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DISTAL COMPACTIFICATIONS OF GROUP EXTENSIONS

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ABSTRACT. Let N and K be topological groups, and let G be a topological group extension of N by K. We show that if N or K is compact then, under suitable conditions, the distal compactification of G is a canonical extension of a group compactification of N by the distal compactification of K. An analogous result is shown to hold for the universal point distal G-flow.

1. Introduction. Let N and K be groups with identity e. A group G_0 is an extension of N by K if there exists a short exact sequence

$$e \longrightarrow N \longrightarrow G_0 \longrightarrow K \longrightarrow e.$$

A result of Schreier [**12**] asserts that G_0 is canonically isomorphic to $G := N \times K$ with multiplication in G given by

$$(s,t)(s',t') = (st(s')[t,t'], tt'), \quad s, s' \in N, t, t' \in K, \tag{1}$$

where the mappings $(t,t') \mapsto [t,t'] : K \times K \to N$ and $t \mapsto t(\cdot) : K \to \operatorname{Aut}(N)$ satisfy the Schreier extension formulation conditions

$$\text{(SEF)} \qquad \begin{cases} e(s) = s \quad \text{and} \quad [t,e] = [e,t] = e, \\ [t,t'](tt')(s) = t(t'(s))[t,t'], \quad \text{and} \\ [t,t'][tt',t''] = t([t',t'']), [t,t't''], \end{cases}$$

see [**13**]. To indicate this we shall write $G = N \times K$ (SEF).

Now suppose that N and K are topological groups and that the Schreier mappings $[\cdot,\cdot] : K \times K \to N$ and $(s,t) \mapsto t(s) : N \times K \to N$

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are jointly continuous, so that G is a topological group. In [**6**, Corollary 4.2] we showed that the almost periodic compactification G^{AP} of G is a canonical group extension of a topological group compactification of N by K^{AP} if and only if the mapping $[\cdot,\cdot]$ enjoys a certain relative compactness condition. In the present paper we prove distal, point distal and LC analogs of this result. For example, we prove in Theorem 3.1 that, if K is compact and if either G is a central extension or the left and right uniformities of N coincide, then the LC compactification G^{LC} of G is a canonical extension of N^{LC} by K if and only if the mapping $t \to g(t(\cdot))$ is norm continuous for each $g \in LC(N)$. In the same section we also prove that, if N is compact, then G^{LC} is a canonical extension of N by K^{LC} if and only if the mapping $[\cdot, K]$ is equicontinuous. In Section 4 we give sufficient conditions for the distal compactification G^{DLC} of G to be a canonical extension of N by K^{DLC} in case N is compact, and a canonical extension of N^{DLC} by K if K is compact. In each case we characterize the Furstenberg structure of G^{DLC} in terms of that of the component compactifications (Section 5). Finally, in Section 6 we give sufficient conditions for the universal (jointly continuous) point distal G-flow G^{PDLC} to be canonically isomorphic to a G-flow $N \times K^{PLDC}$ if N is compact, and canonically isomorphic to a G-flow $N^{PDLC} \times K$ if K is compact.

2. Preliminaries. Let N and K be topological groups with identity e, and let $G = N \times K$ (SEF) be a topological group extension of N by K, as described in the introduction. G is said to be a *split extension* or a *semi-direct product* if the range of $[\cdot,\cdot]$ is $\{e\}$, and a *central extension* if the range of $[\cdot,\cdot]$ is contained in the center of N. For central extensions the middle equation of (SEF) asserts that K acts on N via the mapping $t \mapsto t(\cdot)$. We let $q_1 : N \to G$ and $q_2 : K \to G$ denote the canonical injections, $p_1 : G \to N$ and $p_2 : G \to K$ the projection mappings, and we set $r_i = q_i \circ p_i$. In general, q_1 and p_2 are homomorphisms, and in the semi-direct product case q_2 is a homomorphism. As usual, $C(G)$ denotes the C^*-algebra of bounded, continuous, complex-valued functions on G and $R(\cdot)$ and $L(\cdot)$ the right and left translation operators on $C(G)$, $R(s)f(t) = f(ts) = L(t)f(s)$.

We shall need some notions from the theory of semigroup compactifications; for details the reader is referred to Chapters 3 and 4 of [**1**].

A *right topological semigroup* is a semigroup S with a topology relative to which the right translation mappings $s \mapsto st : S \to S$ are continuous. A (*right topological semigroup*) *compactification* of G is a compact, Hausdorff, right topological semigroup G' together with a continuous homomorphism $\varepsilon_{G'} : G \to G'$, the *compactification map* such that the image $\varepsilon_{G'}(G)$ is dense in G' and the mappings $s' \mapsto \varepsilon_{G'}(s)s' : G' \to G'$ and $s' \mapsto s't' : G' \to G'$ are continuous for each $s \in G$ and $t' \in G'$. A compactification G'' is a *factor* of a compactification G' if there exists a continuous function $\varphi : G' \to G''$, the *compactification homomorphism* such that $\varphi \circ \varepsilon_{G'} = \varepsilon_{G''}$. If X is a compact Hausdorff space and $\phi : G \to X$, $\phi' : G' \to X$ are continuous functions such that $\phi' \circ \varepsilon_{G'} = \phi$, then ϕ' is said to be an *extension* of ϕ.

A compactification with a given property $\mathcal{P}$ is called a $\mathcal{P}$*-compactification*. A *universal* $\mathcal{P}$*-compactification* of G is a $\mathcal{P}$-compactification of which every $\mathcal{P}$-compactification of G is a factor. We denote the universal $\mathcal{P}$-compactification of G by $G^{\mathcal{P}}$ and the *function space* $\varepsilon^*_{G^{\mathcal{P}}}(C(G^{\mathcal{P}}))$ of $G^{\mathcal{P}}$ by $\mathcal{P}(G)$. $\mathcal{P}(G)$ is easily seen to be *m-admissible*, i.e., it is a translation invariant C^*-subalgebra of $C(G)$ containing the constant functions and the functions $x \mapsto \mu(L(x)f)$, where $f \in \mathcal{P}(G)$ and μ is a member of the spectrum of $\mathcal{P}(G)$. Conversely, if F is an m-admissible C^*-subalgebra of $C(G)$, then the spectrum G^F of F is a compactification of G with compactification map $x \to \hat{x}$, where $\hat{x}(f) = f(x)$.

