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SOURCE-TYPE SOLUTIONS TO POROUS MEDIUM EQUATIONS WITH CONVECTION II

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ABSTRACT. Existence and uniqueness of nonnegative source-type solutions to porous medium equations with convection are investigated in one space dimension. We consider the case when both diffusion and convection nonlinearities are pure powers, namely, r^m, $m > 1$, and r^q, $q \in (1, m)$, respectively. Results on the behavior of the source-type solution for small times and estimates of its support are also provided.

1. Introduction. We investigate existence and uniqueness of nonnegative source-type solutions to degenerate convection-diffusion equations in one space dimension. More precisely, we consider the following problem:

$$u_t + (u^q)_x - (u^m)_{xx} = 0 \quad \text{in} \quad \mathbf{R} \times (0, +\infty), \tag{1.1}$$

with initial data

$$u(0) = M\delta, \tag{1.2}$$

where M is a positive real number, δ denotes the Dirac mass at $x = 0$, and q and m are nonnegative real numbers satisfying

$$m > 1, \quad q \in (1, m). \tag{1.3}$$

This paper is a continuation of [**10**], where existence and uniqueness of the nonnegative source-type solution to (1.1)–(1.2) are proved for $m > 1$ and $q \geq m$, while the case $m = 1$ and $q > 1$ is completely solved in [**3**]. These investigations were motivated by the study of the long-time behavior of the solutions to (1.1) with nonnegative and integrable initial data u_0 satisfying $|u_0|_{L^1} = M$. Indeed, when $q = m+1$, the long-time profile of these solutions is described by the source-type solution

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to (1.1)–(1.2) which turns out to be self-similar in this particular case, [**4, 11**]. However, the methods used in [**3**] and [**10**] do not seem to apply when (m, q) are in the whole range given by (1.3). It is the purpose of this paper to prove existence and uniqueness of the source-type solution to (1.1)–(1.2) when (m, q) satisfies (1.3).

We now state precisely the definition of a source-type solution to (1.1) we use in this paper.

Definition 1.1. Let M be a positive real number. A source-type solution of mass M to (1.1) is a function

$$u \in \mathcal{C}((0,+\infty), L^1(\mathbf{R})) \cap \mathcal{C}(\mathbf{R}\times(0,+\infty)) \cap L^\infty(\mathbf{R}\times(\tau,+\infty)), \quad \tau > 0,$$

such that, for each $\tau > 0$, $t \mapsto u(t+\tau)$ is a mild solution to (1.1) in the sense of the nonlinear semigroups theory in $L^1(\mathbf{R})$ and

$$\lim_{t\to 0} \int u(x,t)\zeta(x)\,dx = M\zeta(0), \tag{1.4}$$

for each $\zeta \in \mathcal{C}_b(\mathbf{R})$ (here, $\mathcal{C}_b(\mathbf{R})$ denotes the space of bounded and continuous functions in $\mathbf{R}$).

Let $M > 0$ and consider a source-type solution u of mass M to (1.1) in the sense of Definition 1.1. Then $u(\tau)$ belongs to $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ for each $\tau > 0$, and it follows from [**2**, Théorème 2.2] that u satisfies (1.1) in $\mathcal{D}'(\mathbf{R}\times(\tau,+\infty))$. This fact and (1.4) yield that

$$\int u(x,t)\,dx = M, \quad t > 0. \tag{1.5}$$

In order to state our results, we need the following notation. For $M > 0$, we denote by E_M the source-type solution of mass M to the porous medium equation, namely, the solution to

$$E_{Mt} - (E_M^m)_{xx} = 0 \quad \text{in } \mathbf{R}\times(0,+\infty), \tag{1.6}$$

$$\lim_{t\to 0} \int E_M(x,t)\zeta(x)\,dx = M\zeta(0), \quad \zeta \in \mathcal{C}_b(\mathbf{R}), \tag{1.7}$$

see [**7**] and the references therein.

Our main result then reads

Theorem 1.2. *Let M be a positive real number and (m, q) be real numbers satisfying* (1.3). *There exists a unique nonnegative source-type solution S_M of mass M to* (1.1). *In addition,*

$$\lim_{t\to 0} |S_M(t) - E_M(t)|_{L^1} = 0, \tag{1.8}$$

where E_M is given by (1.6)–(1.7).

Remark 1.3. In fact, (1.8) also holds true for $m > 1$ and $q \in [m, m+1)$, see Corollary 3.4 below. As far as we know, the behavior as $t \to 0$ of the source-type solution to (1.1)–(1.2) is not known for $m > 1$ and $q > m + 1$. Nevertheless, some scaling arguments seem to indicate that it is described by the source-type solution to the nonlinear conservation law $z_t + (z^q)_x = 0$.

Next, it follows from [**5**] that solutions to (1.1) with compactly supported nonnegative initial data remain compactly supported through time evolution. Our last result gives an estimate of the size of the support of nonnegative source-type solutions to (1.1).

