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# ON THE ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM WITH A REMAINDER TERM

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ABSTRACT. In this paper we have improve the remainder term of the prime number theorem by using the elementary method, Selberg's method, and obtained

$$\pi(x) = \operatorname{li} x + O\{x \exp(-\log^{(1/2)-\varepsilon} x)\},$$

where $\varepsilon > 0$ is an arbitrarily small constant, and the $O$ is dependent on $\varepsilon$.

**1. Introduction.** Let $\pi(x)$ denote the number of prime numbers not exceeding $x$. The elementary proof of the prime number theorem was obtained by Selberg in 1949 and his method was modified by several mathematicians. In 1962 and 1964, Bombieri [**2**] and Wirsing [**6**] respectively and independently proved

$$\pi(x) = \operatorname{li} x + O\left(\frac{x}{\log^A x}\right),$$

where $A$ is an arbitrary positive constant. In 1970, Diamond and Steinig [**3**] proved

$$\pi(x) = \operatorname{li} x + O\{x \exp(-(\log x)^{1/7} (\log\log x)^{-2})\}.$$

In 1973, А.Ф. Лаврик and А.Ш. Собиров [**5**] proved

$$\pi(x) = \operatorname{li} x + O\{x \exp(-(\log x)^{1/6} (\log\log x)^{-3})\}.$$

In this paper we have modified Selberg's method again and obtained

$$\pi(x) = \operatorname{li} x + O(x \exp(-\log^{(1/2)-\varepsilon} x)),$$

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where $\varepsilon > 0$ is an arbitrarily small constant and the $O$ is dependent on $\varepsilon$.

**2. The path of the proof.** In this paper $n, m, i, j$ and $k$ always denote positive integral numbers and the rest denote positive real numbers (except where otherwise noted).

In this paper the integral part of $x$ is denoted by $[x]$ and the convolution of $f$ and $g$ is defined by

$$(f * g)(n) = \sum_{m|n} f(m) g\left(\frac{n}{m}\right).$$

Let $\mu$ be the Möbius function and $\Lambda$ the Mangoldt function. Define

$$\Lambda_i = \mu * \log^i, \quad i \geq 1, \tag{1}$$

$$R(x) = \sum_{n \leq x} \{\Lambda(n) - 1\} + 2\gamma, \quad x > 0, \tag{2}$$

where $\gamma$ is Euler's constant.

The proof of the paper is based on the application of Balog's identity

$$R(x) \log^k x + \sum_{i=1}^{k} \binom{k}{i} \sum_{n \leq x} \Lambda_i(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right) = \sum_{n \leq x} \mu(n) \log^k \frac{x}{n} \tilde{R}\left(\frac{x}{n}\right), \tag{3}$$

where

$$\tilde{R}(x) = \sum_{n \leq x} R\left(\frac{x}{n}\right).$$

The difficulty existing in (3) is how to deal with the coefficient $\Lambda_i(n)$ in the second term on the left side; here the estimated result of $\Lambda_i(n)$ in short intervals will determine the main result researched in the paper.

Define

$$m_i(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \log^i \frac{x}{n}, \quad i \geq 0. \tag{4}$$

Via simple calculation we can obtain

$$\sum_{y<n\le x}\{\Lambda_i(n)-m_i(n)\}\ll 3^i(x-y)(\log y)^{\sigma(i+1)}, \tag{5}$$

when $(y/2)\exp(-\log^\sigma y)\le x-y\le y\exp(-\log^\sigma y)$ and $2\le i\le(\log\log x)^{-1}\log^\sigma y$, where $\sigma$ is an arbitrary constant, $0<\sigma<1$ and the $\ll$ depends on $\sigma$.

Now (5) is effective only when the exponent of $\log y$ is less than $i-1$, namely, $\sigma(i+1)<i-1$ or $\sigma<(i-1)/(i+1)=1-2/(i+1)$. From this we see that, when $i=1$, (5) is not effective, and, when $i\ge 2$, the smaller $i$ is, the more $\sigma$ is restricted. On the other hand, the more $\sigma$ is restricted, the poorer the result is. And, hence, the result is determined by whether we can get a better estimate than (5) when $i$ is small.