We shall be concerned primarily with the universal compactifications $G^{\mathcal{P}}$ where $\mathcal{P}$ is either the *right topological group* property, i.e., the property that the compactification is a right topological group, or the *LC property*, i.e., the property that the natural left action of G on the compactification is jointly continuous. The function spaces $\mathcal{P}(G)$ corresponding to these properties are, respectively, $D(G)$, the algebra of distal functions on G, and $LC(G)$, the algebra of left norm continuous, equivalent to bounded, right uniformly continuous [**4**, p. 21], functions on G, see Sections 3 and 4. We shall also need to consider the algebra $RC(G)$ of right norm continuous functions, which is generally not m-admissible, and the m-admissible algebra $LMC(G)$, which consists of all functions $f \in C(G)$ such that the pointwise closure of $R(G)f = \{R(x)f : x \in G\}$ is a subset of $C(G)$. Details concerning these and related spaces may be found in Chapter 4 of [**1**]. We note in particular that $LMC(G) = LC(G)$ if G is locally compact or complete metric [**1**].

Our goal is to find conditions under which there exists a compactification isomorphism $G^{\mathcal{P}} \cong N' \times K'$ (SEF), where N' and K' are compactifications of N and K and multiplication on $N' \times K'$ is defined as in (1) by Schreier maps $[\cdot,\cdot] : K' \times K' \to N'$ and $(x, y) \mapsto y(x) : N' \times K' \to N'$. These maps satisfy (SEF) but, because $G^{\mathcal{P}}$ may not be a group, $y(\cdot)$ need not be an automorphism. Also, the Schreier maps need not be jointly continuous, although they do satisfy certain separate continuity conditions, see [**6**, Section 2]. The compactification map $\varepsilon_{N' \times K'}$ is assumed to be the product map $\varepsilon_{N'} \times \varepsilon_{K'} : N \times K \to N' \times K'$; this is equivalent to requiring that the Schreier maps of $N' \times K'$ be extensions of the Schreier maps of $N \times K$. Under these circumstances we shall call $N' \times K'$ an (SEF) *compactification* of G.

The main results of the paper rely on the following version of Theorem 3.4 in [**6**].

Theorem 2.1. *Let F, A and B be m-admissible C^*-subalgebras of $C(G)$, $C(N)$ and $C(K)$, respectively. Suppose that the following conditions hold*:

(a) $p_1^*(A) \subset F$;

(b) $p_2^*(B) \subset F$;

(c) $q_1^*(F) \subset A$;

(d) $q_2^*(F) \subset B$;

(e) *either N is compact and $F \subset LC(G)$ or K is compact and the map $t \mapsto f(\cdot, t)$: $K \to C(N)$ is norm continuous for each $f \in F$.*

Then $G^F \cong N^A \times K^B$ (SEF), $A = q_1^*(F)$ and $B = q_2^*(F)$. Moreover, G^F is a split extension if G is a split extension and a central extension if G is a central extension and N^A is a topological semigroup.

Conversely, if $G^F \cong N' \times K^{|}p$ (SEF) for some compactifications N' and K', then $N' \cong N^A$ and $K' \cong K^B$, where $A = q_1^*(F)$ and $B = q_2^*(F)$. Moreover, conditions (a)–(d) *hold, and the maps $s \mapsto f(s, \cdot)$ and $t \mapsto f(\cdot, t)$ are norm continuous for $f \in F$.*

Remark 2.2. If $F = \mathcal{P}(G)$, $A = \mathcal{P}(N)$ and $B = \mathcal{P}(K)$ for a property $\mathcal{P}$ which is inherited by subcompactifications, see [**7**] then, because p_2 and q_1 are homomorphisms, conditions (b) and (c) hold automatically.

3. ***LC*-Compactifications.** The space $LC(G)$ consists of all functions $f \in C(G)$ such that $x \mapsto L(x)f : G \mapsto C(G)$ is norm continuous. The space $RC(G)$ is defined analogously. In this section we consider the problem of expressing G^{LC} as a canonical (right topological semigroup) extension of some compactification of N by K^{LC}. Simple examples show that one cannot expect this to occur in general unless N or K is compact (e.g., $\mathbf{Z} \times \mathbf{Z}$). We consider first the case K compact.

Theorem 3.1. *Let K be compact and suppose either that G is a central extension or that $LC(N) = RC(N)$. Then $G^{LC} = N^{LC} \times K$ (SEF) if and only if the family $\{t \mapsto \varepsilon_{N^{LC}}(t(s)) : s \in N\}$ is equicontinuous (equivalently the function $t \to g(t(\cdot)) : K \mapsto LC(N)$ is norm continuous for each $g \in LC(N)$). In particular, this holds if $\{t \mapsto t(s) : s \in N\}$ is equicontinuous.*

Proof. For the necessity, let $t_\alpha \to t_0$ in K and set $x_\alpha := \varepsilon_{N^{LC}}([t_\alpha, t_\alpha^{-1}])$ and $x_0 := \varepsilon_{N^{LC}}([t_0, t_0^{-1}])$. By the LC property of G^{LC}, $\varepsilon_{G^{LC}}(e, t_\alpha)\varepsilon_{G^{LC}}(s, t_\alpha^{-1}) = (\varepsilon_{N^{LC}}(t_\alpha(s))x_\alpha, e)$ converges uniformly in s to $\varepsilon_{G^{LC}}(e, t_0)e_{G^{LC}}(s, t_0^{-1}) = (\varepsilon_{N^{LC}}(t_0(s))x_0, e)$. The hypotheses then imply that $\varepsilon_{N^{LC}}(t_\alpha(s)) \to \varepsilon_{N^{LC}}(t_0(s))$ uniformly in s.

