*:

$$m_1 + m_0 = bn \quad \text{with } b = \max\deg(bn) \geq 3 \tag{3.1}$$

or

$$m_2 + m_0 = cn \quad \textit{with } c = \max\deg(cn) \geq 3 \tag{3.2}$$

or

(3.3)

$$n + m_0 = m_1 + a_2 m_2 \quad \textit{where } a_2 \geq 2, 1 + a_2 = \max\deg\ (m_1 + a_2 m_2)$$

or

$$n + m_0 = a_2 m_2 \quad \textit{where } a_2 = \max\deg(a_2 m_2) \geq 3 \tag{3.4}$$

or

$$B_1 = \varnothing. \tag{3.5}$$

Further, when (3.1) *or* (3.2) *holds, one has* $n < m_1$*; when* (3.3) *or* (3.4) *holds, then* $n > m_2$*.*

Proof. When $B_1 \neq \varnothing$ and $\{m_1, m_2\} \cap B_1 \neq \varnothing$, then (3.1) or (3.2) holds, according to Corollary 2.6. In each of these cases, one can see that $n < m_1$. Further, when $n \in B_1$, one has $n + m_0 = a_1 m_1 + a_2 m_2$ with $a_1 + a_2 \geq 3$. This implies that $a_1 \leq 1$, since $2m_1 = m_0 + m_2$, so $a_2 \geq 2$ and one has (3.3) or (3.4). In these cases it is clear that $n > m_2$. So, if (3.1) or (3.2) holds, one has $n < m_1$, while $n > m_2$ if (3.3) or (3.4) holds. Then the result follows. $\square$

Lemma 3.2. *Let* $m_0 < n$*. If* (3.1) *or* (3.2) *holds, then* $2m_0, m_0 + n, 2n$ *belong to* $Y_2 - B_2$*.*

Proof. Suppose (3.1) or (3.2) holds. Since $m_0 < n < m_1 < m_2$, according to the assumption and Lemma 3.1, one obviously has $2m_0, m_0 + n \in Y_2 - B_2$. Further, $2n \in Y_2$, since $b = \max\deg\ (bn)$ (respectively, $c = \max\deg(cn)$). Finally it is clear that $2n \notin B_2$. $\square$

Lemma 3.3. *Let* $m_0 < n$*. If* (3.1) *holds, then*

(1) *if* $n + m_2$*, respectively* $2m_2$*,* $\in Y_2$*, then* $n + m_2$*, respectively* $2m_2$*,* $\notin B_2$*;*

(2) $B_2 \subset \{m_0 + m_2 = 2m_1, n + m_1, m_1 + m_2\}$.

Proof. (1) Suppose (3.1) holds. If $n + m_2 \in Y_2$, then $n + m_2 \notin B_2$; otherwise $n + m_2 + m_0 = am_1$ with $a \geq 4$, since (3.1) implies m_2, $n \notin B_1$ by Lemma 3.1. It follows that $n \in \langle m_1 \rangle$. If $2m_2 \in Y_2$, then $2m_2 \notin B_2$; otherwise, $2m_2 + m_0 = a_0 n + a_1 m_1$ with $a_0 + a_1 \geq 4$, since (3.1) implies $m_2 \notin B_1$ by Lemma 3.1, i.e., $m_2 + 2m_1 = a_0 n + a_1 m_1$, so that $a_1 \leq 1$. According to (3.1), if $a_1 = 1$, one obtains $m_2 \in \langle m_0, n\rangle$; if $a_1 = 0$, then $2m_2 = m_1 + (a_0 - b)n$ with $(a_0 - b) \geq 2$, since $n < m_i$ for $i = 1, 2$, so $2m_2 \notin Y_2$.

(2) follows immediately from Lemmas 3.2, 3.3 and the fact that $m_1 + m_0 \in Y_b$ for $b \geq 3$ according to (3.1). □

Lemma 3.4. *Let $m_0 < n$. If* (3.1) *holds, one has*

(i) *if $n + m_1 \in Y_2$,then $3n \in S_3 \cup C_3$;*

(ii) *if $2m_1 \in Y_2$, then $m_1 + 2n \in S_3 \cup C_3$;*

(iii) *if $m_1 + m_2 \in Y_2$, one has $m_2 + 2n \in S_3 \cup C_3$.*

Proof. By (3.1), one has $m_1+n+m_0 = (b+1)n$, $2m_1+m_0 = m_1+bn$, $m_1+m_2+m_0 = m_2+bn$, where $b \geq 3$. So, according to Lemma 2.1, in order to obtain (i), respectively (ii), (iii), it is enough to show that $3n$, respectively m_1+2n, m_2+2n, $\in Y_3$. One can easily prove it, according to the assumptions. □