In this paper we have used a recursive method to estimate $\Lambda_i(n)$ in short intervals when $i$ is small and obtained a more refined result than (5).

Let us define

$$\begin{aligned}R_1(x)&=R(x);\\ R_i(x)&=\sum_{n\le x}\{\Lambda_i(n)-i\log^{i-1}n\},\quad i\ge 2.\end{aligned}\tag{6}$$

If $i$ is a constant, then $m_i(n)=i\log^{i-1}n+O(\log^{i-2}n)$, $i\ge 1$, so that the estimate of $\sum_{y<n\le x}\{\Lambda_i(n)-m_i(n)\}$ is equivalent to the estimate of $R_i(x)-R_i(y)$.

We first derive a recurrence formula

$$\begin{aligned}R_i(x)-R_i(y)&=\{R_{i-1}(x)-R_{i-1}(y)\}\log x\\ &\quad+\sum_{n\le b}\Lambda(n)\left\{R_{i-1}\left(\frac{x}{n}\right)-R_{i-1}\left(\frac{y}{n}\right)\right\}\\ &\quad+O\left\{|R_{i-1}(x)-R_{i-1}(y)|+(x-y)\frac{\log^{i-1}x}{\log\log x}\right\}\end{aligned}\tag{7}$$

from the identity $\Lambda_i=\Lambda_{i-1}\log+\Lambda*\Lambda_{i-1}$, $i\ge 2$, under the conditions $i\ge 2$, $y\ge(x/2)$ and $x-y\ge x\exp(-\log^{1-\sigma}x)$ where $\sigma$ is an arbitrary constant, $0 < \sigma < 1$, $b = x\exp(-\log x/(\log\log x))$ and the $O$ depends on $\sigma$ and $i$.

Next let $i \geq 2$. By applying (7) we can prove that if, when $x\exp(-(\log x)^{1-\rho}(\log\log x)^{-h}) \leq x-y \leq x\exp(-\sqrt{\log x})$, the inequality

$$|R_{i+1}(x) - R_{i+1}(y)| \leq \frac{N}{M}(x-y)\log^i x + O\left\{(x-y)\frac{\log^i x}{\log\log x}\right\}$$

holds, then when $x\exp(-(\log x)^{1-\rho}(\log\log x)^{-(h+M)}) \leq x-y \leq x\exp(-\sqrt{\log x})$, the inequality

$$|R_i(x) - R_i(y)| \leq \frac{3N}{M}(x-y)\log^{i-1} x + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\}$$

holds too, where $\rho$ is an arbitrary constant, $0 < \rho < (1/2)$, $h$ is an integer, $h \geq i$, $M$ and $N$ are arbitrary integers satisfying $M > N \geq 1$ and both $O$'s depend on $\rho$, $i$ and $M$.

That is to say, if we can get a good estimate of $R_i(x) - R_i(y)$ when $i$ is greater, then we can also get a good estimate when $i$ is smaller.

By induction we finally obtain, when $i \geq 2$ and $x\exp(-\log^{1-2\rho} x) \leq x-y \leq x\exp(-\sqrt{\log x})$,

$$|R_i(x) - R_i(y)| \leq \frac{1}{A}(x-y)\log^{i-1} x + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\},$$

where $\rho$ is the same as above and $A$ is an arbitrarily large constant and the $O$ depends on $\rho$, $i$ and $A$.

In the case $i = 2$, we take $\rho = (\delta/4)$ in the last inequality and $\sigma = (\delta/2)$ in (7), then substitute the last inequality in (7); we get

$$\begin{aligned}\{R(x) - R(y)\}\log x = &-\sum_{n\leq b}\Lambda(n)\left\{R\left(\frac{x}{n}\right) - R\left(\frac{y}{n}\right)\right\}\\ &+\frac{\theta}{A}(x-y)\log x + O\left\{(x-y)\frac{\log x}{\log\log x}\right\},\end{aligned}$$

where $\theta$ satisfies $|\theta| \leq 1$ and $x, y$ satisfy $y \geq (x/2)$, $x-y \geq x\exp(-\log^{1-(\delta/2)} x)$, $\delta$ being an arbitrary constant, $0 < \delta < 1$ and the $O$ depends on $\delta$ and $A$.