For the sufficiency we need to verify that conditions (a) and (e) of Theorem 2.1 hold for $F = LC(G)$, $A = LC(N)$ and $B = C(K)$ (see Remark 2.2). For 2.1 (a), let $g \in LC(N)$, $f = p_1^*(g)$, and let $(s_\alpha, t_\alpha) \to (s_0, t_0)$ in G. To see that $f \in LC(G)$, consider the inequality

$$\begin{aligned}
&|L(s_\alpha, t_\alpha)f(s,t) - L(s_0, t_0)f(s,t)| \\
&\qquad = |g(s_\alpha t_\alpha(s)[t_\alpha, t]) - g(s_0 t_0(s)[t_0, t])| \\
&\qquad \le |g(s_\alpha t_\alpha(s)[t_\alpha, t]) - g(s_0 t_\alpha(s)[t_\alpha, t])| \\
&\qquad\quad + |g(s_0 t_\alpha(s)[t_\alpha, t]) - g(s_0 t_0(s)[t_\alpha, t])| \\
&\qquad\quad + |g(s_0 t_0(s)[t_\alpha, t]) - g(s_0 t_0(s)[t_0, t])|.
\end{aligned}$$

The first term on the right side of the inequality obviously converges to zero uniformly in (s, t). To see that the second term on the right also converges to zero uniformly in (s, t), note that the equicontinuity hypothesis implies that $g(s_0 t_\alpha(\cdot)r) \to g(s_0 t_0(\cdot)r)$ in norm for each $r \in N$, so if $g \in RC(N)$ or G is central then the convergence is uniform in r on the compact set $[K, K]$. Finally, because $[\cdot, K]$ is trivially equicontinuous, the third term converges to zero uniformly in (s,t) if either G is central or $LC(N) = RC(N)$.

It remains to show that, for a function $f \in LC(G)$, the map $t \mapsto f(\cdot, t)$ is continuous in the uniform norm. Let $t_\alpha \to t_0$ in K and set $r_\alpha = [t_\alpha^{-1}, t_\alpha]$, $r_0 = [t_0^{-1}, t_0]$ and $h = q_1^*(L(e,t_0)f)$. Then

$$\begin{aligned}|f(s,t_\alpha) - f(s,t_0)| &= |f((e,t_\alpha)(r_\alpha^{-1}t_\alpha^{-1}(s)r_\alpha, e)) \\ &\qquad - f((e,t_0)(r_0^{-1}t_0^{-1}(s)r_0, e))| \\ &\le \|L(e,t_\alpha)f - L(e,t_0)f\| \\ &\quad + \|h(r_\alpha^{-1}t_\alpha^{-1}(\cdot)r_\alpha) - h(r_0^{-1}t_0^{-1}(\cdot)r_0)\| \\ &= a_\alpha + b_\alpha,\end{aligned}$$

where, obviously, $a_\alpha \to 0$. Since q_1 is a homomorphism, $h \in LC(N)$, and it follows from the equicontinuity hypothesis that $b_\alpha \to 0$ if G is central and that $h(ut_\alpha^{-1}(\cdot)v) \to h(ut_0(\cdot)v)$ in norm for each u and v in N. If $LC(N) = RC(N)$, then the latter convergence is uniform in u and v on compacta, hence again, $b_\alpha \to 0$. □

An illustration of Theorem 3.1 is given in Example 4.4 below.

For the case N compact the equicontinuity requirement shifts from the action to the cocycle.

Theorem 3.2. *If N is compact, then $G^{LC} \cong N \times K^{LC}$ (SEF) if and only if $[\cdot, K]$ is equicontinuous.*

Proof. If $G^{LC} = N \times K^{LC}$ (SEF) and $t_\alpha \to t_0$ in K, then

$$\begin{aligned}([t_\alpha, t], \varepsilon_{K^{LC}}(t_\alpha t)) &= \varepsilon_{G^{LC}}(e,t_\alpha)\varepsilon_{G^{LC}}(e,t) \to \varepsilon_{G^{LC}}(e,t_0)\varepsilon_{G^{LC}}(e,t) \\ &= ([t_0,t], \varepsilon_{K^{LC}}(t_0 t))\end{aligned}$$

uniformly in t, hence $[\cdot, K]$ is equicontinuous.

Conversely, suppose that $[\cdot, K]$ is equicontinuous. We use Theorem 2.1 with $F = LC(G)$, $A = C(N)$ and $B = LC(K)$ to show that $G^{LC} \cong N \times K^{LC}$ (SEF). By Remark 2.2 it suffices to show that Theorem 2.1 (a) and (d) hold. For (a), let $g \in C(N)$, $f = p_1^*(g)$, and let $(s_\alpha, t_\alpha) \to (s_0, t_0)$ in G. Each bracketed term on the right side of the equality

$$\begin{aligned}L(s_\alpha,t_\alpha)f(s,t) &- L(s_0,t_0)f(s,t)\\ &= [g(s_\alpha t_\alpha(s)[t_\alpha,t]) - g(s_\alpha t_\alpha(s)[t_0,t])]\\ &\quad + [g(s_\alpha t_\alpha(s)[t_0,t]) - g(s_0 t_0(s)[t_0,t])]\end{aligned}$$

converges to zero uniformly in (s,t), the first by the equicontinuity of $[\cdot,K]$ and the second by the joint continuity of $(s,t)\to t(s)$. Therefore, $f\in LC(G)$.

It remains to show that $q_2^*(LC(G))\subset LC(K)$. Let $f\in LC(G)$, $g=q_2^*(f)$, and let $t_\alpha\to t_0$ in K. Then, for any compact subset C of G, $f(c(e,t_\alpha)(e,t))\to f(c(e,t_0)(e,t))$ uniformly in $t\in K$ and $c\in C$, so the first bracketed term on the right side of the following equality converges to zero uniformly in t:

$$\begin{aligned}L(t_\alpha)g(t) - L(t_0)g(t) &= [f(([t_\alpha,t],e)^{-1}(e,t_\alpha)(e,t))\\ &\quad - f(([t_\alpha,t],e)^{-1}(e,t_0)(e,t))]\\ &\quad + [f(([t_\alpha,t],e)^{-1}(e,t_0)(e,t))\\ &\quad - f(([t_0,t],e)^{-1}(e,t_0)(e,t))].\end{aligned}$$

To see that the second bracketed term on the right also converges to zero uniformly in t, and hence that $g\in LC(K)$, choose a neighborhood V of (e,e) in G such that $xy^{-1}\in V$ implies $|f(x)-f(y)|<\varepsilon$, and let W be a neighborhood of e in N such that $q_1^*(W)\subset V$. By the equicontinuity of $[\cdot,K]$, $[t_\alpha,t]^{-1}[t_0,t]\in W$ for all sufficiently large α and all $t\in K$, and this implies that the absolute value of the second bracketed term on the right in the above equality is less than ε. □

Corollary 3.3 [**9**, Theorem 10]. *If N is compact and K is discrete, then $G^{LC}\cong N\times\beta(K)$ (SEF).*

