GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A SEMILINEAR PARABOLIC SYSTEM

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ABSTRACT. We discuss the initial-boundary value problem \( (u_t)_t = \Delta u + f_i(u_1, \ldots, u_m), \) with \( u_i|_{\partial \Omega} = 0 \) and \( u_i(x, 0) = \phi_i(x), \) \( i = 1, \ldots, m, \) in a bounded domain \( \Omega \in \mathbb{R}^n, \) with \( n \geq 1 \) and \( m \geq 1. \) Under suitable assumptions on the nonlinear terms \( f_i, \) we will prove that, if \( 0 \leq \phi < \lambda \psi \) with \( \lambda < 1, \) then the solutions are global; while if \( \phi > \lambda \psi \) with \( \lambda > 1, \) then the solutions must blow up in finite time, where the \( \psi_i \) are positive solutions of \( \Delta \psi_i + f_i(\psi_1, \ldots, \psi_m) = 0 \) with \( \psi_i|_{\partial \Omega} = 0. \)

In this paper we study the initial boundary-value problem

\[
\begin{align*}
\frac{du}{dt} &= \Delta u + f(u), & x \in \Omega, t > 0, \\
\frac{du}{dt} &= 0, & x \in \partial \Omega, t > 0, \\
\frac{du}{dt} &= \phi(x), & x \in \Omega,
\end{align*}
\]

(1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary and \( n \geq 1, \) and \( u = (u_1, \ldots, u_m), \) \( f = (f_1, \ldots, f_m) \) are vectors with \( m \geq 1. \) It is well-known that, for some small initial values, the solution may exist globally, while for some large initial values the solution may blow-up in finite time if the nonlinear term \( f(u) \) increases superlinearly, see [2–6, 8, 14] and [17]. For a large class of nonlinearities, considerably less is known as to when solutions exist globally or blow-up in finite time.

If \( m = 1 \) and \( f(u) = |u|^{p-1}u \) with \( p > 1, \) Levine [14] proved that solutions of (1) must blow-up in finite time, if \( \phi(x) \) is large enough in the sense that its “energy”

\[
E(\phi) = \frac{1}{2} \| \nabla \phi \|^2 - \frac{1}{p + 1} \| \phi \|^{p+1}_{p+1}
\]

is negative. Weissler [18] proved that for \( n = 1 \) blow-up occurs only at the point \( x = 0 \) if \( \phi = k\psi \) and \( p, k \) are sufficiently large, where \( \psi \)

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is a positive solution of $\psi_{xx} + \psi^p = 0$, $\psi|_{\partial \Omega} = 0$ with $\Omega = (-1, 1)$. In [19], he also proved that, for $n \leq 2$ or $p < (n + 2)/(n - 2)$ there exists a constant $c > 0$ such that

$$u(x, t) \leq c(T - t)^{-1/(p - 1)},$$

if $\phi \geq 0$ is radially symmetric and $\Delta \phi + \phi^p \geq 0$. For more general cases, see [4].

If $m = 2$ and $f(u) = (u_1^\beta u_2, -u_1^\beta u_2)$ with $\beta \geq 1$, Hollis, Martin and Pierre [13] have shown that solutions of (1) exist globally. If $f(u) = ((1 - u_2)g(u_1), (1 - u_2)g(u_1))$ with $g(u_1) = e^{u_1}$, Bebendorf and Lacey [2] have shown that solutions of (1) blow up in finite time. Later, in [3] they extended their results to more general $g(u_1)$. If $f(u) = (u_2^\beta, u_1^\beta)$, Escobedo and Herrero [8] have proved that all solutions of (1) are global if $pq < 1$, while if $pq > 1$, both global solutions and solutions that blow-up in finite time can occur, depending on the initial values. Later Caristi and Mitidieri [5] obtained the following estimates:

$$u_1(x, t) \leq c(T - t)^{-(p + 1)/(pq - 1)}, \quad u_2(x, t) \leq c(T - t)^{-(q + 1)/(pq - 1)},$$

if $pq > 1$, where $T$ is the blow-up time.

