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THE TRANSFORMATION FORMULA FOR THE SZEGŐ KERNEL

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ABSTRACT. Let Ω_i be smoothly bounded domains in $\mathbf{C}$, $i = 1, 2$. Then we shall show that the Szegő kernel functions associated to Ω_i transform under proper holomorphic functions and proper holomorphic correspondences from Ω_1 onto Ω_2 via a new formula.

1. Introduction. Suppose that Ω is a bounded domain in $\mathbf{C}$. The Bergman projection associated to the domain Ω is the orthogonal projection P of the space $L^2(\Omega)$ of L^2 functions on Ω onto the closed subspace $H^2(\Omega)$ of holomorphic functions. The Bergman kernel function $K_\Omega(z, w)$ associated to Ω is the reproducing kernel for the Bergman projection P, i.e.,

$$Ph(z) = \int_\Omega K_\Omega(z, w) h(w)\, dV_w \quad \text{for} \quad h \in L^2(\Omega) \quad \text{and} \quad z \in \Omega.$$

Now we suppose that Ω_1 and Ω_2 are two bounded domains in $\mathbf{C}$ and that f is a proper holomorphic mapping of Ω_1 onto Ω_2. (Note that f is said to be proper if the inverse image $f^{-1}(\mathcal{K})$ of a compact subset $\mathcal{K}$ of Ω_2 is a compact subset of Ω_1.) Then it is a well-known fact, the Remmert proper holomorphic mapping theorem (see [**11**]), that f is a branched cover of some finite order m and that the set $\mathcal{V} = \{f(z) |\ \mathrm{Det}\,[f'](z) = 0\}$ is a complex variety in Ω_2. There are m local inverses $F_1, F_2, \cdots, F_m$ to f which are defined locally on $\Omega_2 - \mathcal{V}$. Bell [**2, 4**] proved that the Bergman kernel functions transform under proper holomorphic mappings exactly as under biholomorphic mappings as follows:

$$\sum_{j=1}^{m} K_{\Omega_1}(z, F_j(w))\, \overline{\mathrm{Det}\,[F_j'](w)} = K_{\Omega_2}(f(z), w)\ \mathrm{Det}[f'](z)$$

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for $z \in \Omega_1$ and $w \in \Omega_2$ where K_{Ω_i} denotes the Bergman kernel function associated to Ω_i for $i = 1, 2$. Also it holds for the domains in $\mathbf{C}^n$, $n > 1$. Using the above formula, Bell gave many important applications. In particular, see [**3, 4, 5, 6**].

It is natural to ask if a similar transformation rule in the Szegő kernel function holds. Contrary to the Bergman kernel function, there are few known results about the Szegő kernel function, at least about boundary behavior of the Szegő kernel. Let Ω be a bounded domain in $\mathbf{C}$ with C^∞ boundary. Let $H^2(b\Omega)$ be the space consisting of the closure in the $L^2(b\Omega)$ topology of the restrictions to $b\Omega$ of functions which are continuous on $\overline{\Omega}$ and which are holomorphic on Ω. The Szegő projection associated to Ω is the orthogonal projection Q of the space $L^2(b\Omega)$ onto the closed subspace $H^2(b\Omega)$. The Szegő kernel function $S_\Omega(z, w)$ is the reproducing kernel for the Szegő projection Q. Namely,

$$Qh(z) = \int_{b\Omega} S_\Omega(z, w) h(w)\, d\sigma(w) \quad \text{for } h \in L^2(b\Omega) \quad \text{and} \quad z \in \Omega,$$

where the $\sigma(w)$ is the induced Euclidean measure on $b\Omega$. Unlike the Bergman kernel function, there is no known transformation rule for the Szegő kernel under proper holomorphic mappings in $\mathbf{C}^n$, $n > 1$. In dimension one, we have the following transformation formula for the Szegő kernel function under proper holomorphic mappings of a domain onto a *simply connected* domain which is proved in [**10**].