Lemma 3.5. *Let $m_0 < n$. If* (3.2) *holds, then*

(1) *$m_0 + m_1 \notin B_2$;*

(2) *if $n + m_1 \in Y_2$, then $n + m_1 \notin B_2$;*

(3) *$B_2 \subset \{n + m_2, m_1 + m_2, 2m_2\}$.*

Proof. (1) If $m_0 + m_1 \in B_2$, one has $m_1 + 2m_0 = a_1n + a_2m_2$ with $a_1 + a_2 \geq 4$, since $2m_0 \in Y_2$ and (3.2) implies $m_1 \notin B_1$, see Lemma 3.1. First we note that $a_2 \leq 1$; otherwise, $m_0 = a_1n + 3d + \cdots$, so a_1 must be zero, since $m_0 < n$, then $a_2 \geq 4$, that implies $0 \geq m_0 + 7d$, a contradiction. Then $m_1 + 2m_0 = a_1n$ or $m_1 + 2m_0 = a_1n + m_2$, with $a_1 \geq 4$, respectively, $a_1 \geq 3$. In any case we have contradictions, according to (3.2), respectively, since $m_0 < n$, $m_1 < m_2$.

(2) Suppose $n + m_1 \in Y_2$. One has $n + m_1 \notin B_2$; otherwise, $n + m_1 + m_0 = a_2m_2$ with $a_2 \geq 4$ (since (3.2) implies $m_1, n \notin B_1$, see Lemma 3.1) a contradiction.

(3) follows immediately from Lemmas 3.2, 3.5 (1), (2) and the fact that $m_2 + m_0 = 2m_1 \in Y_c$ with $c \geq 3$, according to (3.2). □

Lemma 3.6. *Let $m_0 < n$. If (3.2) holds, one has*

(i) *if $n + m_2 \in Y_2$, then $3n \in S_3 \cup C_3$;*

(ii) *if $m_1 + m_2 \in Y_2$, then $m_1 + 2n \in S_3 \cup C_3$;*

(iii) *if $2m_2 \in Y_2$, one has $m_2 + 2n \in S_3 \cup C_3$.*

Proof. It is completely similar to the proof of Lemma 3.4. □

Corollary 3.7. *Let $m_0 < n$. If (3.1) or (3.2) holds, then $h_3 \geq 0$.*

Lemma 3.8. *Let $m_0 < n$. If (3.3) or (3.4) holds, then*

(1) $2m_0$, $m_0 + m_1$, $2m_2$ belong to $Y_2 - B_2$;

(2) *if $2m_1$, respectively m_1+m_2, belongs to Y_2, then $2m_1$, respectively $m_1 + m_2$, does not belong to B_2.*

Proof. (1) Suppose (3.3) or (3.4). Since $n > m_2$, see Lemma 3.1, one has $2m_0$, $m_0 + m_1 \in Y_2 - B_2$. Further, $2m_2 \in Y_2$, since $1 + a_2 = \max\deg(m_1 + a_2m_2)$, respectively, $a_2 = \max\deg(a_2m_2)$. Finally, $2m_2 \notin B_2$, otherwise $2m_2 + m_0 = b_1m_1 + b_2n$ with $b_1 + b_2 \geq 4$ (since $m_2 \notin B_1$ when (3.3) or (3.4) holds, see Lemma 3.1), then $b_2 \leq 1$, $b_1 \geq 3$ (since $n > m_2$) with $2m_1 = m_0 + m_2$, a contradiction.

(2) Let $2m_1 \in Y_2$. If $2m_1 \in B_2$, then $2m_1 + m_0 = b_2m_2 + b_3n$ with $b_2 + b_3 \geq 4$ (since $m_0 + m_1 \in Y_2$, see (1)), a contradiction. Let $m_1 + m_2 \in Y_2$. Then $m_1 + m_2 \notin B_2$; otherwise, $m_1 + m_2 + m_0 = bn$ with $b \geq 4$ (since m_1 and m_2 do not belong to B_1 when (3.3) or (3.4) holds, see Lemma 3.1), a contradiction, since $n > m_2$. □

Corollary 3.9. *Let $m_0 < n$.*

(1) *If (3.3) holds, one has $B_2 \subset \{m_2 + n, 2n\}$;*

(2) *If (3.4) holds, then $B_2 \subset \{m_1 + n, m_2 + n, 2n\}$.*

Proof. First we note that $m_0 + n \in Y_q$ with $q \geq 3$ when (3.3) or (3.4) holds. Further, one can show that $\max\deg(m_1 + n) \geq 3$ when (3.3) holds. So the result follows from Lemma 3.8. □