Lu and Sleeman [15] gave several sufficient conditions to get the blow-up property for the one-dimensional parabolic system with $m = 2$

$$\frac{\partial u_i}{\partial t} = \alpha_i \frac{\partial^2 u_i}{\partial x^2} + f_i(u_1, u_2), \quad -a < x < a, \quad \alpha_i > 0, \quad i = 1, 2.$$

However, in order to get blow-up solutions, many authors, see [9], for example, need to assume that $\partial u / \partial t \geq 0$ or $\Delta \phi + f(\phi) \geq 0$, for $n > 1$ (where $u > 0$ means $u_i > 0$ for all $i$). So there is a gap between the global solutions and the blow-up solutions. For the Cauchy problem,

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^n, \quad t > 0,$$

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}^n,$$

with $p \geq (n + 2)/(n - 2)$ and $n \geq 3$, Gui, Ni and Wang [11] have obtained the following perfect results:

(i) if $\phi \leq u_\alpha$ and $\phi \neq u_\alpha$ for some $\alpha$, then $\|u\|_\infty \to 0$ as $t \to \infty$;
(ii) if \( \phi \geq u_\alpha \) and \( \phi \neq u_\alpha \) for some \( \alpha \), then \( u \) must blow-up in finite time, where \( u_\alpha \) is a positive solution of \( \Delta \psi + \psi^p = 0 \), \( \psi(0) = \alpha \).

The purpose of this paper is to fill in this gap for the initial-boundary problem (1). We assume that

(i) \( \psi(x) \) is a positive solution of \( \Delta \psi + f(\psi) = 0 \), with \( \psi|_{\partial \Omega} = 0 \).

(ii) \( \phi(x) \in C^m(\overline{\Omega}, R^m) \), with \( \phi|_{\partial \Omega} = 0 \) for \( 0 < \alpha < 1 \).

(iii) \( f : R^n \to R^n \) is locally Lipschitz continuous, \( f(0) = 0 \), and

\[
f_i(u)/u_i > f_i(v)/v_i \quad \text{for any} \quad u > v \geq 0 \quad \text{and} \quad i = 1, \ldots, m.
\]

(iv) \( f_i(u)/u_i^\sigma \geq c_0 > 0 \) for some \( \sigma > 1 \) and all \( u > 0 \), \( i = 1, \ldots, m \).

Our result is:

**Theorem.** If \( 0 \leq \phi_i < \lambda \psi_i \) for all \( i \), with \( \lambda < 1 \), and the assumptions (i)–(iii) hold, then the solution \( u \) of (1) is global with exponential decay. If \( \phi_i > \lambda \psi_i \) for all \( i \), with \( \lambda > 1 \), and the assumptions (i)–(iv) hold, then the solution \( u \) of (1) must blow-up in finite time.

**Proof.** First we prove global existence. Since \( \phi \in C^2_0(\overline{\Omega}) \), from standard parabolic PDE theory, there exists a unique solution \( u(x,t) \in C(\overline{\Omega} \times [0,\tau]) \cap C^2(\overline{\Omega} \times (0,\tau]) \) for some \( \tau > 0 \), see [1 or 10] and \( u \geq 0 \), see [17]. Set \( v_i(x,t) = (\lambda + ct)\psi_i(x) - u_i(x,t) \) with \( c \) large enough such that

\[
|((\lambda + 1) \frac{f_i(\psi)}{\psi_i} - \frac{f_i(u)}{\psi_i})| < c \quad \text{in} \quad \Omega \times (0,\tau).
\]