Theorem 1. *Let Ω_1 be a smoothly bounded domain in $\mathbf{C}$ and Ω_2 be a smoothly bounded, simply-connected domain in $\mathbf{C}$. Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic mapping of order m. Let $F_1, \ldots, F_m$ denote the m local inverses to f. Let $S_{\Omega_i}(z, w)$ denote the Szegő kernel function associated to Ω_i for $i = 1, 2$. The Szegő kernels transform according to*

$$\sum_{i=1}^{m} S_{\Omega_1}(z, F_i(w))^2 \overline{F_i'(w)} = f'(z) S_{\Omega_2}(f(z), w)^2$$

for all $z \in \Omega_1$ and $w \in \Omega_2$.

Remark. Bell [**7**] used this result to prove that the Szegő kernel associated to a smoothly bounded domain of connectivity, greater than

1, is not rational. There is another application to the above result that the derivative of the Ahlfors map, which will be introduced in Section 2, is expressed explicitly in terms of the Szegő kernel functions, see [8].

We can use the above result to prove the following which is more explicit form than that.

Corollary 2. *Under the same hypothesis as Theorem* 1, *we get the following transformation formula for the Szegő kernel*:

$$S_{\Omega_2}(\tilde{w}, w)^2 = \frac{1}{m} \sum_{i,j=1}^{m} F_j'(\tilde{w}) S_{\Omega_1}(F_j(\tilde{w}), F_i(w))^2 \overline{F_i'(w)}$$

for all $\tilde{w}, w \in \Omega_2$.

Proof. By Theorem 1, we get

$$\sum_{i=1}^{m} S_{\Omega_1}(z, F_i(w))^2 \overline{F_i'(w)} = f'(z) S_{\Omega_2}(f(z), w)^2$$

for all $z \in \Omega_1$ and $w \in \Omega_2$. The set $V = \{f(z) : f'(z) = 0\}$ is an analytic subvariety. For each $\tilde{w} \in \Omega_2 - V$, there are a neighborhood $B_\varepsilon(\tilde{w})$ in Ω_2, holomorphic functions F_i defined on $B_\varepsilon(\tilde{w})$ and disjoint neighborhoods U_i of $F_i(\tilde{w})$ such that $F_i \circ f$ =identity on U_i and $f \circ F_i$ =identity on $B_\varepsilon(\tilde{w})$ for each $i = 1, \ldots, m$. Therefore by setting $z = F_j(\tilde{w})$ in the above formula, we have

$$\sum_{j=1}^{m} \sum_{i=1}^{m} S_{\Omega_1}(F_j(\tilde{w}), F_i(w))^2 \overline{F_i'(w)} \frac{1}{f'(F_j(\tilde{w}))} = \sum_{j=1}^{m} S_{\Omega_2}(\tilde{w}, w)^2.$$

Since $f'(F_j(\tilde{w})) F_j'(\tilde{w}) = 1$,

$$S_{\Omega_2}(\tilde{w}, w)^2 = \frac{1}{m} \sum_{i,j=1}^{m} F_j'(\tilde{w}) S_{\Omega_1}(F_j(\tilde{w}), F_i(w))^2 \overline{F_i'(w)}$$

for all $\tilde{w} \in \Omega_2 - V$ and $w \in \Omega_2$. By the L^2 removable singularity theorem, it holds for all $w, \tilde{w} \in \Omega_2$. □

In Section 2 we would like to generalize the above theorem to the case when Ω_1 and Ω_2 are multiply connected planar domains. We shall also give the transformation rule under proper holomorphic correspondences and hope to return to this topic.

2. Main results. Observe that if Ω is a bounded simply connected domain in $\mathbf{C}$, then the Riemann mapping function associated to a point $a \in \Omega$ is the unique function that makes $h'(a)$ real and as large as possible, among all holomorphic functions h mapping Ω into the unit disc. If Ω is multiply connected, the Ahlfors map associated to the point $a \in \Omega$ has the same extremal property as the Riemann mapping function does. In fact, if Ω is n-connected, the Ahlfors map is a branched n-to-one covering map of Ω onto the unit disc. Moreover, it has a good relationship with the Szegő kernel in that it is the quotient of the Szegő kernel and the Garabedian kernel, see [**9**]. Recall that the Garabedian kernel $L(z,w)$ associated to the domain Ω is defined by

$$L(z,w) = \frac{1}{2\pi(z-w)} - iQ(\frac{1}{2\pi(\cdot - w)})(z).$$

From such a relationship, we can also derive the fact that the Ahlfors map is related to the Bergman kernel function, see [**8**].