We can use this formula to estimate $R(x)-R(y)$ and we see that its result is mainly determined by the condition $x-y \geq x\exp(-\log^{1-(\delta/2)} x)$. Transforming the last formula into a suitable form and applying Wirsing's lemma found in the proof of [**6**, Lemma 1], we get a very good estimate of $R(x)-R(y)$. When $x-y \geq x\exp(-\log^{1-\delta} x)$,

$$|R(x)-R(y)| < \frac{19}{20}(x-y) + O\bigg(\frac{x-y}{\log\log x}\bigg),$$

where $\delta$ is an arbitrary constant, $0<\delta<1$, and the $O$ depends on $\delta$.

Thus we obtain a good estimate of $\Lambda_i(n)$ in short intervals for $i \geq 1$, so that we can obtain the main theorem in the paper.

**3. A recurrence formula of $\Lambda_i(n)$.** In this section we shall give a recurrence formula for $\Lambda_i(n)$ as the base for estimating $\Lambda_i(n)$ in short intervals.

The function $\Lambda_i$ given by (1) has the following recurrence relation, see [**1**, p. 288].

$$\Lambda_i = \Lambda_{i-1}\log + \Lambda * \Lambda_{i-1}, \quad \text{for } i \geq 2.$$

It follows that, for $0<y<x$ and $i \geq 2$,

$$\sum_{y<n\leq x} \Lambda_i(n) = \sum_{y<n\leq x} \Lambda_{i-1}(n)\log n + \sum_{y<nm\leq x} \Lambda(n)\Lambda_{i-1}(m). \tag{8}$$

Defining $\Psi_i, \Psi$ by

$$\Psi_i(x) = \sum_{n\leq x} \Lambda_i(n), \quad i \geq 1 \tag{9}$$

$$\Psi(x) = \sum_{n\leq x} \Lambda(n), \tag{10}$$

and substituting $\log n = \log x - \log(x/n) = \log x + O(\log(x/y)) = \log x + O(1)((x/2) \leq y < x; y < n \leq x)$ in (8), we have, when $(x/2) \leq y < x$ and $i \geq 2$,

$$\begin{aligned} \Psi_i(x) - \Psi_i(y) = {} & \{\Psi_{i-1}(x) - \Psi_{i-1}(y)\}\log x \\ & + O\bigg\{\sum_{y<n\leq x} |\Lambda_{i-1}(n)|\bigg\} + \sum_{y<nm\leq x} \Lambda(n)\Lambda_{i-1}(m), \end{aligned} \tag{11}$$

and the $O$ is absolute.

Write

$$b = x \exp\biggl(-\frac{\log x}{\log\log x}\biggr). \tag{12}$$

Considering the last term of (11) and using (9) again we get, when $x/2 \le y < x$ and $i \ge 2$,

$$\begin{aligned} \Psi_i(x) - \Psi_i(y) = \{\Psi_{i-1}(x) - \Psi_{i-1}(y)\}\log x + O\biggl\{\sum_{y<n\le x} |\Lambda_{i-1}(n)|\biggr\} \\ + \sum_{n\le b} \Lambda(n)\biggl\{\Psi_{i-1}\biggl(\frac{x}{n}\biggr) - \Psi_{i-1}\biggl(\frac{y}{n}\biggr)\biggr\} \\ + \sum_{m\le (x/b)} \Lambda_{i-1}(m) \sum_{\max((y/m),b)<n\le (x/m)} \Lambda(n), \end{aligned} \tag{13}$$

and the $O$ is absolute.

Now we shall give several formulae which will be used for the estimate of the last term of (13).

By Balog [**1**, p. 288],

$$\Lambda_i(n) \ge 0 \quad \text{for} \quad i \ge 1,\ n \ge 1, \tag{14}$$

and by [**1**, p. 290],

$$\Psi_i(x) \ll x \log^{i-1} x \quad \text{for } i \ge 2, \tag{15}$$

and the $\ll$ depends on $i$.