Then

\[
(v_i)_t - \Delta v_i = c\psi_i - (u_t)_i - (\lambda + ct)\Delta \psi_i + \Delta u_i
\]

\[
= \psi_i \left[ c + (\lambda + ct) \frac{f_i(\psi)}{\psi_i} - \frac{f_i(u)}{\psi_i} \right] > 0,
\]

for \( t \leq \min(1/c, \tau) \). By the maximum principle, \( u_i \geq 0 \). Choose \( t_1 \) sufficiently small such that \( \lambda + ct_1 \leq 1 \). Then \( (\lambda + ct)\psi_i(x) \geq u_i(x,t) \) for \( t \leq t_1 \). Now, for any number \( n \), set

\[
g^n_i(t) = \int_{\Omega} u^{n+2}_i(x,t)\psi^{-n}_i(x) \, d\Omega, \quad \text{for} \quad t \in [0,t_1].
\]
Then \( g^n(t) \) is well defined, and \( g^n(t) \in C[0, t_1] \), because \( u_i(x, t) \in C(\overline{\Omega} \times [0, \tau]) \). Differentiating (2), substituting in the equations (1) and integrating by parts (notice that the boundary values are always zero), we have

\[
\frac{d}{dt} g^n(t) = (n + 2) \int_\Omega u_i^{n+1} \psi_i^{-n} (\Delta u_i + f_i(u)) \, d\Omega \\
= (n + 2) \int_\Omega u_i^{n+1} \psi_i^{-n} f_i(u) \, d\Omega \\
+ (n + 2) \left\{ n \int_\Omega u_i^{n+1} \psi_i^{-n-1} \nabla u_i \nabla \psi_i \, d\Omega \\
- (n + 1) \int_\Omega u_i^n \psi_i^{-n} |\nabla u_i|^2 \, d\Omega \right\} \\
= (n + 2) \int_\Omega u_i^{n+2} \psi_i^{-n} \frac{f_i(u)}{u_i} \, d\Omega \\
- (n + 1)(n + 2) \int_\Omega u_i^n \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
- (n + 2)^2 \int_\Omega u_i^{n+1} \psi_i^{-n-1} \nabla u_i \nabla \psi_i \, d\Omega \\
+ (n + 1)(n + 2) \int_\Omega u_i^{n+2} \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
\leq (n + 2) \int_\Omega u_i^{n+2} \psi_i^{-n} \frac{f_i(u)}{u_i} \, d\Omega \\
+ (n + 1)(n + 2) \int_\Omega u_i^{n+2} \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
- (n + 2) \left\{ (n + 1) \int_\Omega u_i^{n+2} \psi_i^{-n-2} |\nabla \psi_i|^2 \, d\Omega \\
- \int_\Omega u_i^{n+2} \psi_i^{-n-1} \Delta \psi_i \, d\Omega \right\} \\
= -(n + 2) \int_\Omega u_i^{n+2} \psi_i^{-n} \left( \frac{f_i(\psi)}{\psi_i} - \frac{f_i(u)}{u_i} \right) \, d\Omega \leq 0,
\]

for \( t \leq t_1 \). Thus, we get

\[
g^n(t) \leq g^n(0) \quad \text{for } t \in (0, t_1].
\]

Taking the \( n \)th roots and letting \( n \to \infty \), we have

\[
\frac{u_i(x, t)}{\psi_i(x)} \leq \sup_\Omega \frac{u_i}{\psi_i} \leq \sup_\Omega \frac{\phi_i}{\psi_i} \leq \lambda, \quad \text{for } t \in (0, t_1].
\]
We can extend \( u_i \) from \( t_1 \) to \( t_2 \) by the same method since \( u_i(x,t)/\psi_i(x) \leq \lambda \). Hence, (4) holds for all \( t > 0 \) for which the solution \( u_i \) exists. By [1] or [12], we can extend \( u_i \) to \( \infty \) as long as \( u_i \) is bounded.

Now we prove that \( u_i(x,t) \) decays exponentially. For any \( \varepsilon > 0 \), let \( \Omega_\varepsilon = \{ x \in \Omega \mid \text{dist} (x, \partial \Omega) < \varepsilon \} \). Since \( \lambda < 1 \), by assumption (iii), there exists a \( c_\varepsilon > 0 \) such that

\[
\frac{f_i(\psi)}{\psi_i} - \frac{f_i(u)}{u_i} = f_i(\psi) \left( 1 - \frac{f_i(u)/u_i}{f_i(\psi)/\psi_i} \right) \geq c_\varepsilon
\]

for \( x \in \Omega - \Omega_\varepsilon \). Using (3), we define \( g^n_{\varepsilon,i}(t) \) by

\[
\frac{d}{dt} g^n_{\varepsilon,i}(t) \leq -(n+2)c_\varepsilon \int_{\Omega - \Omega_\varepsilon} u_i^{n+2} \psi_i^{-n} \, d\Omega \equiv -(n+2)c_\varepsilon g^n_{\varepsilon,i}(t).
\]