Corollary 3.4 [**10**, Theorem 3.5]. *If N is compact and G is a semi-direct product of N and K, then G^{LC} is a semi-direct product of N and K^{LC}.*

Example 3.5. Let $G=N\times K=\mathbf{C}^*\times\mathbf{R}^2$ have multiplication

$$(z,x,y)(z',x',y')=(zz'e^{iyx'},x+x',y+y').$$

Here $[(x,y),(x',y')] = e^{iyx'}$, and the automorphism determined by (x,y) is the identity map. For the subgroups $G_1 = \mathbf{T}\times(\mathbf{R}\times\mathbf{Z})$ and $G_2 = \mathbf{T}\times(\mathbf{Z}\times\mathbf{R})$, Theorem 3.2 implies that $G_1^{LC} \cong \mathbf{T}\times(\mathbf{R}\times\mathbf{Z})^{LC}$ (SEF) and $G_2^{LC} \ncong \mathbf{T}\times(\mathbf{Z}\times\mathbf{R})^{LC}$ (SEF).

4. Distal compactifications. A function $f \in LMC(G)$ is *distal* if the pointwise closure of the right orbit $R(G)f$ is a distal flow under right translation. This is equivalent to the function $\hat{f} = (\varepsilon^*_{G^{LMC}})^{-1}(f)$ satisfying

$$\hat{f}(uvw) = \hat{f}(uw), \quad u, v = v^2, w \in G^{LMC},$$

[**1**]. The algebra of distal functions on G is denoted by $D(G)$, and we set $DLCL(G) := D(G)\cap LC(G)$. Note that $DLC(G) = D(G)$ if G is locally compact or complete metric. We let $\varphi : G^{LC} \to G^{DLC}$ denote the compactification homomorphism.

Lemma 4.1. *Let H be a topological group and $\theta : H \to G$ a continuous function such that $\theta^*(LC(G)) \subset LC(H)$ (this implies that θ has an extension $\bar{\theta} : H^{LC} \to G^{LC}$). The following statements are equivalent*:

(a) *$\theta^*(DLC(G)) \subset DLC(H)$, hence θ has an extension $\tilde{\theta} : H^{DLC} \to G^{DLC}$.*

(b) *$(\varphi\circ\bar{\theta})(uvw) = (\varphi\circ\bar{\theta})(uw)$, $u, v = v^2$, $w \in H^{LC}$.*

Proof. We have $\varepsilon_{G^{DLC}} \circ \theta = \varphi \circ \bar{\theta} \circ \varepsilon_{H^{LC}}$, so $\theta^*(DLC(G)) = \varepsilon^*_{H^{LC}} \circ (\varphi\circ\bar{\theta})^* C(G^{DLC})$. Thus, (a) is equivalent to the identity

$$(\varphi\circ\bar{\theta})^* f(uvw) = (\varphi\circ\bar{\theta})^* f(uw),$$
$$f \in C(G^{DLC}), \quad u, v = v^2, w \in H^{LC}$$

and hence to (b). $\square$

Lemma 4.2. *Let K be compact, and suppose that $G^{LC} = N' \times K$ (SEF) for some compactification N' of N. Then $G^{DLC} = N^A \times K$ (SEF) where $A = q_1^*(DLC(G))$.*

Proof. By Theorem 2.1, $r_1^*(LC(G)) \subset LC(G)$. We show that Lemma

4.1(b) holds for $\theta = r_1$. The desired conclusion will then follow from Lemma 4.1, Theorem 2.1 and Remark 2.2.

Let $u = (x, y)$, $v = (x', y')$ and $w = (x'', y'')$ be members of G^{LC} with $v^2 = v$. Then $x'^2 = x'$ and $y' = e$ so

$$\begin{aligned}\varphi(\bar{r}_1(uvw)) &= \varphi(xy(x')y(x'')[y, y''], e) \\ &= \varphi(x, e)\varphi(y(x'), e)\varphi(x, e)^{-1}\varphi(\bar{r}_1(uw)).\end{aligned}$$

Since x' is an idempotent, so is $(y(x'), e)$. Therefore, $\varphi(y(x'), e) = (e, e)$ and (b) of Lemma 4.1 follows. □

We may now prove the following distal analog of Theorem 3.1.

Theorem 4.3. *Let K be compact and $\{t \mapsto \varepsilon_{N^{LC}}(t(s)) : s \in N\}$ equicontinuous. If G is a central extension or if $LC(N) = RC(N)$, then $G^{DLC} \cong N^{DLC} \times K$ (SEF).*

Proof. By Theorem 3.1 and Lemma 4.2, $G^{DLC} \cong N' \times K$ (SEF) for some factor N' of N^{DLC}. To show that N^{DLC} is a factor of N' and hence that $N' \cong N^{DLC}$, it suffices to show that multiplication can be extended to $G' := N^{DLC} \times K$ so that G' is a right topological group compactification of G.

Define multiplication in G' by $(x, t)(x', t') = (xt(x')\varepsilon_{N^{DLC}}([t, t']), tt')$, where the mapping $x \mapsto t(x) : N^{DLC} \to N^{DLC}$ is the extension of $t(\cdot) : N \to N$. Conditions (SEF) are obviously satisfied, so it remains only to show that G' is a right topological compactification of G. But this follows easily from the fact that G' is the continuous homomorphic image of G^{LC} under $\theta \times \mathrm{id}_K$, where $\theta : N^{LC} \to N^{DLC}$ is the compactification map. □

Example 4.4. Let $\mathbf{Z}_p$ denote the p-adic integers, and let $G = \mathbf{C}^* \times \mathbf{Z}_p^2$ have multiplication

$$(z, x, y)(z', x', y') = (zz'e^{2\pi i x_0 y x'}, x + x', y + y'),$$

where x_0 is a fixed p-adic number. Then $G^{LC} \cong \mathbf{C}^{*LC} \times \mathbf{Z}_p^2$ (SEF) (Theorem 3.1) and $G^D \cong \mathbf{C}^{*D} \times \mathbf{Z}_p^2$ (SEF) (Theorem 4.3).

Lemma 4.5. *Let N be compact, and suppose that $G^{LC} \cong N' \times K^{LC}$ (SEF) for some compactification N' of N. Consider the following identities in the variables x', $x'' \in N'$ and $y, y' = y'^2$, $y'' \in K^{LC}$:*

(a) $\varphi(y(x')[y,y'](yy')(x'')[yy',y''],e) = \varphi(y(x'')[y,y''],e)$ *whenever* $y'(x')[y',y'] = e$.