Theorem 3. *Let Ω_1 be a smoothly bounded domain in $\mathbf{C}$ and Ω_2 be a smoothly bounded, n-connected domain in $\mathbf{C}$. Suppose $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping of order m. Let $g : \Omega_2 \to \Delta$ be the Ahlfors map associated to some point $a \in \Omega_2$ where Δ is the unit disk in $\mathbf{C}$. Let $G_1, \dots, G_n$ denote the n local inverses to g defined locally on $\Delta - W$ where $W = \{g(\eta) : g'(\eta) = 0\}$. Let $S_{\Omega_i}(\cdot,\cdot)$ denote the Szegő kernel associated to Ω_i for $i = 1, 2$. Then for given $w_0 \in \Delta$, there exists a neighborhood of w_0 such that the Szegő kernels transform according to*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} S_{\Omega_1}(z, F_{i,j}(G_i(w))^2 \overline{F'_{i,j}(G_i(w))}\, \overline{G'_i(w)} = \sum_{k=1}^{n} S_{\Omega_2}(f(z), G_k(w))^2 f'(z)\overline{G'_k(w)}$$

for all $z \in \Omega_1$ *and* w *in the neighborhood of* w_0*, where* $F_{i,j}$*,* $j = 1, \ldots, m$*, are the local inverses to* f *defined in a neighborhood of* $G_i(w_0)$ *in* Ω_2*.*

Remark. There are n zeroes $a_1 = a, a_2, a_3, \ldots, a_n$ of the Ahlfors map g associated to the point a. As a result, it turns out that if $S_{\Omega_1}(z,w)$ and $S_{\Omega_2}(f(z), a_j)$ are rational functions, then so is f'.

Proof. Since $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping of order m and $g : \Omega_2 \to \Delta$ is the Ahlfors map, $g \circ f : \Omega_1 \to \Delta$ is a proper holomorphic map of order mn. Let $w_0 \in \Delta - W$ be fixed. There exist a neighborhood $B_\varepsilon(w_0)$ in Δ, disjoint open subsets $D_1, D_2, \ldots, D_n$ of Ω_2, and local inverses $G_1, G_2, \ldots, G_n$ to the map g on $B_\varepsilon(w_0)$ such that $g^{-1}(B_\varepsilon(w_0)) = \cup_{i=1}^n D_i$ and $G_i \circ g = id$ on D_i. We may assume that all of $G_1(w_0), G_2(w_0), \ldots, G_n(w_0)$ belong in the set $\Omega_2 - V$ where $V = \{f(z) \mid f'(z) = 0\}$. Now for each $i = 1, \ldots, n$, there exist a neighborhood $B_\delta(G_i(w_0))$ in Ω_2, disjoint open sets $E_{i,1}, E_{i,2}, \ldots, E_{i,m}$ contained in Ω_1, holomorphic functions $F_{i,1}, F_{i,2}, \ldots, F_{i,m}$ on $B_\delta(G_i(w_0))$ such that $F_{i,j} \circ f = id$ on $E_{i,j}$ and $f^{-1}(B_\delta(G_i(w_0))) = \cup_{j=1}^m E_{i,j}$. We can assume that each $G_i(B_\varepsilon(w_0))$ is contained in $B_\delta(G_i(w_0))$ by shrinking the ball $B_\varepsilon(w_0)$ if necessary. Hence, there are mn local inverses $\{F_{i,j} \circ G_i\}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, to $g \circ f$ defined on $B_\varepsilon(w_0)$. Let $S_\Delta(\cdot,\cdot)$ denote the Szegő kernel associated to Δ. By applying Theorem 1 twice, we get

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} S_{\Omega_1}(z, (F_{i,j} \circ G_i)(w))^2 \overline{(F_{i,j} \circ G_i)'(w)} \\
&\qquad = (g \circ f)'(z) S_\Delta((g \circ f)(z), w)^2 \\
&\qquad = g'(f(z)) S_\Delta(g(f(z), w)^2 f'(z) \\
&\qquad = \sum_{k=1}^{n} S_{\Omega_2}(f(z), G_k(w))^2 \overline{G_k'(w)} f'(z)
\end{aligned}$$

for all $z \in \Omega_1$ and $w \in B_\varepsilon(w_0)$. Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{m} S_{\Omega_1}(z,(F_{i,j}\circ G_i)(w))^2\overline{F'_{i,j}(G_i(w))}\,\overline{G'_i(w)} = \sum_{k=1}^{n} S_{\Omega_2}(f(z),G_k(w))^2\overline{G'_k(w)}f'(z)$$

for all $z \in \Omega_1$ and $w \in B_\varepsilon(w_0)$.