Proposition 1.3. *Let M be a positive real number and (m, q) real numbers satisfying* (1.3). *We denote by S_M the unique nonnegative source-type solution of mass M to* (1.1), *and put*

$$\xi_i(t) = \inf\{x \in \mathbf{R},\ S_M(x,t) > 0\},$$
$$\xi_s(t) = \sup\{x \in \mathbf{R},\ S_M(x,t) > 0\}.$$

There exist positive real numbers γ_1 and γ_2 depending only on m, q and M such that, for $t > 0$,

$$-\gamma_1 t^{1/(m+1)} \le \xi_i(t) \le 0 \le \xi_s(t) \le \gamma_2 (t^{1/(m+1)} + t^{(m+2-q)/(m+1)}). \tag{1.9}$$

We now briefly describe the contents of the paper. In Section 2 we recall some basic properties of (1.1) which we use in the sequel and derive an L^∞-estimate for $(u^{m-1})_x$. In Section 3 we use a scaling method to prove that any nonnegative source-type solution of mass M to (1.1) in the sense of Definition 1.1 satisfies (1.8). A similar method was used in [**8**] to identify the behavior as $t \to 0$ of source-type solutions to porous medium equations with absorption. Uniqueness then follows, since (1.1) generates a nonlinear semigroup of contractions in $L^1(\mathbf{R})$. The existence part of Theorem 1.2 is proved in Section 4, and Proposition 1.3 in Section 5.

From now on, we assume that m and q are given real numbers satisfying (1.3).

2. Preliminaries. The well-posedness of the Cauchy problem for (1.1) has been investigated by several authors. If $u(0) \in L^1(\mathbf{R})$, (1.1) has a unique mild solution u in the sense of the nonlinear semigroups theory in $L^1(\mathbf{R})$ [**2**]. Existence and uniqueness of generalized solutions to (1.1) have also been obtained when $u(0)$ is a nonnegative continuous and bounded function, see [**6**] and the references therein. In fact, both notions of solution coincide, provided $u(0)$ lies in a suitable class of functions.

Hereafter, we shall work with mild solutions. The following result is a consequence of [**2**].

Proposition 2.1. *For each $u_0 \in L^1(\mathbf{R})$, there is a unique mild solution $u \in \mathcal{C}([0,+\infty), L^1(\mathbf{R}))$ to* (1.1) *in the sense of the nonlinear semigroups theory with $u(0) = u_0$, which we denote by $t \mapsto S_t u_0$, $t \geq 0$. Moreover, if $u_0 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $t \mapsto S_t u_0$ satisfies* (1.1) *in $\mathcal{D}'(\mathbf{R} \times (0,+\infty))$, and*

$$\int (S_t u_0)(x)\, dx = \int u_0(x)\, dx, \tag{2.1}$$

$$|S_t u_0|_{L^\infty} \leq |u_0|_{L^\infty}. \tag{2.2}$$

Finally, if $u_0 \in L^1(\mathbf{R})$ and $\hat{u}_0 \in L^1(\mathbf{R})$, the following holds

$$|(S_t u_0 - S_t \hat{u}_0)^+|_{L^1} \leq |(u_0 - \hat{u}_0)^+|_{L^1}, \quad t \geq 0, \tag{2.3}$$

where $r^+ = \max(r, 0)$.

It follows from (2.3) that, if $u_0 \in L^1(\mathbf{R})$ is nonnegative, $S_t u_0 \geq 0$ for $t \geq 0$.

We next recall an L^∞-estimate for mild solutions to (1.1) [**11**, Section 2.2].

Lemma 2.2. *Let u_0 be a nonnegative function in $L^1(\mathbf{R})$ and put $u(t) = S_t u_0$, $t \geq 0$. There exists a positive constant κ_0 $(|u_0|_{L^1})$ depending only on m and $|u_0|_{L^1}$ such that*

$$0 \leq u(x,t) \leq \kappa_0(|u_0|_{L^1}) t^{-1/(m+1)}, \quad (x,t) \in \mathbf{R} \times (0,+\infty). \tag{2.4}$$

We next derive an L^∞-bound for $(u^{m-1})_x$.

Lemma 2.3. *Let u_0 be a nonnegative function in $L^1(\mathbf{R})$, and put $u(t) = S_t u_0$, $t \geq 0$. There exists a positive constant κ_1 $(|u_0|_{L^1})$ depending only on m, q and $|u_0|_{L^1}$ such that, for $t > 0$,*

$$\begin{aligned} -\kappa_1(|u_0|_{L^1}) t^{-m/(m+1)} &\leq V(x,t) \\ &\leq \kappa_1(|u_0|_{L^1})(t^{-m/(m+1)} + t^{-(q-1)/(m+1)}), \end{aligned} \tag{2.5}$$

for almost every $x \in \mathbf{R}$, where

$$V(x,t) = -\left(\frac{m}{m-1}(u^{m-1})_x - u^{q-1}\right)(x,t),$$

and

$$|(u^{m-1})_x(t)|_{L^\infty} \leq \kappa_1(|u_0|_{L^1})(1 + t^{(m+1-q)/(m+1)}) t^{-m/(m+1)}, \tag{2.6}$$

$$|(u^m)_x(t)|_{L^\infty} \leq \kappa_1(|u_0|_{L^1})(1 + t^{(m+1-q)/(m+1)}) t^{-1}. \tag{2.7}$$

The L^∞-estimate of $(u^{m-1})_x$ similar to (2.6) is well-known for the porous medium equation, see, e.g., [**1**], and is obtained in [**12**] for (1.1) when $q \geq m+1$. However, their proofs are different from the one we give below.