(b) $\varphi([y,y'][yy',y''],e) = \varphi(y([y',y'])[y,y''],e)$.

(c) $y(x')[y,y'](yy')(x'')[yy',y''] = y(x'')[y,y'']$ *whenever* $y'(x')[y',y'] = e$.

(d) $[y,y'][yy',y''] = y([y',y'])[y,y'']$.

Then

(i) $G^{DLC} \cong N'' \times K^{DLC}$ (SEF) *for some factor N'' of N' if and only if* (a) *and* (b) *hold*;

(ii) $G^{DLC} \cong N' \times K^{DLC}$ (SEF) *if and only if* (c) *and* (d) *hold.*

Proof. (a) is essentially a restatement of Lemma 4.1 (b) with $\theta = r_1$. We show that (b) is equivalent to Lemma 4.1 (b) with $\theta = q_2$; (i) will then follow from Lemma 4.1 and Theorem 2.1. With y, y', y'' as above, we have $\varphi([y',y'],e) = \varphi(e,y')$ so

$$\begin{aligned}\varphi([y,y'][yy',y''],e)(\varphi\circ\bar{q}_2)(yy'y'') &= \varphi(e,y)\varphi(e,y')\varphi(e,y'')\\ &= \varphi(y([y',y']),e)\varphi(e,y)\varphi(e,y'')\\ &= \varphi(y([y',y'])[y,y''],e)(\varphi\circ\bar{q}_2)(yy''),\end{aligned}$$

from which the desired equivalence follows.

For (ii), note that $G^{DLC} \cong N' \times K^{DLC}$ (SEF) if and only if (a) and (b) hold and $q_1^*(DLC(G)) = q_1^*(LC(G))$, in which case $\varphi(\cdot,e)$ is the identity map. The preceding equality is implied by the inclusion $r_1^*(LC(G)) \subset DLC(G)$, which is equivalent to (c). $\square$

Theorem 4.6. *Let G be a central extension with N compact and $[t,\cdot]$ a homomorphism for each $t \in K$. Then $G^{DLC} \cong N \times K^{DLC}$ (SEF) if and only if $[\cdot,K]$ is equicontinuous and the action of K on N is distal.*

Proof. The necessity is clear. For the sufficiency, note first that, by Theorem 3.2, $G^{LC} \cong N \times K^{LC}$ (SEF). Consider the Schreier maps

$[\cdot,\cdot] : K^{LC} \times K^{LC} \to N$ and $(s,y) \mapsto y(s) : N \times K^{LC} \to N$ for G^{LC}. By continuity, $[y,\cdot] : K^{LC} \to N$ is a homomorphism for each $y \in K^{LC}$. Let $y, y', y'' \in K^{LC}$ with $y'^2 = y'$. Then $[y,y'] = e$ and, by (SEF), we have

$$[yy', y''] = y([y', y''])[y, y'']. \tag{2}$$

If the flow (K, N) is distal, then $y'(\cdot)$ is the identity map, and taking $y = y'$ in (2) we see that $[y', y''] = y'([y', y'']) = e$. Hence (2) reduces to $[yy', y''] = [y, y'']$, from which (c) and (d) of Lemma 4.5 readily follow. □

Corollary 4.7. *Let N be compact, and let G be a semi-direct product, so that, by Corollary* 3.4, *G^{LC} is a semi-direct product $N \times K^{LC}$. The following statements are equivalent*:

(a) *G^{DLC} is a split extension of N by K^{DLC}.*

(b) *The action of K on N is distal.*

(c) *$(e,y)N_1 = N_1(e,y)$, $y \in K^{LC}$, where $N_1 = N \times e$.*

(d) *$y(N) = N$, $y \in K^{LC}$.*

(e) *$y(\cdot) = \mathrm{id}_N$, $y^2 = y \in K^{LC}$.*

Proof. Clearly $(e,y)N_1 \subset N_1(e,y)$, and the reverse inclusion is equivalent to $y(N) = N$. Thus (c) and (d) are equivalent. The equivalence of (a) and (b) follows directly from Theorem 4.6.

To see that (a) implies (d), let $\theta : K^{LC} \to K^{DLC}$ and $\varphi : G^{LC} \to G^{DLC}$ denote the compactification homomorphisms, so that $\varphi(s,y) = (s, \theta(y))$. Then

$$\begin{aligned}(\theta(y)(s), \theta(y)) &= (e, \theta(y))(s,e) = \varphi((e,y)(s,e)) \\ &= \varphi(y(s), y) = (y(s), \theta(y)),\end{aligned}$$

so $y(\cdot) = \theta(y)(\cdot)$ and hence $y(\cdot)$ is surjective.

Clearly (d) implies (e). That (e) implies (a) is a consequence of Lemma 4.5, which asserts in the present setting that G^{DLC} is a split extension if and only if $y(x')(yy')(x'') = y(x'')$ for all $x', x'' \in N$ and $y, y' = y'^2 \in K^{LC}$ with $y'(x') = e$. □

Example 4.8 (Wreath product). Let H be a compact topological group, Λ a nonempty set, let $N := H^{\Lambda}$ with the product topology and group structure, and let K be a discrete group acting on Λ on the right. G is a semi-direct product of N and K where, for $t \in K$, $t(\cdot) : N \to N$ is defined by $t(s)(\lambda) = s(\lambda t)$, $s \in N$, $\lambda \in \Lambda$. By Corollary 3.4, G^{LC} is a semi-direct product $N \times \beta(K)$. If each orbit λK is finite, then the action of K on N is equicontinuous, hence distal, and G^D $(= G^{DLC})$ is a semi-direct product $N \times K^D$ (Corollary 4.7). For a concrete example, take $K = \mathbf{Z}$, $\Lambda = \{0, 1, \dots , p-1\}$, and let the action be $\lambda n = \lambda + n$ (mod p).

On the other hand, it is easy to give examples for which G^D is not a semi-direct product $N \times K^D$. For instance, take $H = \mathbf{T}$, $\Lambda = K = \mathbf{Z}$, and let the (nondistal) action of K on Λ be $\lambda n = \lambda + n$. If $y \neq 0$ is an idempotent in K^{LC}, then $y(\cdot)$ cannot be the identity function, hence G^D cannot be a semi-direct product of N and K^D (Corollary 4.7).