Lemma 3.10. *Let $m_0 < n$, and suppose that (3.3) or (3.4) holds.*

(i) *if $2n \in Y_2$, then $n + 2m_2 \in S_3 \cup C_3$;*

(ii) *if $m_1 + n \in Y_2$, then $m_1 + 2m_2 \in S_3 \cup C_3$;*

(iii) *if $m_2 + n \in Y_2$, then $3m_2 \in S_3 \cup C_3$.*

Proof. It is similar to the proof of Lemma 3.4. As regards (ii), it is useful to note that when (3.4) holds, if $m_1 + n \in Y_2$, then also $m_1 + m_2 \in Y_2$. $\square$

Corollary 3.11. *Let $m_0 < n$. If (3.3) or (3.4) holds, one has $h_3 \geq 0$.*

Lemma 3.12. *Let $m_0 < n$. Suppose (3.5) holds. If B_2 contains some elements of the type $m_i + m_j$, $i, j \in \{0, 1, 2\}$, then only one of the following facts holds*:

$$\begin{gathered} 2m_2 + m_0 = m_1 + bn \quad \text{with } 2m_2 \in Y_2, \\ 1 + b = \max\deg\,(m_1 + bn) \geq 4 \end{gathered} \tag{3.6}$$

or

$$\begin{gathered} (m_0 + m_1) + m_0 = c_{01}n \quad \text{with } m_0 + m_1 \in Y_2, \\ c_{01} = \max\deg\,(c_{01}n) \geq 4 \end{gathered} \tag{3.7}$$

or

$$\begin{gathered} (m_0 + m_2) + m_0 = c_{02}n \quad \text{with } m_0 + m_2 \in Y_2, \\ c_{02} = \max\deg\,(c_{02}n) \geq 4 \end{gathered} \tag{3.8}$$

or

$$\begin{gathered} (m_1 + m_2) + m_0 = c_{12}n \quad \text{with } m_1 + m_2 \in Y_2, \\ c_{12} = \max\deg\,(c_{12}n) \geq 4 \end{gathered} \tag{3.9}$$

or

$$\begin{gathered} 2m_2 + m_0 = c_{22}n \quad \text{with } 2m_2 \in Y_2, \\ c_{22} = \max\deg\,(c_{22}n) \geq 4. \end{gathered} \tag{3.10}$$

Proof. Let $m_i + m_j \in B_2$. Then, by Corollary 2.5, one has

$$m_i + m_j + m_0 = m_a + b_0 n \quad \text{with } b_0 \geq 3 \tag{α_{ij}}$$

or

$$(\beta_{ij}) \qquad m_i + m_j + m_0 = c_{ij}n \quad \text{with } c_{ij} \geq 4$$

where one can see that $1 + b_0 = \max\deg(m_a + b_0 n)$ and $c_{ij} = \max\deg(c_{ij}n)$.

Now, according to the assumptions, it is easy to verify that one cannot have (α_{ij}) when $i = j = 0$, $(i,j) = (0,1)$, $(i,j) = (0,2)$, $(i,j) = (1,1)$, $(i,j) = (1,2)$. So, one can have only the following situation of "type (α_{ij})":

$$(\alpha_{22}) \qquad \begin{gathered} 2m_2 + m_0 = m_1 + bn \quad \text{with } b \geq 3, \\ 1 + b = \max\deg(m_1 + bn). \end{gathered}$$

Further, one cannot have (β_{ij}) for $(i,j) = (0,0)$ and, if each of the situations (β_{ij}) with $(i,j) \neq (0,0)$ holds, then (α_{22}) cannot hold. Besides, each (β_{ij}) with $(i,j) \neq (0,0)$ excludes the other ones; we show the proof only for (β_{01}) and (β_{22}) since the other proofs are very simple.

If one has (β_{01}) and (β_{22}), then $3d = (c_{22} - c_{01})n$ where $c_{22} - c_{01} \notin 3\mathbf{Z}$ since $m_i \notin \langle m_0, n\rangle$ for $i \in \{1,2\}$, then $c_{22} - c_{01} = 3q + r$ with $r \in \{1,2\}$ and $n = 3n'$ with $n' \in \mathbf{N} - \{0\}$. Then $d = (3q + r)n'$, so $3c_{01}n' = 3m_0 + 3qn' + rn'$, that implies $n' = 3n''$, since $r \in \{1,2\}$, and further $n''(3c_{01} - 3q - r) = m_0$. It follows that $n'' = 1$, since $\gcd(n, d, m_0) = 1$, $n = 9$, $m_0 = 4c_{01} - c_{22}$, $m_1 = c_{01} + 2c_{22}$, $m_2 = 5c_{22} - 2c_{01}$. This implies $m_2 = 4m_0 + (c_{22} - 2c_{01})n$, where $c_{22} - 2c_{01} \geq 0$, otherwise $d < (3/2)m_0$; so, according to (β_{01}), one has $2c_{01} < m_0$, so $m_0 \geq 9$ $(= n)$, a contradiction. $\square$