Proof of Lemma 2.3. The proof of (2.5) relies on a modification of the Bernstein technique due to Bénilan, which has been further developed in [**6**] and [**14**]. In the following, κ_* denotes any positive constant depending only on m.

Step 1. We first consider a smooth and bounded nonnegative function v_0 satisfying $0 < \eta \leq v_0(x)$, $x \in \mathbf{R}$, for some $\eta > 0$. Classical results then ensure the existence of a unique classical solution v to (1.1), satisfying $v(0) = v_0$ and

$$0 < \eta \leq v(x,t) \leq |v_0|_{L^\infty}, \quad (x,t) \in \mathbf{R} \times [0,+\infty).$$

We put

$$w(x,t) = \frac{(v^m)_x(x,t) - v^q(x,t)}{k(v(x,t))}, \quad (x,t) \in \mathbf{R} \times [0,+\infty),$$

where $k(r) = 2|v_0|_{L^\infty}^{m-1} r - r^m$, $r \geq 0$. Then $k(v) > 0$ and

$$\mathcal{L}w = 0 \quad \text{in } \mathbf{R} \times (0,+\infty),$$

where $\mathcal{L}$ is the parabolic operator given by

$$\begin{aligned}
\mathcal{L}p = p_t &- m v^{m-1} p_{xx} - (2k'(v) + (m-1)k(v)v^{-1}) p p_x \\
&- \left(2v^q \frac{k'}{k}(v) + (m-1-q)v^{q-1} \right) p_x \\
&- \frac{k(v)k''(v)}{m} v^{1-m} p^3 - \frac{2}{m} k''(v) v^{q+1-m} p^2 \\
&- \frac{k''(v)}{mk(v)} v^{2q+1-m} p.
\end{aligned}$$

We first estimate w from above. For that purpose, we put

$$W_1(t) = C_1 t^{-1/2}, \qquad C_1 = (2(m-1)|v_0|_{L^\infty}^{m-1})^{-1/2}.$$

Since $v, k(v)$, W_1 are nonnegative and $k''(v)$ is negative, the choice of C_1 yields $\mathcal{L}W_1 \geq 0$ in $\mathbf{R} \times (0,+\infty)$. Moreover, since $W_1(0) = +\infty$, the comparison principle yields

$$w(x,t) \leq W_1(t), \quad (x,t) \in \mathbf{R} \times (0,+\infty),$$

hence, for $(x,t) \in \mathbf{R} \times (0,+\infty)$,

$$\left(\frac{m}{m-1}(v^{m-1})_x - v^{q-1}\right)(x,t) \le \kappa_* |v_0|_{L^\infty}^{(m-1)/2} t^{-1/2}. \tag{2.8}$$

Next, let $T > 0$, and put

$$W_2(t) = -C_2 t^{-1/2},$$
$$C_2 = 4T^{1/2}|v_0|_{L^\infty}^{q-m} + (m-1)^{-1/2}|v_0|_{L^\infty}^{(1-m)/2}.$$

Then, if $t \in (0,T)$, we have

$$\begin{aligned}\mathcal{L}W_2 &\le \frac{C_2}{2}(1 + 4C_2(m-1)v^{q-1}t^{1/2} - 2C_2^2(m-1)v^{-1}k(v))t^{-3/2} \\ &\le \frac{C_2}{2}(1 + 4C_2(m-1)|v_0|_{L^\infty}^{q-1}T^{1/2} \\ &\quad - 2C_2^2(m-1)|v_0|_{L^\infty}^{m-1})t^{-3/2} \le 0.\end{aligned}$$

Since $W_2(0) = -\infty$, the comparison principle yields

$$w(x,t) \ge W_2(t), \quad (x,t) \in \mathbf{R} \times (0,T);$$

hence, for $(x,t) \in \mathbf{R} \times (0,T)$,

$$\begin{aligned}&\left(\frac{m}{m-1}(v^{m-1})_x - v^{q-1}\right)(x,t) \\ &\qquad \ge -\kappa_*(T^{1/2}|v_0|_{L^\infty}^{q-1} + |v_0|_{L^\infty}^{(m-1)/2})t^{-1/2}.\end{aligned}$$

Now, for $t > 0$, we choose $T = 4t > t$. The above estimate then yields

$$\begin{aligned}&\left(\frac{m}{m-1}(v^{m-1})_x - v^{q-1}\right)(x,t) \\ &\qquad \ge -\kappa_*(|v_0|_{L^\infty}^{q-1} + |v_0|_{L^\infty}^{(m-1)/2} t^{-1/2}),\end{aligned} \tag{2.9}$$

for $x \in \mathbf{R}$.