Example 4.9. Let $G = N \times K = \mathbf{T}^2 \times \mathbf{Z}^2$ have multiplication

$$(z_1, z_2, m, n)(z_1', z_2', m', n') = (z_1 z_1' z_2'^n \lambda^{m'n(n-1)/2}, z_2 z_2' \lambda^{m'n}, m + m', n + n'),$$

where λ is a fixed member of $\mathbf{T}$. Here $[(m, n), (m', n')] = (\lambda^{m'n(n-1)/2}, \lambda^{nm'})$, and the automorphism determined by (m, n) is the map $(z_1, z_2) \mapsto (z_1 z_2^n, z_2)$. By Theorem 4.6, $(\mathbf{T}^2 \times \mathbf{Z}^2)^D \cong \mathbf{T}^2 \times \mathbf{Z}^{2D}$ (SEF). Similarly, $(\mathbf{T} \times \mathbf{Z}^2)^D \cong \mathbf{T} \times \mathbf{Z}^{2D}$ (SEF) for the subgroup $\mathbf{T} \times \mathbf{Z}^2$ of the group G in Example 3.5.

Example 4.10. Let N be an abelian topological group, K a topological group, and let $\psi : K \to N$ be a continuous function such that $\psi(e) = e$. Then $[t, t'] = \psi(t)\psi(t')\psi(tt')^{-1}$ satisfies the cocycle identity in (SEF); in fact, it is a coboundary. Hence, taking $t(\cdot)$ to be the identity map, we have $G = N \times K$ (SEF).

Now take $N = \mathbf{T}$ and $K = \mathbf{Z}$. By Corollary 3.3, $(\mathbf{T} \times \mathbf{Z})^{LC} \cong \mathbf{T} \times \beta(\mathbf{Z})$, and if ψ is distal, then Lemma 4.5 implies that $(\mathbf{T} \times \mathbf{Z})^D \cong \mathbf{T} \times \mathbf{Z}^D$ (SEF). On the other hand, $\chi(z, t) = z\psi(t)$ defines a continuous character of G such that $q_2^*(\chi) = \psi$, so if $\mathcal{P}$ is a property of compactifications such that $\mathcal{P}(G)$ contains all continuous characters and $\psi \notin \mathcal{P}(K)$, then $q_2^*(\mathcal{P}(G)) \not\subset \mathcal{P}(K)$ and hence $G^{\mathcal{P}}$

cannot be isomorphic to $N' \times K'$ (SEF) for any compactifications N' and K', Theorem 2.1. For example, if we take $\psi(n) = e^{in/(1+n^2)}$, then $(\mathbf{T} \times \mathbf{Z})^D \ncong \mathbf{T}' \times \mathbf{Z}'$ (SEF).

5. The Furstenberg structure of G^{DLC}. A refinement, due to Milnes and Pym [**10**, Theorem 11], of Namioka's version [**11**] of the Furstenberg structure theorem [**3, 2**] asserts the existence of an ordinal ξ_0 and a family $\{V_\xi : 0 \le \xi \le \xi_0\}$ of closed normal subgroups of G^{DLC} such that the following hold:

(a) $V_0 = G^{DLC}$ and $V_{\xi_0} = \{e\}$;

(b) for $\xi < \xi_0$, $V_{\xi+1}$ is a subset of V_ξ, and the function

$$(uV_{\xi+1}, vV_{\xi+1}) \longmapsto uvV_{\xi+1}, \quad G^{DLC}/V_{\xi+1} \times V_\xi/V_{\xi+1} \longrightarrow G^{DLC}/V_{\xi+1}$$

is continuous in the Hausdorff quotient topologies;

(c) for each limit ordinal $\xi \le \xi_0$, $V_\xi = \bigcap_{\nu<\xi} V_\nu$.

One can restate this result in terms of the compactifications $G_\xi := G^{DLC}/V_\xi$ of G as follows:

(a$'$) $G_0 = \{e\}$ and $G_{\xi_0} = G^{DLC}$;

(b$'$) for $\xi < \xi_0$, G_ξ is a factor of $G_{\xi+1}$ with compactification homomorphism φ_ξ, and $G_{\xi+1}$ satisfies the *G_ξ-relative joint continuity property*: $u_\alpha v_\alpha \to uv$ for any pair of nets $u_\alpha \to u$ and $v_\alpha \to v$ in $G_{\xi+1}$ with $\varphi_\xi(v_\alpha) = e$;

(c$'$) for each limit ordinal $\xi \le \xi_0$, the compactification G_ξ is the projective limit of the compactifications G_ν, $\nu < \xi$.

One may further assume that $G_{\xi+1}$ is maximal with respect to the relative joint continuity property in (b$'$), in which case we shall say that $G_{\xi+1}$ is the *G_ξ-relatively almost periodic right topological group compactification* (or simply, *r.a.p. compactification*) of G and call the system of compactifications $\{G_\xi : \xi \le \xi_0\}$ of G an *r.a.p. chain* for G^{DLC}. The algebra $AP_{G_\xi}(G) := \varepsilon^*_{G_{\xi+1}}(C(G_{\xi+1}))$ is the space of $\varepsilon^*_{G_\xi}(C(G_\xi))$-relatively almost periodic functions considered in [**8**], see also [**7**].

In this section we show that, under suitable conditions, a r.a.p. chain $\{G_\xi : \xi \le \xi_0\}$ for G^{DLC} is necessarily of the form $\{N_\xi \times K : \xi \le \xi_0\}$ if K is compact, and $\{N \times K_\xi : \xi \le \xi_0\}$ if N is compact, where $\{N_\xi\}$ and

$\{K_\xi\}$ are r.a.p. chains for N^{DLC} and K^{DLC}, respectively. We begin with the case K compact.