Notice that on a small disc in the w plane, the expressions of both sides are independent of the order in which we chose to label the local inverses and agree on the intersection of the discs if they overlap. Hence the proof of the theorem is finished by Riemann's removable singularity theorem. □

We want to get a transformation formula for the Szegő kernel under proper holomorphic correspondence. Before we proceed to get it, we shall mention the notion of proper holomorphic correspondence. A proper holomorphic correspondence is a generalized notion of a proper holomorphic map. Elementary properties of holomorphic correspondences can be found in [**12**].

Let Ω_1 and Ω_2 be domains in $\mathbf{C}$. Let $f : \Omega_1 \multimap \Omega_2$ be a proper holomorphic correspondence given by $f(z) = \{w \in \Omega_2 : (z,w) \in V\}$ where V is an analytic subvariety of $\Omega_1 \times \Omega_2$. The projections $\pi_1 : V \to \Omega_1$ and $\pi_2 : V \to \Omega_2$ are the proper maps. There are subvarieties V_1 and V_2 of Ω_1 and Ω_2, respectively and positive integers p and q satisfying the following conditions:

(1) Near a point $z \in \Omega_1 - V_1$, there are exactly p holomorphic maps $\{f_i\}_{i=1}^p$ defined near z that represent the multi-valued mapping $\pi_2\circ\pi_1^{-1}$.

(2) Near a point $w \in \Omega_2 - V_2$, there are q holomorphic mappings $\{F_j\}_{j=1}^q$ that represent $\pi_1 \circ \pi_2^{-1}$.

We can find the following Lemma in [**1**].

Lemma 4. *Let $f : \Omega_1 \multimap \Omega_2$ be a proper holomorphic correspondence between two bounded domains in $\mathbf{C}^n$. Let f be represented locally by $\{f_i\}_{i=1}^p$ and f^{-1} be represented by $\{F_j\}_{j=1}^q$. Let $K_{\Omega_i}(\cdot,\cdot)$ denote the*

Bergman kernels associated to Ω_i, $i = 1, 2$. Then the Bergman kernels transform according to

$$\sum_{i=1}^{p} u_i(z) K_{\Omega_2}(f_i(z), w) = \sum_{j=1}^{q} \overline{U_j(w)} K_{\Omega_1}(z, F_j(w))$$

which holds for all $z \in \Omega_1$, $w \in \Omega_2$ where $u_i(z)$, respectively $U_j(w)$, is the Jacobian determinant of $f_i(z)$, respectively $F_j(w)$.

Now we can get the following transformation rule for the Szegő kernel under proper holomorphic correspondence.

Theorem 5. *Let Ω_1 be a smoothly bounded, n_1- connected domain in $\mathbf{C}$ and Ω_2 be a smoothly bounded, n_2-connected domain in $\mathbf{C}$. Suppose that $f : \Omega_1 \multimap \Omega_2$ is a proper holomorphic correspondence. Let f be represented by $\{f_i\}_{i=1}^{p}$ locally and f^{-1} be represented by $\{F_j\}_{j=1}^{q}$ locally. Let $g : \Omega_1 \to \Delta$ and $h : \Omega_2 \to \Delta$ be the Ahlfors maps with $g^{-1} = \{G_k\}_{k=1}^{n_1}$ and $h^{-1} = \{H_l\}_{l=1}^{n_2}$ local inverses to g and h, respectively when Δ is the unit disk in $\mathbf{C}$. Then the Szegő kernels transform via*

$$\begin{aligned}\sum_{i,k,s} (f_i \circ G_k)'(z) \overline{H_s'(w)} S_{\Omega_2}((f_i \circ G_k)(z), H_s(w))^2 \\ = \sum_{j,l,t} \overline{(F_j \circ H_l)'(w)} G_t'(z) S_{\Omega_1}(G_t(z), (F_j \circ H_l)(w))^2\end{aligned}$$

for all $z, w \in \Delta$ where $\sum_{i,k,s} = \sum_{i=1}^{p} \sum_{k=1}^{n_1} \sum_{s=1}^{n_2}$ and $\sum_{j,l,t} = \sum_{j=1}^{q} \sum_{l=1}^{n_2} \sum_{t=1}^{n_1}$.