Step 2. We now consider a nonnegative function $u_0 \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ such that $u_0^m \in W^{1,\infty}_{\text{loc}}(\mathbf{R})$ and $((u_0^m)_x - u_0^q) \in W^{1,1}(\mathbf{R})$ and put

$u(t) = S_t u_0$. It follows from [**2**] that u is nonnegative, continuous and bounded in $\mathbf{R} \times [0, +\infty)$, and u is in fact the generalized solution to (1.1) in the sense of [**6**].

Let $\tau > 0$. We consider the approximation of $u(\tau)$ by the sequence $(u_{0,k})_{k\geq 1}$ of nonnegative and smooth functions which is constructed in [**6**, pp. 182–183]. Among other properties, $(u_{0,k})$ converges to $u(\tau)$ in $\mathcal{C}(\mathbf{R})$ and $0 < k^{-1} \leq u_{0,k} \leq 2|u(\tau)|_{L^\infty}$. We denote by u_k the classical solution to (1.1) with initial data $u_{0,k}$. On the one hand, it follows from [**6**] that $(u_k(x,t))$ converges to $u(x,t+\tau)$ for any $(x,t) \in \mathbf{R} \times [0, +\infty)$. On the other hand, we infer from (2.8) and (2.9) that, for each $k \geq 1$, it holds, for $(x,t) \in \mathbf{R} \times (0, +\infty)$,

$$-\kappa_*(|u(\tau)|_{L^\infty}^{q-1} + |u(\tau)|_{L^\infty}^{(m-1)/2} t^{-1/2}) \leq \left(\frac{m}{m-1}(u_k^{m-1})_x - u_k^{q-1}\right)(x,t),$$

$$\kappa_*|u(\tau)|_{L^\infty}^{(m-1)/2} t^{-1/2} \geq \left(\frac{m}{m-1}(u_k^{m-1})_x - u_k^{q-1}\right)(x,t).$$

We then let $k \to +\infty$ in the above estimate to obtain, for any $t > \tau$ and almost every $x \in \mathbf{R}$,

$$-\kappa_*(|u(\tau)|_{L^\infty}^{q-1} + |u(\tau)|_{L^\infty}^{(m-1)/2} (t-\tau)^{-1/2}) \leq \left(\frac{m}{m-1}(u^{m-1})_x - u^{q-1}\right)(x,t),$$

$$\kappa_*|u(\tau)|_{L^\infty}^{(m-1)/2} (t-\tau)^{-1/2} \geq \left(\frac{m}{m-1}(u^{m-1})_x - u^{q-1}\right)(x,t).$$

Finally, for $t > 0$, we take $\tau = t/2$ in the above formula and use (2.4) to obtain (2.5). Then (2.6) and (2.7) are straightforward consequences of (2.4) and (2.5).

Step 3. The general case $u_0 \in L^1(\mathbf{R})$, $u_0 \geq 0$, then follows from Step 2 and [**2**, Théorème 1.3] by a density argument. □

3. Behavior as $t \to 0$.

Proposition 3.1. *Consider $M > 0$ and let u be a nonnegative source-type solution of mass M to* (1.1)*, in the sense of Definition* 1.1*. Then,*

$$\lim_{t \to 0} |u(t) - E_M(t)|_{L^1} = 0, \tag{3.1}$$

where E_M is given by (1.6)–(1.7)*.*

Proof of Proposition 3.1. For $\lambda \in (0,1)$, we put

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^{m+1} t), \quad (x,t) \in \mathbf{R} \times (0, +\infty).$$

Then, for each $\tau > 0$, $t \mapsto u_\lambda(t+\tau)$ is the mild solution to

$$u_{\lambda t} + \lambda^{m+1-q} (u_\lambda^q)_x - (u_\lambda^m)_{xx} = 0 \quad \text{in} \quad \mathbf{R} \times (0, +\infty) \tag{3.2}$$

with initial data $u_\lambda(\tau)$ and satisfies, thanks to (1.5),

$$|u_\lambda(t)|_{L^1} = M, \quad t > 0. \tag{3.3}$$

In the following we denote by $(C_i)_{i \geq 1}$ any positive constant depending only on m, q and M. Additional dependence on other parameters will be indicated explicitly.

Since $\lambda \in (0,1)$, we infer from (1.3), Lemma 2.2 and Lemma 2.3 that

$$0 \leq u_\lambda(x,t) \leq C_1 t^{-1/(m+1)}, \quad (x,t) \in \mathbf{R} \times (0, +\infty), \tag{3.4}$$

$$|(u_\lambda^m)_x(t)|_{L^\infty} \leq C_1 t^{-1} (1 + t^{(m+1-q)/(m+1)}), \quad t > 0. \tag{3.5}$$

We next use (3.5) and parabolic regularity results of [**9**] to obtain Hölder estimates in time in $L^1_{\text{loc}}(\mathbf{R})$.