Corollary 3.13. *Let $m_0 < n$. Suppose (3.5) holds. If $2m_2$, respectively $m_0 + m_1$ or $m_0 + m_2$ or $m_1 + m_2$, $\in B_2$ and (3.6) or (3.10), respectively, (3.7) or (3.8) or (3.9), holds, then $3n \in S_3$.*

Proof. According to (3.6) or (3.10), respectively (3.7) or (3.8) or (3.9), one has $3n \in Y_3$; the result then follows from assumption (3.5) and Corollary 2.2. $\square$

Lemma 3.14. *Let $m_0 < n$, and suppose (3.5). Then $m_0 + n \in Y_2$. Further, if B_2 contains elements of the type $m_i + n$, then $i \in \{0,1\}$ and* *the following facts can happen*:

$$(m_0+n)+m_0=a_0m_2 \quad \textit{with } a_0=\max\deg(a_0m_2)\geq 4 \tag{3.11}$$

or

$$\begin{gathered} m_1+n\notin Y_1, \quad (m_0+n)+m_0=m_1+c_2m_2 \\ \textit{with } 1+c_2=\max\deg(m_1+c_2m_2)\geq 4 \end{gathered} \tag{3.12}$$

or

(3.13)

$$\begin{gathered} m_1+n\in Y_2, \quad (m_0+n)+m_0=m_1+c_2m_2 \quad \textit{and} \\ (m_1+n)+m_0=a_1m_2 \quad \textit{with } 1+c_2=\max\deg\ (m_1+c_2m_2)\geq 4 \\ \textit{and } a_1=\max\deg(a_1m_2)\geq 4. \end{gathered}$$

Proof. Since $B_1=\varnothing$, it is clear that $m_0+n\in Y_2$. Suppose $m_i+n\in B_2$. Then one has $m_i+n+m_0=c_1m_1+c_2m_2$ (since $m_i+n\in Y_2$, $m_i\notin B_1$) with $c_1+c_2\geq 4$ and $i\neq 2$, otherwise, $c_2=0$, since $n\notin B_1$, which implies that $c_1\geq 4$ and $n\in\langle m_1\rangle$. Further, since $n\notin B_1$, one has $c_1\leq 1$ if $i=0$, $c_1=0$ if $i=1$. So we have to consider only the following cases:

(i) $(m_0+n)+m_0=a_0m_2$ with $a_0\geq 4$,

(ii) $(m_0+n)+m_0=m_1+c_2m_2$ with $1+c_2\geq 4$,

(iii) $(m_1+n)+m_0=a_1m_2$ with $a_1\geq 4$,

where $a_0=\max\deg(a_0m_2)$, $1+c_2=\max\deg\ (m_1+c_2m_2)$ when $m_0+n\in Y_2$, $a_1=\max\deg\ (a_1m_2)$ when $m_1+n\in Y_2$.

Now if (ii) holds, one has (by adding d):

$$(m_1+n)+m_0=(1+c_2)m_2 \quad \text{with } 1+c_2\geq 4;$$

on the other hand, if (iii) holds, one has (by subtracting d)):

$$(m_0+n)+m_0=m_1+(a_1-1)m_2 \quad \text{with } a_1\geq 4.$$

So each of the facts (ii) and (iii) implies the other one.

Further, (i) $\cap$ (ii) $=\varnothing$. $\square$

Corollary 3.15. *Let* $m_0 < n$ *and suppose* (3.5). *If* (3.11) *holds, then* $3m_2 \in S_3$. *If* (3.12) *holds, one has* $m_1 + 2m_2 \in S_3$. *If* (3.13) *holds, then* $3m_2, m_1 + 2m_2 \in S_3$.

Proof. According to (3.11), respectively, (3.12) or (3.13), one has $3m_2$, respectively $m_1 + 2m_2$ or $3m_2$ and $m_1 + 2m_2, \in Y_3$. Then the result follows from the assumption (3.5) and from Corollary 2.2. □

Lemma 3.16. *Let* $m_0 < n$*, and suppose* (3.5). *If* $2n \in B_2$*, then*

$$\begin{gathered} 2n + m_0 = b_1 m_1 + b_2 m_2 \quad \textit{with } 2n \in Y_2, \\ b_1 + b_2 = \max\deg\,(b_1 m_1 + b_2 m_2) \geq 4 \quad \textit{and } b_1 \in \{0, 1\}. \end{gathered} \tag{3.14}$$

Further,

(1) *if* $b_i \neq 0$ *for* $i = 1, 2$*, one has* $m_1 + 2m_2 \in S_3$;

(2) *if* $b_1 = 0$*, then* $3m_2 \in S_3$;

(3) *if* $b_1 = 1$*, then* $m_1 + 2m_2$, $3m_2 \in S_3$.