In fact, Balog has proved that $\Psi_i(x) = ix\log^{i-1} x + O(i(i-1)x\log^{i-2} x)$ for $2 \le i \le \log x$ in [**1**, p. 290], but in this paper we do not need this strong result and only need (15).

From (9), (10) and (1), we see that $\Psi_1(x) = \sum_{n\le x} \Lambda_1(n) = \sum_{n\le x} \Lambda(n) = \Psi(x) \ll x$, that is to say, inequality (15) is also valid when $i = 1$, and, therefore, we can get from (15) via partial summation

$$\sum_{n\le x} \frac{\Lambda_i(n)}{n} \ll \log^i x \quad \text{for } i \ge 1, \tag{16}$$

and the $\ll$ depends on $i$.

Finally we give a well-known formula

$$\pi(A+M)-\pi(A)\le \frac{2M}{\log M}\left\{1+O\left(\frac{\log\log M}{\log M}\right)\right\}, \tag{17}$$

and the $O$ is independent of $A$ and $M$. This formula is obtained by Selberg's sieve, see Geng [**4**].

Using (17) with $A=y$ and $M=x-y$, we get

$$\begin{aligned}\Psi(x)-\Psi(y) &= \sum_{y<n\le x}\Lambda(n)=\sum_{y<p\le x}\log p+\sum_{\substack{y<p^m\le x\\ m\ge 2}}\log p\\ &\le \{\pi(x)-\pi(y)\}\log x+O(\sqrt{x}\log^2 x)\\ &\le \frac{2(x-y)\log x}{\log(x-y)}\left\{1+O\left(\frac{\log\log x}{\log(x-y)}\right)\right\}\\ &\quad +O(\sqrt{x}\log^2 x),\end{aligned} \tag{18}$$

where $p$ denotes prime number and both $O$'s are absolute.

When $x-y\ge x^{2/3}$, we have from (18),

$$\Psi(x)-\Psi(y)\ll x-y, \tag{19}$$

and the $\ll$ is absolute.

When $y\ge(x/2)$ and $x-y\ge x\exp(-\log^\sigma x)$, where $\sigma$ is a constant, $0<\sigma<1$, we have from (18) again

$$\Psi(x)-\Psi(y)\le 2(x-y)\{1+O(\log^{-1+\sigma}x)\},$$

and this gives

$$|R(x)-R(y)|\le (x-y)\{1+O(\log^{-1+\sigma}x)\}, \tag{20}$$

and the $O$ depends on $\sigma$.

Let $x$ be suitably large, such that when $m\le(x/b)$,

$$m\le(x/b)=\exp\left(\frac{\log x}{\log\log x}\right)\le x^{1/6},$$

the second step following from (12); then, if $x-y \geq x^{5/6}$ and $m \leq (x/b)$,

$$\frac{x}{m} - \frac{y}{m} \geq x^{(5/6)-(1/6)} = x^{2/3} \geq \left(\frac{x}{m}\right)^{2/3},$$

that is to say, in this case the condition of (19) is satisfied by $x/m$ and $y/m$, and so we can apply (19) and get, when $x - y \geq x^{5/6}$ and $m \leq (x/b)$,

$$\Psi\left(\frac{x}{m}\right) - \Psi\left(\frac{y}{m}\right) \ll \frac{x-y}{m},$$

and the $\ll$ is absolute.

Now we estimate the last term of (13). Using the last inequality and inequalities (14) and (16), we have, when $x - y \geq x^{5/6}$, $i \geq 2$ and $x$ is suitably large,

$$\begin{aligned}
\Bigg| \sum_{m \leq (x/b)} \Lambda_{i-1}(m) \sum_{\max((y/m),b)<n\leq(x/m)} \Lambda(n)\Bigg| &\leq \sum_{m\leq(x/b)} \Lambda_{i-1}(m)\left\{\Psi\left(\frac{x}{m}\right) - \Psi\left(\frac{y}{m}\right)\right\} \\
&\ll (x-y) \sum_{m\leq(x/b)} \frac{\Lambda_{i-1}(m)}{m} \\
&\ll (x-y)\log^{i-1}\frac{x}{b} \qquad (21)\\
&= (x-y)\left(\frac{\log x}{\log\log x}\right)^{i-1} \\
&\leq (x-y)\frac{\log^{i-1} x}{\log\log x},
\end{aligned}$$

and the latter $\ll$ depends on $i$.