Lemma 3.2. *For each $t_1 > 0$, $t_2 \in (t_1, +\infty)$ and $R > 0$, there exists $C_2(R, t_1, t_2)$ such that, for each $\lambda \in (0,1)$, $h \in (0,1)$ and $t \in [t_1, t_2 - h]$, the following holds*

$$\int_{-R}^{R} |u_\lambda^m(x, t+h) - u_\lambda^m(x,t)| \, dx \leq C_2(R, t_1, t_2) h^{1/(2m+1)}. \tag{3.6}$$

Proof of Lemma 3.2. We fix $t_1 > 0$, $t_2 > t_1$ and $R > 0$. We infer from (3.5) that, for $\lambda \in (0,1)$, $h \in \mathbf{R}$ and $t \in [t_1, t_2]$,

$$\int_{-R}^{R} |u_\lambda(x+h,t) - u_\lambda(x,t)|\,dx \le C_2(R,t_1,t_2)|h|^{1/m}. \tag{3.7}$$

Thanks to (3.3), (3.4) and (3.7), we are in a position to apply [**9**, Theorem 2]. We thus obtain for $\lambda \in (0,1)$, $h \in (0,1)$ and $t \in [t_1, t_2]$,

$$\int_{-R}^{R} |u_\lambda(x,t+h) - u_\lambda(x,t)|\,dx \le C_2(R,t_1,t_2)h^{1/(2m+1)}.$$

The above estimate and (3.4) then yield (3.6). □

We next investigate the behavior of u_λ for large values of x. We fix $\rho \in \mathcal{C}^\infty(\mathbf{R})$ such that $0 \le \rho \le 1$,

$$\rho(x) = 0 \quad \text{if } |x| \le 1, \qquad \rho(x) = 1 \quad \text{if } |x| \ge 2,$$

and put $\rho_R(x) = \rho(x/R)$, $R > 0$.

Lemma 3.3. *There exists $C_3 > 0$ such that, for each $R \ge 1$ and $\lambda \in (0,1)$ the following holds*

$$\int u_\lambda(x,t)\rho_R(x)\,dx \le \frac{C_3}{R}(1+t). \tag{3.8}$$

Proof of Lemma 3.3. Let $\lambda \in (0,1)$ and $R \ge 1$. For $t > 0$, we have

$$\int u_\lambda(x,t)\rho_R(x)\,dx = \int u(x,\lambda^{m+1}t)\rho_R(x/\lambda)\,dx.$$

Since $x \mapsto \rho_R(x/\lambda) \in \mathcal{C}_b(\mathbf{R})$, we may let $t \to 0$ in the above equality and use (1.4) to obtain

$$\lim_{t\to 0} \int u_\lambda(x,t)\rho_R(x)\,dx = 0. \tag{3.9}$$

Now let $t > 0$ and $\tau \in (0,t)$. Since $s \mapsto u(s+\tau)$ is a mild solution to (1.1) with initial data $u(\tau)$ in $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, we have

$$\int u_\lambda(x,t)\rho_R(x)\,dx = \int u_\lambda(x,\tau)\rho_R(x)\,dx + \frac{1}{R^2}\int_\tau^t \int u_\lambda^m(x,s)\rho''\left(\frac{x}{R}\right)dx\,ds + \frac{\lambda^{m+1-q}}{R}\int_\tau^t \int u_\lambda^q(x,s)\rho'\left(\frac{x}{R}\right)dx\,ds.$$

It then follows from (1.3), (3.3) and (3.4) that

$$\left|\int u_\lambda(x,t)\rho_R(x)\,dx - \int u_\lambda(x,\tau)\rho_R(x)\,dx\right| \le \frac{C_3}{R}|\rho|_{W^{2,\infty}}(t^{2/(m+1)} + t^{(m+2-q)/(m+1)}).$$

We then let $\tau \to 0$ in the above inequality and use (3.9) to obtain (3.8). □

We are now in a position to complete the proof of Proposition 3.1. We infer from (3.4), (3.5), (3.6), [**13**, Theorem 5] and the compactness of the embedding of $W^{1,2}_{\mathrm{loc}}(\mathbf{R})$ in $\mathcal{C}(\mathbf{R})$ that (u_λ^m) is relatively compact in $\mathcal{C}([-R,R]\times[t_1,t_2])$ for each $R > 0$ and $0 < t_1 < t_2$.

Consequently there is a subsequence $(u_{\lambda'})$ of (u_λ) and a nonnegative function $v \in \mathcal{C}(\mathbf{R}\times(0,+\infty))$ such that $(u_{\lambda'}^m)$ converges to v uniformly on any compact subset of $\mathbf{R}\times(0,+\infty)$ as $\lambda' \to 0$. Consequently, since $m > 1$, it follows from the above analysis that, for $t > 0$,

$$u_{\lambda'}(t) \longrightarrow u_\infty(t) \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}), \tag{3.10}$$

where $u_\infty = v^{1/m} \in \mathcal{C}(\mathbf{R}\times(0,+\infty))$. Owing to (3.4), (3.8) and (3.10), we may apply the Vitali convergence theorem and obtain that, for each $t > 0$,

$$u_{\lambda'}(t) \longrightarrow u_\infty(t) \quad \text{in } L^1(\mathbf{R}). \tag{3.11}$$

Now, let $\tau > 0$. Since $t \mapsto u_{\lambda'}(t+\tau)$ is the mild solution to (3.2) with initial data $u_{\lambda'}(\tau)$ and $r \mapsto (\lambda')^{m+1-q}r$ converges to zero in $\mathcal{C}(\mathbf{R})$ as

$\lambda' \to 0$, we infer from (3.11) and [**2**, Théorème 1.3] that $t \mapsto u_\infty(t+\tau)$ is the mild solution to the porous medium equation $z_t - (z^m)_{xx} = 0$ with initial data $u_\infty(\tau)$.