Proof. Follows from the assumptions and Corollary 2.2. □

Corollary 3.17. *Let* $m_0 < n$ *and suppose* (3.5). *Then* $h_3 \geq 0$.

Proof. First we remark

$$(3.14) \cap (3.11) = (3.14) \cap (3.13) = \varnothing$$

if $b_1 = 0$;

$$(3.14) \cap (3.12) = (3.14) \cap (3.13) = \varnothing$$

if $b_1 = 1$. Moreover, when $2n \notin B_2$, one easily sees that $\operatorname{card}(S_3) \geq \operatorname{card}(B_2)$ according to Corollaries 3.13 and 3.15. □

Now we consider the case $\Gamma = \langle m_0, m_1, m_2, n\rangle$ with $n < m_0$. As regards B_1, one can prove

Lemma 3.18. *Let* Γ *be as above,* $n < m_0$*. Only one of the following situations can happen*:

$$\begin{aligned} &m_1 + n = cm_0, \quad m_2 + n = (c-1)m_0 + m_1 \\ &\textit{with } c = \max\deg(cm_0) = \max\deg\ ((c-1)m_0 + m_1) \geq 3 \end{aligned} \tag{3.15}$$

or

$$m_2 + n = c'm_0 \quad \textit{with } c' = \max\deg(c'm_0) \geq 3 \quad \textit{and } m_1 \notin B_1 \tag{3.16}$$

or

$$B_1 = \varnothing. \tag{3.17}$$

Proof. Since $n < m_0$, one obviously has $n, m_0 \notin B_1$. So, when $B_1 \neq \varnothing$, one has $m_1 \in B_1$ or $m_1 \notin B_1$. Then, according to Corollary 2.10, if $m_1 \in B_1$, respectively if $m_1 \notin B_1$, then (3.15), respectively (3.16), holds. $\square$

Lemma 3.19. *Let* $n < m_0$*. If* (3.15) *or* (3.16) *holds, then*

(i) $2n$*,* $n + m_0$ *belong to* $Y_2 - B_2$;

(ii) $2m_0 \in Y_2 - B_2$;

(iii) $n + m_1$*,* $n + m_2$ *do not belong to* B_2*.*

Proof. (i) Since $n < m_i$, $m_0 < m_i$ for $i = 1, 2$, one immediately has $2n \in Y_2 - B_2$, $n + m_0 \in Y_2 - B_2$.

(ii) Suppose (3.15) or (3.16) holds. Then $2m_0 \in Y_2$, since $c = \max\deg((c-1)m_0 + m_1)$, respectively $c' = \max\deg(c'm_0)$. Further, $2m_0 \notin B_2$; otherwise, (since $2m_0 \in Y_2$, $n \notin B_1$) $2m_0 + n = a_1m_1 + a_2m_2$ with $a_1 + a_2 \geq 4$, a contradiction.

(iii) We have only to show that $n + m_1 \notin B_2$ if (3.16) holds. In this case $n + m_1 \in Y_2$, since $m_1 \notin B_1$; further,if $n + m_1 \in B_2$, one has $2n + m_1 = a_0m_0 + a_2m_2$ with $a_0 + a_2 \geq 4$, that implies $a_2 = 0$, so $n + c'm_0 = (a_0 - 1)m_0 + m_1$ with $c' \geq 3$, $a_0 - 1 \geq 3$, a contradiction. $\square$

Corollary 3.20. *Let $n < m_0$. If* (3.15) *or* (3.16) *holds, then*

$$B_2 \subset \{m_0 + m_1, m_0 + m_2 = 2m_1, m_1 + m_2, 2m_2\}.$$

Remark. When (3.15) holds, one has that if $2m_2 \in Y_2$ then $m_1+m_2 \in Y_2$.

Suppose $2m_2 \in Y_2$. If $m_1 + m_2 \notin Y_2$, one has $m_1 + m_2 = an + bm_0$ with $a+b \geq 3$; if $b \geq 1$, then $2m_2 = an+(b-1)m_0+m_1$; if $b = 0$, then $a \geq 3$ and $2m_2 = (a-1)n + (c-1)m_0$, so that $\max\deg(2m_2) \geq 3$.