Lemma 5.1. *Let K be compact, $\{t \mapsto t(s) : s \in N\}$ equicontinuous, and suppose that $G^{DLC} = N^{DLC} \times K$ (SEF). Let G' be a right topological group compactification of G such that $G' \cong N' \times K$ (SEF) for some compactification N' of N. Then $G^\sharp \cong N^\sharp \times K$ (SEF), where $N^\sharp$ is the N'-r.a.p. compactification of N and $G^\sharp$ is the G'−r.a.p. compactification of G.*

Proof. We use Theorem 2.1 with $F = AP_{G'}(G)$, $A = AP_{N'}(N)$ and $B = C(K)$ to show that $G^\sharp \cong N^\sharp \times K$ (SEF). It is enough to verify 2.1 (a), the remaining conditions of Theorem 2.1 being trivial. We have the compactification homomorphisms

$$G^{DLC} \xrightarrow{\varphi} G^\sharp \xrightarrow{\phi} G', \qquad N^{DLC} \xrightarrow{\psi} N^\sharp \xrightarrow{\theta} N',$$

where $\phi \circ \varphi = (\theta \circ \psi) \times \mathrm{id}_K$. Condition 2.1(a) will hold if, for nets $(x_\alpha, t_\alpha) \to (x, t)$ and $(x'_\alpha, t'_\alpha) \to (x', t')$ in G^{DLC} with $(\phi \circ \varphi)(x'_\alpha, t'_\alpha) = (e, e)$, we have

$$\psi(x_\alpha t_\alpha(x'_\alpha)[t_\alpha, t'_\alpha]) \longrightarrow \psi(xt(x')[t, t']). \tag{3}$$

Now by equicontinuity, $t_\alpha(x'_\alpha) \to t(x)$. Moreover,

$$(\theta \circ \psi)(t_\alpha(x'_\alpha)) = t_\alpha((\theta \circ \psi)(x'_\alpha)) = t_\alpha(e) = e.$$

Since $t'_\alpha = t' = e$, (3) follows immediately from the N'-relative joint continuity property of $N^\sharp$. Therefore, Theorem 2.1(a) holds and $G^\sharp \cong N^\sharp \times K$ (SEF). □

Theorem 5.2. *Let K be compact and $\{t \mapsto t(s) : s \in N\}$ equicontinuous. Suppose that either G is a central extension or $LC(N) = RC(N)$. If $\{G_\xi : \xi \leq \xi_0\}$ is an r.a.p. chain for G^{DLC}, then there exists a r.a.p. chain $\{N_\xi : \xi \leq \xi_0\}$ for N^{DLC} such that $G_\xi \cong N_\xi \times K$ (SEF).*

Proof. By Theorem 4.3 and Lemma 5.1, $G_\xi \cong N_\xi \times K$ (SEF) where, for $\xi < \xi_0$, $N_{\xi+1}$ is the N_ξ-r.a.p. compactification of N. It remains to show that, if $\xi \le \xi_0$ is a limit ordinal then N_ξ is the projective limit of the system $\{(N_\nu, \phi_{\nu\eta}) : \nu < \eta < \xi\}$, where $\phi_{\nu\eta} : N_\eta \to N_\nu$ is the compactification homomorphism. But this follows easily from the fact that G_ξ is the projective limit of the system $\{(G_\nu, \phi_{\nu\eta} \times \mathrm{id}_K) : \nu < \eta < \xi\}$. □

The case N compact follows directly from Theorem 4.6 and the next lemma, which may be proved by arguing as in Lemma 5.1.

Lemma 5.3. *Let N be compact and suppose that $G^{DLC} \cong N \times K^{DLC}$ (SEF). Let G' be a right topological group compactification of G such that $G' \cong N \times K'$ (SEF) for some compactification K' of K. Then $G^\sharp \cong N \times K^\sharp$ (SEF), where $K^\sharp$ denotes the K'-r.a.p. compactification of K and $G^\sharp$ the G'-r.a.p. compactification of G.*

Theorem 5.4. *Suppose that N is compact, $[\cdot, K]$ is equicontinuous, the action of K on N is distal, and $[t, \cdot]$ is a homomorphism for each $t \in K$. If $\{G_\xi : \xi \le \xi_0\}$ is a r.a.p. chain for G^{DLC}, then there exists a r.a.p. chain $\{K_\xi : \xi \le \xi_0\}$ for K^{DLC} such that $G_\xi \cong N \times K_\xi$ (SEF).*

6. Point distal compactifications. A function $f \in LMC(G)$ is *point distal* if, under the action of right translation $R(x)$, the pointwise closure of $R(G)f$ is a point distal flow with distal point f. This is equivalent to the function $\hat{f} = (\varepsilon^*_{G^{LMC}})^{-1}(f)$ satisfying

$$\hat{f}(uv) = \hat{f}(u), \quad u, v = v^2 \in G^{LMC}$$

[**1**]. The algebra of point distal functions on G is denoted by $PD(G)$, and we set $PDLC(G) := PD(G) \cap LC(G)$. Since $PDLC(G)$ is left translation invariant, its spectrum G^{PDLC}, while generally not a semigroup, is a flow under $(s,t)x = L(x,t)^*(x)$. Viewing G^{LC} similarly, we see that the canonical map $\varphi : G^{LC} \to G^{PDLC}$ is equivariant and satisfies

$$\varphi(uv) = \varphi(u), \quad u, v = v^2 \in G^{LC}. \tag{4}$$

The flow G^{PDLC} can be characterized as the maximal such factor of G^{LC}. It is also the universal jointly continuous point distal G-flow.

We shall call a continuous function ψ from G^{LC} to a compact Hausdorff topological space X *point distal* if it has property (4) or, equivalently, if $g \circ \psi \circ \varepsilon_{G^{LC}} \in PDLC(G)$ for every $g \in C(X)$. For such a map there always exists a continuous function $\bar{\psi} : G^{PDLC} \to X$ such that $\bar{\psi} \circ \varphi = \psi$.

The following is a point distal analog of Theorem 4.3.