Finally, consider $\zeta \in \mathcal{D}(\mathbf{R})$. We infer from (1.4) that, for each $\lambda \in (0,1)$,

$$\lim_{t\to 0} \int u_\lambda(x,t)\zeta(x)\,dx = M\zeta(0). \tag{3.12}$$

Let $t \in (0,1)$ and $\tau \in (0,t)$. It follows from (3.2), (3.3) and (3.4) that

$$\left| \int u_{\lambda'}(x,t)\zeta(x)\,dx - \int u_{\lambda'}(x,\tau)\zeta(x)\,dx \right| \le C_4 |\zeta|_{W^{2,\infty}} t^{2/(m+1)}. \tag{3.13}$$

We first let $\tau \to 0$ in (3.13) and use (3.12). We then let $\lambda' \to 0$ in the resulting estimate and use (3.11) to obtain

$$\left| \int u_\infty(x,t)\zeta(x)\,dx - M\zeta(0) \right| \le C_4 |\zeta|_{W^{2,\infty}} t^{2/(m+1)}.$$

Therefore, $u_\infty(0) = M\delta$, and u_∞ is a nonnegative solution to (1.6)–(1.7); hence $u_\infty = E_M$ [**7**]. Since the only possible limit of $(u_\lambda(t))$ as $\lambda \to 0$ is $E_M(t)$ for $t > 0$, we conclude that in fact the whole sequence $(u_\lambda(t))$ converges to $E_M(t)$ in $L^1(\mathbf{R})$ as $\lambda \to 0$ for $t > 0$. Thus,

$$\lim_{\lambda\to 0} |u_\lambda(1) - E_M(1)|_{L^1} = 0. \tag{3.14}$$

We put $\lambda = t^{1/(m+1)}$. Since $t^{-1/(m+1)} E_M(xt^{-1/(m+1)}, 1) = E_M(x,t)$, (3.14) yields (3.1). □

In fact, the above proof is valid when $q \in (1, m+1)$. Also, it follows from [**10**] that (1.1) has a unique nonnegative source-type solution of mass M when $m > 1$ and $q \in [m, m+1)$. We have thus obtained the behavior as $t \to 0$ of this source-type solution.

Corollary 3.4. *Assume that $m > 1$ and $q \in [m, m+1)$. Let $M > 0$ and u be the nonnegative source-type solution of mass M to* (1.1). *Then*

$$\lim_{t\to 0} |u(t) - E_M(t)|_{L^1} = 0.$$

4. Proof of Theorem 1.2.

Uniqueness. Let $M > 0$ and u_1, u_2 be two nonnegative source-type solutions of mass M to (1.1). It follows from (2.1) and (2.3) that, for $t > 0$ and $\tau \in (0, t)$,

$$\begin{aligned}|u_1(t) - u_2(t)|_{L^1} &\le |u_1(\tau) - u_2(\tau)|_{L^1} \\ &\le |u_1(\tau) - E_M(\tau)|_{L^1} + |E_M(\tau) - u_2(\tau)|_{L^1}.\end{aligned}$$

We then let $\tau \to 0$ in the above estimate and use Proposition 3.1. This gives

$$|u_1(t) - u_2(t)|_{L^1} = 0,$$

hence $u_1 = u_2$.

Existence. Let $\varphi \in \mathcal{D}(\mathbf{R})$ be a nonnegative function such that $0 \le \varphi \le 1$, $\operatorname{Supp}\varphi \subset (-1, +1)$ and $|\varphi|_{L^1} = 1$.

Let $M > 0$. For any integer $n \ge 1$, we put

$$u_{0,n}(x) = Mn\varphi(nx), \quad x \in \mathbf{R},$$

and denote by u_n the mild solution to (1.1) with initial data $u_{0,n}$.

In the following we denote by $(C_i)_{i\ge1}$ any positive constant depending only on q, m and M. Additional dependence on other parameters will be indicated explicitly.