Lemma 3.21. *Let $n < m_0$. If* (3.15) *holds, then*

(i) *if $m_1 + m_0 \in Y_2$, then $3m_0 \in S_3 \cup C_3$;*

(ii) *if $2m_1 \in Y_2$, then $2m_0 + m_1 \in S_3 \cup C_3$;*

(iii) *if $m_1 + m_2 \in Y_2$, then $2m_0 + m_2 \in S_3 \cup C_3$;*

(iv) *if $2m_2 \in Y_2$, then $3m_1 \in S_3 \cup C_3$.*

Proof. If (3.15) holds, one has that $m_1 + m_0 + n = (c+1)m_0$, $2m_1+n = m_1+cm_0$, $m_1+m_2+n = m_2+cm_0$, $2m_2+n = (c-2)m_0+3m_1$, where $c \geq 3$. So, according to Lemma 2.1, in order to have the result it is enough to show that $3m_0$, respectively $2m_0+m_1$, $2m_0+m_2$, $3m_1$, belongs to Y_3. One can prove it, according to the assumptions. □

Lemma 3.22. *Let $n < m_0$. If* (3.16) *holds, one has*

(i) *if $m_0 + m_2 \in Y_2$, then $3m_0 \in S_3 \cup C_3$;*

(ii) *if $m_1 + m_2 \in Y_2$, then $2m_0 + m_1 \in S_3 \cup C_3$;*

(iii) *if $2m_2 \in Y_2$, then $2m_0 + m_2 \in S_3 \cup C_3$;*

(iv) $m_0 + m_1 \notin B_2$.

Proof. The proof of (i), (ii) and (iii) is similar to the proof of Lemma 3.21. Let us show (iv). Suppose (3.16) and $m_0 + m_1 \in Y_2$. Then, if $m_0 + m_1 \in B_2$, one has, by Lemma 2.9, $m_0 + m_1 + n = (b-1)m_0 + m_r$ with $b \geq 4$, $0 \leq r < 2$, then $m_1 + n = (b-2)m_0 + m_r$, which contradicts (3.16). □

From Corollary 3.20, Lemma 3.21 and Lemma 3.22, one has

Corollary 3.23. *Let Γ be as above, $n < m_0$. If $B_1 \neq \varnothing$, then $h_3 \geq 0$.*

Lemma 3.24. *Let $n < m_0$ and suppose (3.17) holds. If $m_i + m_j \in B_2$, $i, j \in \{0,1,2\}$, then $(i,j) = (1,2)$ or $i = j = 2$ and only one of the following facts holds*:

$$\begin{gathered} m_1 + m_2 \in B_2, \quad 2m_2 \notin Y_2, \quad m_1 + m_2 + n = b_1 m_0 \\ \text{with } b_1 = \max\deg(b_1 m_0) \geq 4 \end{gathered} \tag{3.18}$$

or

$$\begin{gathered} m_1 + m_2 \in B_2, \quad 2m_2 \in Y_2, \quad m_1 + m_2 + n = b_1 m_0 \\ \text{and } 2m_2 + n = (b_1 - 1)m_0 + m_1 \quad \text{with } b_1 = \max\deg(b_1 m_0) \\ = \max\deg((b_1 - 1)m_0 + m_1) \geq 4 \end{gathered} \tag{3.19}$$

or

$$2m_2 \in B_2, \quad 2m_2 + n = b_2 m_0 \quad \text{with } b_2 = \max\deg(b_2 m_0) \geq 4. \tag{3.20}$$

Further, if (3.19) holds, then $3m_0$, $2m_0 + m_1 \in S_3 \cup C_3$; if (3.18) or (3.20) holds, one has $3m_0 \in S_3 \cup C_3$.

Proof. According to Lemma 2.9 and the assumption $B_1 = \varnothing$, it is easy to see that $m_0 + m_1$, $m_0 + m_2 = 2m_1$ do not belong to B_2. Further, if $2m_0 \in Y_2$, one has $2m_0 \notin B_2$; otherwise, $2m_0 + n = a_1 m_1 + a_2 m_2$ with $a_1 + a_2 \geq 4$, a contradiction. So, if $m_i + m_j \in B_2$, one has necessarily $(i,j) = (1,2)$ or $i = j = 2$. If $m_1 + m_2 \in B_2$, then (since $B_1 = \varnothing$) only the following

(i) $m_1 + m_2 + n = b_1 m_0$ (with $b_1 \geq 4$)

can happen, so $b_1 = \max\deg(b_1 m_0)$. From (i) it also follows that $2m_2 + n = (b_1 - 1)m_0 + m_1$ where $b_1 \geq 4$ and $\max\deg((b_1 - 1)m_0 + m_1)$ when $2m_2 \in Y_2$.

When $2m_2 \in B_2$, if (i) does not hold, according to Lemma 2.9 only the following situation can happen:

(ii) $2m_2 + n = b_2 m_0$ (with $b_2 \geq 4$) and one can see $b_2 = \max\deg(b_2 m_0)$.