Then

\[
g^n_{\varepsilon,i}(t) < g^n_i(0) - (n+2)c_\varepsilon \int_0^t g^n_{\varepsilon,i}(\tau) \, d\tau,
\]

which implies that, for fixed \( n \),

\[
\frac{1}{\max \psi_i^n} \int_{\Omega - \Omega_\varepsilon} u_i^{n+2} d\Omega < g^n_{\varepsilon,i}(t) \to 0 \quad \text{as } t \to \infty.
\]

Since \( u_i(x,t) = 0 \) on the boundary, we have \( \int_{\Omega} u_i^{n+2} d\Omega \to 0 \) as \( t \to \infty \).

If \( \lim_{\psi_i \to 0} f_i(\psi)/\psi_i = 0 \) by [12], \( u_i(x,t) \) decays exponentially. If \( f_i(\psi)/\psi_i \geq c_0 > 0 \) for any \( x \in \Omega \), applying (5) to (3), we obtain

\[
\frac{d}{dt} g^n_i(t) \leq -c(n+2)g^n_i(t)
\]

which also implies that \( u_i(x,t) \) decays exponentially.

Now we prove the blow-up property. Similar to the argument above, we set

\[
h^n_i(t) = \int_{\Omega} \psi_i^{n+2}(x) u_i^{-n}(x,t) \, d\Omega.
\]

Then

\[
\frac{d}{dt} h^n_i(t) \leq -n \int_{\Omega} \psi_i^{n+2} u_i^{-n} \left( \frac{f_i(u)}{u_i} - \frac{f_i(\psi)}{\psi_i} \right) \, d\Omega
\]

\[
- \, n(n+1) \int_{\Omega} \psi_i^nu_i^{-n-2} |\nabla u_i - u_i \nabla \psi_i|^2 \, d\Omega \leq 0.
\]
So \( u_i(x,t) \geq \lambda \psi_i(x) \) for \( \lambda > 1 \) and all \( t > 0 \) such that \( u(x,t) \) exists. By assumption (iii), there exists \( c_1 > 0 \) such that

\[
1 - \frac{f_i(\psi)/\psi_i}{f_i(u)/u_i} \geq c_1.
\]

From (6) with \( n = \sigma - 1 \) and assumption (iv), we get

\[
\frac{d}{dt} h_i^{\sigma - 1}(t) \leq -(\sigma - 1) \int_{\Omega} \psi_i^{\sigma + 1} \frac{f_i(u)}{u_i^2} \left( 1 - \frac{f_i(\psi)/\psi_i}{f_i(u)/u_i} \right) d\Omega
\]

\[
\leq -c \int_{\Omega} \psi_i^{\sigma + 1} d\Omega.
\]

Hence

\[
0 \leq h_i^{\sigma - 1}(t) \leq h_i^{\sigma - 1}(0) - ct \int_{\Omega} \psi_i^{\sigma + 1} d\Omega, \quad \text{or} \quad t \leq \frac{h_i^{\sigma - 1}(0)}{c \int_{\Omega} \psi_i^{\sigma + 1} d\Omega},
\]

which means \( t \) cannot increase to infinity. The proof is complete. \( \Box \)

**Example.** It is easy to see that, if \( f_1 = (1 + u_2)u_1^2 \) and \( f_2 = (1 + u_1)u_2^2 \), then \( \mathbf{f} = (f_1, f_2) \) satisfies the conditions (iii) and (iv) and that \( \psi = (\psi_1, \psi_2) \) can be chosen such that \( \psi_1 = \psi_2 \) and \( \psi_i \) is a positive solution of \( \Delta \psi_i + (1 + \psi_i)^2 = 0 \) with \( \psi_i|_{\partial \Omega} = 0 \).