Since $|u_{0,n}|_{L^1} = M$, we infer from Proposition 2.1, Lemma 2.2 and Lemma 2.3 that

$$|u_n(t)|_{L^1} = M, \tag{4.1}$$

$$0 \le u_n(x,t) \le C_1 t^{-1/(m+1)}, \quad (x,t) \in \mathbf{R} \times (0, +\infty), \tag{4.2}$$

$$|(u_n^m)_x(t)|_{L^\infty} \le C_1 t^{-1}(1 + t^{(m+1-q)/(m+1)}), \quad t > 0. \tag{4.3}$$

We then proceed as in Lemma 3.2 to prove that, for each $t_1 > 0$, $t_2 \in (t_1, +\infty)$ and $R > 0$, there exists $C_2(R, t_1, t_2)$ such that, for each $n \ge 1$, $h \in (0,1)$ and $t \in [t_1, t_2 - h]$, the following holds

$$\int_{-R}^{R} |u_n^m(x, t+h) - u_n^m(x,t)|\, dx \le C_2(R, t_1, t_2) h^{1/(2m+1)}. \tag{4.4}$$

We next investigate the behavior of u_n for large values of x. We fix $\rho \in \mathcal{C}^\infty(\mathbf{R})$ such that $0 \le \rho \le 1$,

$$\rho(x) = 0 \quad \text{if } |x| \le 1, \qquad \rho(x) = 1 \quad \text{if } |x| \ge 2,$$

and put $\rho_R(x) = \rho(x/R)$, $R > 0$. Proceeding as in Lemma 3.3, we obtain that there exists $C_3 > 0$ such that, for each $R \ge 1$ and $n \ge 1$, the following holds

$$\int u_n(x,t)\rho_R(x)\,dx \le \frac{C_3}{R}(1+t). \tag{4.5}$$

Arguing as in the proof of Proposition 3.1, we deduce from (4.1)–(4.5) that there exists a subsequence of (u_n), which we still denote by (u_n), and a nonnegative function $S_M \in \mathcal{C}(\mathbf{R} \times (0,+\infty))$ such that, for each $t > 0$,

$$u_n(t) \longrightarrow S_M(t) \quad \text{in } L^1(\mathbf{R}), \tag{4.6}$$

and for each $R > 0$,

$$u_n \longrightarrow S_M \quad \text{in } \mathcal{C}([-R,R] \times [1/R,R]). \tag{4.7}$$

Let $\tau > 0$. Since $t \mapsto u_n(t+\tau)$ is the mild solution to (1.1) with initial data $u_n(\tau)$, we infer from (4.6) and [**2**, Théorème 1.3] that $t \mapsto S_M(t+\tau)$ is the mild solution to (1.1) with initial data $S_M(\tau)$.

It remains to prove that S_M satisfies (1.4). We first consider $\zeta \in \mathcal{D}(\mathbf{R})$. Let $t \in (0,1)$. It follows from (1.1), (4.1) and (4.2) that, for each $n \ge 1$,

$$\left| \int u_n(x,t)\zeta(x)\,dx - \int u_{0,n}(x)\zeta(x)\,dx \right| \le C_4 |\zeta|_{W^{2,\infty}} t^{2/(m+1)}. \tag{4.8}$$

We let $n \to \infty$ in (4.8) and use (4.6) to obtain

$$\left| \int S_M(x,t)\zeta(x)\,dx - M\zeta(0) \right| \le C_4 |\zeta|_{W^{2,\infty}} t^{2/(m+1)}.$$

Therefore, for any $\zeta \in \mathcal{D}(\mathbf{R})$,

$$\lim_{t\to 0} \int S_M(x,t)\zeta(x)\,dx = M\zeta(0).$$

The general case $\zeta \in \mathcal{C}_b(\mathbf{R})$ then follows from (4.5) by a density argument. Consequently, S_M is a nonnegative source-type solution of mass M to (1.1), and the proof of Theorem 1.2 is complete. □

5. Estimates of the support of S_M. In this section we prove Proposition 1.3. Let $M > 0$ and S_M be the nonnegative source-type solution of mass M to (1.1). We also consider a nonnegative function $\varphi \in \mathcal{D}(\mathbf{R})$ such that $0 \le \varphi \le 1$, $\operatorname{Supp} \varphi = [-1, +1]$, $|\varphi|_{L^1} = 1$ and

$$\operatorname{sign}(x\varphi'(x)) \le 0, \quad x \in \mathbf{R}. \tag{5.1}$$

For any integer $n \ge 1$, we denote by u_n the mild solution to (1.1) with initial data $u_{0,n}$ given by

$$u_{0,n}(x) = Mn\varphi(nx), \quad x \in \mathbf{R}.$$

It follows from the proof of Theorem 1.2, see (4.7), and the uniqueness of S_M that

$$\lim_{n\to+\infty} u_n(x,t) = S_M(x,t), \quad (x,t) \in \mathbf{R} \times (0,+\infty). \tag{5.2}$$

Let $n \ge 1$. For $t \ge 0$, we put

$$\begin{aligned} P_n(t) &= \{x \in \mathbf{R}, u_n(x,t) > 0\}, \\ \xi_i^n(t) &= \inf P_n(t), \qquad \xi_s^n(t) = \sup P_n(t), \\ V_n(x,t) &= -\left(\frac{m}{m-1}(u_n^{m-1})_x - u_n^{q-1}\right)(x,t), \quad x \in \mathbf{R}. \end{aligned}$$