Proof. Since $g : \Omega_1 \to \Delta$ and $h : \Omega_2 \to \Delta$ are proper holomorphic mappings, $g^{-1} : \Delta \multimap \Omega_1$ and $h^{-1} : \Delta \multimap \Omega_2$ are proper holomorphic correspondences. Let $\tilde{f} = h \circ f \circ g^{-1}$. Then $\tilde{f} : \Delta \multimap \Delta$ is a proper holomorphic correspondence which is represented locally by $\{\tilde{f}_{ik}\}_{i,k} = \{h \circ f_i \circ G_k\}_{i,k}$, $i = 1, \dots, p, k = 1, \dots, n_1$, and $\tilde{f}^{-1}$ is represented locally by $\{\tilde{F}_{jl}\}_{j,l} = \{g \circ F_j \circ H_l\}_{j,l}$, $j = 1, \dots, q, l = 1, \dots, n_2$.

Since $K_\Delta(z, w) = 4\pi S_\Delta(z, w)^2$ where K_Δ is the Bergman kernel associated to Δ, we get the following formula by Lemma 5:

$$\sum_{i,k} \tilde{f}_{ik}'(z) S_\Delta(\tilde{f}_{ik}(z), w)^2 = \sum_{j,l} \overline{\tilde{F}_{jl}'(w)} S_\Delta(z, \tilde{F}_{jl}(w))^2$$

for all $z, w \in \Delta$ where $\sum_{i,k} = \sum_{i=1}^{p} \sum_{k=1}^{n_1}$ and $\sum_{j,l} = \sum_{j=1}^{q} \sum_{l=1}^{n_2}$. Applying Theorem 1 to the Ahlfors maps $g : \Omega_1 \to \Delta$ and $h : \Omega_2 \to \Delta$,

$$\begin{aligned}
\sum_{i,k} \tilde{f}'_{ik}(z) S_\Delta(\tilde{f}_{ik}(z), w)^2 \\
&= \sum_{i,k} (h \circ f_i \circ G_k)'(z) S_\Delta((h \circ f_i \circ G_k)(z), w)^2 \\
&= \sum_{i,k} h'((f_i \circ G_k)(z))(f_i \circ G_k)'(z) S_\Delta(h((f_i \circ G_k)(z)), w)^2 \\
&= \sum_{i,k} (f_i \circ G_k)'(z) \sum_{s=1}^{n_2} S_{\Omega_2}((f_i \circ G_k)(z), H_s(w))^2 \overline{H'_s(w)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j,l} \overline{\tilde{F}'_{jl}(w)} S_\Delta(z, \tilde{F}_{jl}(w))^2 \\
&= \sum_{j,l} \overline{(g \circ F_j \circ H_l)'(w)} S_\Delta(z, (g \circ F_j \circ H_l)(w))^2 \\
&= \sum_{j,l} \overline{g'((F_j \circ H_l)(w))}\, \overline{(F_j \circ H_l)'(w)} S_\Delta(z, g((F_j \circ H_l)(w)))^2 \\
&= \sum_{j,l} \overline{(F_j \circ H_l)'(w)} \sum_{t=1}^{n_1} S_{\Omega_1}(G_t(z), (F_j \circ H_l)(w))^2 G'_t(z).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{i,k,s} (f_i \circ G_k)'(z) \overline{H'_s(w)} S_{\Omega_2}((f_i \circ G_k)(z), H_s(w))^2 \\
&= \sum_{j,l,t} \overline{(F_j \circ H_l)'(w)} G'_t(z) S_{\Omega_1}(G_t(z), (F_j \circ H_l)(w))^2
\end{aligned}$$

for all $z, w \in \Delta$. □

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