**Remark.** For general \( \mathbf{f} \), the system \( \Delta \psi + \mathbf{f}(\psi) = 0 \) with \( \psi|_{\partial \Omega} = 0 \) might have no positive solutions, see, for example, [16, Theorem 4.1] or the work of [7]. However, when \( n = 1 \), such solutions always exist for many kinds of \( \mathbf{f} \).

**Appendix**

We need some results from the theory of analytic semigroups, see [1] and [12]. Suppose the Laplace operator \( \Delta \) is the infinitesimal generator of an analytic semigroup \( \{ e^{t \Delta} \mid 0 \leq t < \infty \} \). Then there exist positive constants \( c_1, c_2 \) and \( \delta \), independent of \( t \), such that

\[
\| e^{t \Delta} \| \leq c_1 e^{-t \delta}, \quad \| \Delta e^{t \Delta} \| \leq c_2 e^{-t \delta} / t,
\]
where $\| \cdot \|$ is the norm of $X = L^p$ for $p \geq 2$. This implies the existence of the integral

\[
\Delta^{-\mu} = \frac{1}{\Gamma(\mu)} \int_0^\infty \tau^{\mu-1} e^{\tau \Delta} \, d\tau,
\]

for every $\mu > 0$, where $\Gamma(x)$ is the gamma function. It follows that each $\Delta^{-\mu}$ is an injective continuous endomorphism of $X$. Hence $\Delta^\mu = [\Delta^{-\mu}]^{-1}$ is a closed bijective linear operator in $X$. If $\phi \in D(\Delta^\mu)$, the domain of $\Delta^\mu$, then

\[
\| \Delta^\mu e^{\tau \Delta} \phi \| \leq c e^{-\delta \tau} \| \Delta^\mu \phi \|,
\]

\[
\| \Delta^\mu e^{\tau \Delta} \phi \| \leq c^\mu e^{-\delta \tau} \| \phi \|.
\]

Now set $X = [L^p(\Omega)]^m$ with the $L^p$ norm $\| \cdot \|$ and $D(\Delta) = [W^{2,p}_0(\Omega)]^m$. We have the following lemma:

**Lemma.** Suppose that $n/(2p) < \beta < 1/2$. Then $X_{\beta} = (D(\Delta^\beta), \| \cdot \|_{\beta})$ is embedded into $[C^{\alpha}(\overline{\Omega})]^m$ with $0 < \mu < 2\beta - n/p$.

**Proof.** The proof is similar to that of [1]. It follows from Friedman [10, Theorem (I.10.1)] that

\[
\| u \|_{C^\alpha(\overline{\Omega})} \leq c \| u \|_{W^{2,p}} \| u \|_{L^p}^{\beta},
\]

where $\nu = \mu/2 + n/(4p) < \beta$. If we let $u = \Delta^{-\beta} v$ with $v \in [L^p(\Omega)]^m$, we obtain from (7)-(9),

\[
\| \Delta^{-\beta} v \|_{C^\alpha(\overline{\Omega})} \leq \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1} \| e^{\tau \Delta} v \|_{C^\alpha(\overline{\Omega})} \, d\tau
\]

\[
\leq c \int_0^\infty \tau^{\beta-1} \| e^{\tau \Delta} v \|_{L^p} \| e^{\tau \Delta} v \|_{L^p}^{\beta} \, d\tau
\]

\[
\leq c \int_0^\infty \tau^{\beta-\nu-1} e^{-\delta \tau} \, d\tau \| v \|_{L^p}.
\]

Hence $\| u \|_{C^\alpha(\overline{\Omega})} \leq c \| \Delta^\beta u \| = c \| u \|_{\beta}$, and the assertion follows. 

Since $\phi \in C_0^\infty(\overline{\Omega})$, we have $\phi \in D(\Delta^\alpha)$. From [12], $u(t) \in C([0, \tau], D(\Delta^\alpha))$, which implies that $u(x, t) \in C(\Omega \times [0, \tau])$ by the lemma with $\beta = \alpha$. 

REFERENCES


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