Since φ is compactly supported, we infer from [**5**, Theorem 1] that, for each $t \ge 0$,

$$-\infty < \xi_i^n(t) \le \xi_s^n(t) < +\infty.$$

We also infer from (5.1) and [**5**, Theorem 4] that, for each $t_1 \geq 0$ and $t_2 > t_1$, the following holds

$$\begin{aligned} \int_{t_1}^{t_2} \liminf_{\substack{x \to \xi_s^n(s) \\ x \in P_n(s)}} V_n(x,s)\,ds &\leq \xi_s^n(t_2) - \xi_s^n(t_1) \\ &\leq \int_{t_1}^{t_2} \limsup_{\substack{x \to \xi_s^n(s) \\ x \in P_n(s)}} V_n(x,s)\,ds. \end{aligned} \tag{5.3}$$

Let $t \geq 0$. It follows from (2.5) and (5.3) with $t_1 = 0$ and $t_2 = t$ that there exists γ_2 depending only on m, q and M such that

$$\xi_s^n(t) \leq \frac{1}{n} + \gamma_2\left(t^{1/(m+1)} + t^{(m+2-q)/(m+1)}\right), \quad t \geq 0. \tag{5.4}$$

In a similar way, we obtain

$$\xi_i^n(t) \geq -\frac{1}{n} - \gamma_1 t^{1/(m+1)}, \quad t \geq 0, \tag{5.5}$$

where γ_1 depends only on m, q and M.

Now let $t > 0$, and consider $x_0 > \gamma_2(t^{1/(m+1)} + t^{(m+2-q)/(m+1)})$. It follows from (5.4) that, for n large enough, $x_0 \geq \xi_s^n(t)$; hence, $u_n(x_0,t) = 0$. Recalling (5.2), we obtain $S_M(x_0,t) = 0$. Consequently,

$$\xi_s(t) \leq \gamma_2\left(t^{1/(m+1)} + t^{(m+2-q)/(m+1)}\right), \quad t > 0. \tag{5.6}$$

Similarly, (5.5) and (5.2) yield

$$\xi_i(t) \geq -\gamma_1 t^{1/(m+1)}, \quad t > 0. \tag{5.7}$$

Finally, let $t > 0$ and $\tau \in (0,t)$. It follows from Theorem 1.2, (5.6) and (5.7) that $S_M(\tau)$ is a nonnegative and compactly supported function in $\mathcal{C}_b(\mathbf{R})$. We then infer from [**5**, Theorem 2] that $s \mapsto \xi_s(s)$ is nondecreasing on $[\tau, +\infty)$. This fact and (5.7) yield

$$\xi_s(t) \geq \xi_s(\tau) \geq -\gamma_1 \tau^{1/(m+1)}.$$

We let $\tau \to 0$ in the above estimate to obtain that $\xi_s(t) \geq 0$ for each $t > 0$. The proof that $\xi_i(t) \leq 0$ for $t > 0$ is similar. The proof of Proposition 1.3 is thus complete. □

REFERENCES

1. S. Benachour and M.S. Moulay, *Régularité des solutions de l'équation des milieux poreux en une dimension d'espace*, C.R. Acad. Sci. Paris **298** (1984), 107–110.

2. Ph. Bénilan and H. Touré, *Sur l'équation générale* $u_t = a(\cdot, u, \varphi(\cdot, u)_x)_x + v$ *dans* L^1, Ann. Inst. Henri Poincaré, Analyse non linéaire **12** (1995), 727–761.

3. M. Escobedo, J.L. Vazquez and E. Zuazua, *Asymptotic behaviour and source-type solutions for a diffusion-convection equation*, Arch. Rational Mech. Anal. **124** (1993), 43–65.

4. M. Escobedo and E. Zuazua, *Large time behaviour for convection-diffusion equations in* $\mathbf{R}^n$, J. Funct. Anal. **100** (1991), 119–161.

5. B.H. Gilding, *The occurrence of interfaces in nonlinear diffusion-advection processes*, Arch. Rational Mech. Anal. **100** (1988), 243–263.

6. ———, *Improved theory for a nonlinear degenerate parabolic equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **16** (1989), 165–224.

7. S. Kamin, *Source-type solutions for equations of nonstationary filtration*, J. Math. Anal. Appl. **64** (1978), 263–276.

8. S. Kamin and L.A. Peletier, *Source-type solutions of degenerate diffusion equations with absorption*, Israel J. Math. **50** (1985), 219–230.

9. S.N. Kruzhkov, *Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications*, Mat. Z. **6** (1969), 97–108; Math. Notes **6** (1969), 517–523, in English.

10. Ph. Laurençot and F. Simondon, *Source-type solutions to porous medium equations with convection*, Comm. Appl. Anal. **1** (1997), 489–502.

11. ———, *Long-time behaviour for porous medium equations with convection*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 315–336 .

12. G. Reyes, *Asymptotic behaviour in convection-diffusion processes*, Nonlinear Anal., to appear.

13. J. Simon, *Compact sets in the space* $L^p(0, T; B)$, Ann. Mat. Pura Appl. **146** (1987), 65–96.

14. F. Simondon, *Strong solutions for* $u_t = \varphi(u)_{xx} - f(t)\psi(u)_x$, Comm. Partial Differential Equations **13** (1988), 1337–1354.

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