Theorem 6.1. *Let G be a central extension with K compact and $\{t \mapsto t(s) : s \in N\}$ equicontinuous. Then there exists a natural action of G on $N^{PDLC} \times K$ such that G^{PDLC} and $N^{PDLC} \times K$ are canonically isomorphic as flows.*

Proof. For brevity, we write F for $PDLC$. By Theorem 3.1, $G^{LC} \cong N^{LC} \times K$ (SEF). Let $(s, x) \to s \cdot x : N \times N^F \to N^F$ denote the action of the flow (N, N^F), and let $\theta : N^{LC} \to N^F$ be the canonical equivariant map. Since $t(\cdot) : N \to N$ is a homomorphism, it has an extension $t(\cdot) : N^F \to N^F$ which satisfies $\theta(t(\cdot)) = t(\theta(\cdot))$ and $t(s \cdot x) = t(s) \cdot t(x)$, $s \in N$, $x \in N^F$. Now define an action of G on $N^F \times K$ by

$$(s,t)(x,y) = (s[t,y] \cdot t(x), ty), \quad (s,t) \in G, x \in N^F, y \in K.$$

That this is indeed an action follows from centrality and the cocycle identity:

$$\begin{aligned}((s,t)(s',t'))(x,y) &= (st(s')[t,t'], tt')(x,y) \\ &= (st(s')[t,t'][tt',y] \cdot (tt')(x), (tt')y) \\ &= (s[t,t'y]t(s'[t',y]) \cdot t(t'(x)), (tt')y) \\ &= (s[t,t'y]t(s'[t',y] \cdot t'(x)), t(t'y)) \\ &= (s,t)(s'[t',y] \cdot t'(x), t'y) \\ &= (s,t)((s',t')(x,y)).\end{aligned}$$

Since the map $(x,y) \mapsto (\theta(x), y) : G^{LC} \to N^F \times K$ is equivariant and point distal, $N^F \times K$ is a factor of G^F. Let $\eta : G^F \to N^F \times K$ denote the canonical map. We need to show that η is injective.

Define $r_t : G \to G$ by $r_t(s', t') = (s', t)$, and let $\bar{r}_t : G^{LC} \to G^{LC}$ be its extension, $\bar{r}_t(x,y) = (x,t)$. If $u = (x,y)$ and $v^2 = v = (x', y')$ are members of G^{LC}, then $x'^2 = x'$ and $y' = e$, so $w := (t^{-1}(y(x')), e)$ is an idempotent and, by (4),

$$\varphi \circ \bar{r}_t(uv) = \varphi(xy(x'), t) = \varphi((x,t)w) = \varphi(x,t) = \varphi \circ \bar{r}_t(u).$$

Thus, $r_t^*(F(G)) \subset F(G)$, cf. Lemma 4.1, and we get an extension $\tilde{r}_t : G^F \to G^F$ of r_t. It is easy to see that

$$\tilde{r}_t(\varphi(x,e)) = \varphi(x,t), \quad x \in N^{LC}, \quad t \in K. \tag{5}$$

Moreover, since $\varphi(\cdot, e) : N^{LC} \to G^F$ is point distal and equivariant, relative to the obvious action of N on G^F, the maximality of the factor N^F of N^{LC} implies the existence of an equivariant map $\gamma : N^F \to G^F$ such that $\gamma \circ \theta = \varphi(\cdot, e)$. The injectivity of η follows from the last identity, from (5), and from the identity $\eta \circ \varphi = \theta \times \mathrm{id}_K$. □

The point distal analog of Theorem 4.6 takes the following form.

Theorem 6.2. *Let G be a central extension with N compact. If $[\cdot, K]$ is equicontinuous, $[t, \cdot]$ is a homomorphism for each $t \in K$, and the action of K on N is distal, then there is an action of G on $N \times K^{PDLC}$ such that G^{PDLC} and $N \times K^{PDLC}$ are canonically isomorphic as flows.*

Proof. By Theorem 3.2, $G^{LC} \cong N \times K^{LC}$ (SEF). Let $\theta : K^{LC} \to K^F$ denote the canonical map, where $F = PDLC$. Consider the Schreier map $[\cdot,\cdot] : K^{LC} \times K^{LC} \to N$. For a fixed t, $[\varepsilon_{K^{LC}}(t), \cdot] : K^{LC} \to N$ is a homomorphism and is therefore point distal. Hence there exists a map $\psi_t : K^F \to N$ such that $\psi_t \circ \theta = [\varepsilon_{K^{LC}}(t), \cdot]$. This, together with the centrality of the extension, implies that

$$(s,t) \cdot (x,y) := (st(x)\psi_t(y), t \cdot y), \quad (s,t) \in G, \ x \in N, \ y \in K^F$$

defines an action of G on $N \times K^F$, where $t \cdot y := L(t)^*(y)$. Moreover, the function $\mathrm{id}_N \times \theta : G^{LC} \to N \times K^F$ is equivariant, and since $y(\cdot) = \mathrm{id}_N$ for idempotents $y \in K^{LC}$ (because the flow (K, N) is distal) $\mathrm{id}_N \times \theta : G^{LC} \to N \times K^F$ is point distal. By maximality, there exists an equivariant map $\eta : G^F \to N \times K^F$ such that $\eta \circ \varphi = \mathrm{id}_N \times \theta$. To complete the proof, we show that η is injective.

Now, since φ is point distal, so is $\varphi_s := \varphi(s,\cdot) : K^{LC} \to G^F$. Indeed, if y and y' are members of K^{LC} with $y'^2 = y'$ then, because $[y,\cdot]$ is a homomorphism, (e,y') is an idempotent and $(s,y)(e,y') = (s[y,y'],yy') = (s,yy')$, so $\varphi_s(yy') = \varphi_s(y)$. As a consequence, there exists an extension $\bar{\varphi}_s : K^F \to G^F$ such that $\bar{\varphi}_s \circ \theta = \varphi_s$, and the injectivity of η easily follows. □

Corollary 6.3. *Let G be a split extension with N compact. If the action of K on N is distal, then G^{PDLC} and $N \times K^{PDLC}$ are canonically isomorphic as flows.*

The distal compactification results of Examples 4.4, 4.8, 4.9 and 4.10 have straightforward point distal analogs.

Theorem 6.2 and Corollary 6.3 have analogs for the universal jointly continuous minimal G-flow G^M, which may be characterized as the spectrum of a maximal left translation invariant subalgebra of *minimal functions*, functions $f \in LC(G)$ with the property that the pointwise closure of $R(G)f$ is a minimal flow. G^M may also be realized as $G^{LC}v$, where v is a minimal idempotent, i.e., an idempotent in the minimal ideal of G^{LC}. Under the hypotheses of Theorem 6.2, $G^{LC} = N \times K^{LC}$ (SEF) and v is of the form (e,y), where y is a minimal idempotent of K^{LC}, so $G^{LC}v = N \times K^{LC}y$. Thus, we have

Theorem 6.4. *Under the hypotheses of Theorem 6.2 there is an action of G on $N \times K^M$ such that G^M and $N \times K^M$ are canonically isomorphic as flows.*

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