It is immediate to see that (i) $\cap$ (ii) $= \varnothing$.

If (3.19) holds, then $3m_0$, $2m_0 + m_1$ both belong to Y_3, according to the properties of b_1; so $3m_0$ and $2m_0 + m_1$ belong to $S_3 \cup C_3$ according to Lemma 2.1. In a similar way, one can show that $3m_0 \in S_3 \cup C_3$ when (3.18) or (3.20) holds. □

Lemma 3.25. *Let $n < m_0$, and suppose* (3.17) *holds. If $n+m_j \in B_2$, then only one of the following facts holds*:

$$n + m_1 + n = k_1 m_0 \quad \textit{and } n + m_2 + n = (k_1 - 1)m_0 + m_1 \\ \textit{with } n + m_1, n + m_2 \in Y_2 \tag{3.21}$$

or

$$n + m_2 + n = k_2 m_0 \quad \textit{with } n + m_2 \in Y_2 \tag{3.22}$$

where $k_j = \max\deg(k_j m_0) \geq 4$ for $j \in \{1, 2\}$, $k_1 = \max\deg((k_1 - 1)m_0 + m_1)$.

Further, if (3.21) *holds, one has $3m_0$ and $2m_0 + m_1$ belongs to $S_3 \cup C_3$; if* (3.22) *holds, then $3m_0 \in S_3 \cup C_3$.*

Proof. Clearly $n + m_0 \notin B_2$. If $n + m_1 \in B_2$ one has $2n + m_1 = k_1 m_0 + k_1' m_2$ with $k_1 + k_1' \geq 4$; it implies necessarily $k_1' = 0$, so one has $k_1 = k_1 + k_1' = \max\deg\ (k_1 m_0 + k_1' m_2)$ and $n + m_1 + n = k_1 m_0$, then also $n+m_2+n = (k_1-1)m_0+m_1$. Moreover, one can prove $n+m_2 \in Y_2$, so $n + m_2 \in B_2$ and one has $k_1 = \max\deg((k_1 - 1)m_0 + m_1) \geq 4$.

If $n + m_2 \in B_2$ and (3.21) does not hold, then necessarily one has $2n + m_2 = k_2 m_0$ where $k_2 \geq 4$ and one has $k_2 = \max\deg\ (k_2 m_0)$, so (3.22) holds.

Clearly, (3.21) $\cap$ (3.22) $= \varnothing$. If (3.21) holds, then, according to the properties of k_1 and Lemma 2.1, one has $3m_0$, $2m_0 + m_1$ belong to $S_3 \cup C_3$; in a similar way, one can see that $3m_0 \in S_3 \cup C_3$ when (3.22) holds. □

Corollary 3.26. *Let $n < m_0$, and suppose $B_1 = \varnothing$. Then one of the following facts can hold*:

(i) $B_2 = \varnothing$,

or

(ii) $B_2 = \{2m_2\}$ *or* $B_2 = \{n + m_2\}$ *or* $B_2 = \{m_1 + m_2\}$ *and* $3m_0 \in S_3 \cup C_3$ *or*

(iii) $B_2 = \{m_1 + m_2, 2m_2\}$ *or* $B_2 = \{n + m_1, n + m_2\}$ *and* $3m_0$, $2m_0 + m_1$ *belong to* $S_3 \cup C_3$.

In any case, one has $h_3 \geq 0$.

Proof. If $B_2 \neq \varnothing$, according to Lemmas 3.24 and 3.25, one has

$$B_2 \subset \{m_1 + m_2, 2m_2, n + m_1, n + m_2\} \cap Y_2.$$

Further, each of the conditions in $\{(3.18), (3.19), (3.20)\} \cup \{(3.21), (3.22)\}$ excludes the other ones.

Then B_2 cannot contain more than two elements. Now, according to Lemmas 3.24 and 3.25, one has that (ii) or (iii) holds and, in any case, $h_3 \geq 0$. □

4. In this section we show an example for each of the conditions considered in Lemmas 3.1, 3.12, 3.14, 3.16, 3.18, 3.24 and 3.25.

Examples 4.1. Suppose $m_0 < n$.

(1) Let $m_0 = 7$, $m_1 = 20$, $m_2 = 33$, $n = 9$. Then $m_1 + m_0 = 3n$, so (3.1) holds.

(2) If $m_0 = 7$, $m_1 = 18$, $m_2 = 29$, $n = 12$, one has $m_2 + m_0 = 3n$, so (3.2) holds.

(3) When $m_0 = 7$, $m_1 = 10$, $m_2 = 13$, $n = 29$, one has $n + m_0 = m_1 + 2m_2$, so (3.3) holds.

(4) If $m_0 = 7$, $m_1 = 10$, $m_2 = 13$, $n = 32$, one has $n + m_0 = 3m_2$, so (3.4) holds.

(5) When $m_0 = 9$, $m_1 = 13$, $m_2 = 17$, $n = 10$, then $2m_2 \in Y_2$, $B_1 = \varnothing$, $2m_2 + m_0 = m_1 + 3n$, so (3.6) holds.

(6) Let $m_0 = 9$, $m_1 = 22$, $m_2 = 35$, $n = 10$; then $m_0 + m_1 \in Y_2$, $B_1 = \varnothing$, $(m_0 + m_1) + m_0 = 4n$, so (3.7) holds.

(7) If $m_0 = 8$, $m_1 = 14$, $m_2 = 20$, $n = 9$, one has $m_0 + m_2 \in Y_2$, $B_1 = \varnothing$, $(m_0 + m_2) + m_0 = 4n$, so (3.8) holds.

(8) When $m_0 = 7$, $m_1 = 16$, $m_2 = 25$, $n = 12$, one has $m_1 + m_2 \in Y_2$, $B_1 = \varnothing$, $(m_1 + m_2) + m_0 = 4n$, so (3.9) holds.

(9) If $m_0 = 10$, $m_1 = 19$, $m_2 = 28$, $n = 11$, then $2m_2 \in Y_2$, $B_1 = \varnothing$, $2m_2 + m_0 = 6n$, so (3.10) holds.

(10) Let $m_0 = 9$, $m_1 = 13$, $m_2 = 17$, $n = 50$; then $m_0 + n \in Y_2$, $B_1 = \varnothing$, $(m_0 + n) + m_0 = 4m_2$, so that (3.11) holds.

(11) When $m_0 = 8$, $m_1 = 11$, $m_2 = 14$, $n = 37$, one has $m_0 + n \in Y_2$, $m_1 + n = 6m_0 \notin Y_2$, $B_1 = \varnothing$, $(m_0 + n) + m_0 = m_1 + 3m_2$, so (3.12) holds.

(12) If $m_0 = 9$, $m_1 = 13$, $m_2 = 17$, $n = 46$, one has $m_0 + n \in Y_2$, $m_1 + n \in Y_2$, $B_1 = \varnothing$, $(m_0 + n) + m_0 = m_1 + 3m_2$, $(m_1 + n) + m_0 = 4m_2$, so (3.13) holds.

(13) Let $m_0 = 9$, $m_1 = 14$, $m_2 = 19$, $n = 31$; then $2n \in Y_2$, $B_1 = \varnothing$, $2n + m_0 = m_1 + 3m_2$, so (3.14) holds.

Examples 4.2. Suppose $n < m_0$.

(1) Let $n = 6$, $m_0 = 7$, $m_1 = 15$, $m_2 = 23$; one has $m_1 + n = 3m_0$ and $m_2 + n = 2m_0 + m_1$, so (3.15) holds.

(2) If $n = 6$, $m_0 = 7$, $m_1 = 11$, $m_2 = 15$, then $m_2 + n = 3m_0$, $m_1 \notin B_1$ and (3.16) holds.

(3) When $n = 9$, $m_0 = 10$, $m_1 = 17$, $m_2 = 24$, one has $m_1 + m_2 \in Y_2$, $m_1 + m_2 + n = 5m_0$, $2m_2 = 2n + 3m_0 \notin Y_2$, $B_1 = \varnothing$; then (3.18) holds.

(4) If $n = 19$, $m_0 = 20$, $m_1 = 27$, $m_2 = 34$, one has $m_1 + m_2 \in Y_2$, $2m_2 \in Y_2$, $m_1 + m_2 + n = 4m_0$, $2m_2 + n = 3m_0 + m_1$, $B_1 = \varnothing$, so (3.19) holds.

(5) Let $n = 22$, $m_0 = 25$, $m_1 = 32$, $m_2 = 39$; then $2m_2 \in Y_2$, $2m_2 + n = 4m_0$, $B_1 = \varnothing$, so (3.20) holds.

(6) If $n = 8$, $m_0 = 9$, $m_1 = 20$, $m_2 = 31$, one has $n + m_1 \in Y_2$, $n + m_2 \in Y_2$, $n + m_1 + n = 4m_0$, $n + m_2 + n = 3m_0 + m_1$, $B_1 = \varnothing$, so (3.21) holds.

(7) When $n = 11$, $m_0 = 20$, $m_1 = 39$, $m_2 = 58$, one has $n + m_2 \in Y_2$, $n + m_2 + n = 4m_0$, $B_1 = \varnothing$, so (3.22) holds.

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