We shall next discuss the third term on the righthand side of (13). We first give the following.

If $1 \le y < x$, then

$$\begin{aligned} \log^i x - \log^i y &= i \int_y^x \frac{\log^{i-1} t}{t}\, dt \\ &\le i \int_y^x \frac{\log^{i-1} x}{y}\, dt \\ &= i \frac{x-y}{y} \log^{i-1} x \quad \text{for} \quad i \ge 1. \end{aligned} \tag{22}$$

Furthermore we shall use Balog's result, see [**1**, p. 290]:

$$\sum_{n \le x} \frac{\Lambda(n)}{n} \log^i \frac{x}{n} = \frac{1}{i+1} \log^{i+1} x + O(\log^i x) \quad \text{for } i \ge 0, \tag{23}$$

and the $O$ is $i$-uniform.

By (23) and (12) we have, when $i \ge 1$ and $x$ is suitably large,

$$\begin{aligned} \sum_{n \le b} \frac{\Lambda(n)}{n} \log^{i-1} \frac{x}{n} &= \sum_{n \le x} \frac{\Lambda(n)}{n} \log^{i-1} \frac{x}{n} - \sum_{b < n \le x} \frac{\Lambda(n)}{n} \log^{i-1} \frac{x}{n} \\ &= \frac{1}{i} \log^i x + O\biggl\{ \log^{i-1} x \biggl( 1 + \sum_{b < n \le x} \frac{\Lambda(n)}{n} \biggr) \biggr\} \\ &= \frac{1}{i} \log^i x + O\biggl\{ \log^{i-1} x \biggl( 1 + \log \frac{x}{b} \biggr) \biggr\} \\ &= \frac{1}{i} \log^i x + O\biggl( \frac{\log^i x}{\log\log x} \biggr), \end{aligned} \tag{24}$$

and the $O$ is $i$-uniform.

By (22), writing $i-1$ for $i$ and $y = n$, we have, when $(x/2) < n < x$ and $i \ge 2$,

$$\begin{aligned} \log^{i-1} n &= \log^{i-1} x - (\log^{i-1} x - \log^{i-1} n) \\ &= \log^{i-1} x + O\biggl( \frac{x-n}{n} \log^{i-2} x \biggr) \\ &= \log^{i-1} x + O(\log^{i-2} x), \end{aligned}$$

and the $O$ depends on $i$. It is obvious that this formula is also valid when $i=1$. Using this formula we get from the definitions of $\Psi_i$ and $R_i$, see (9), (10), (6) and (2),

$$\begin{aligned}\Psi_i(x)-\Psi_i(y) &= R_i(x)-R_i(y)+\sum_{y<n\le x} i\log^{i-1} n \\ &= R_i(x)-R_i(y)+(x-y)i\log^{i-1} x \\ &\quad +O\{(x-y)\log^{i-2} x\},\end{aligned} \tag{25}$$

when $i\ge 1$, $y\ge (x/2)$ and $x-y\ge \log x$, and the $O$ depends on $i$.

If $y\ge (x/2)$, then $(y/n)\ge x/(2n)$ for $n\ge 1$, and if $x-y\ge b\log x$, then $(x/n)-(y/n)\ge (b/n)\log x\ge \log x\ge \log(x/n)$ for $1\le n\le b$. That is to say, when $y\ge (x/2)$, $x-y\ge b\log x$ and $1\le n\le b$, the condition of (25) is satisfied by $(x/n)$ and $(y/n)$ and so, in this case, we can apply (25) and get

$$\begin{aligned}\Psi_i\left(\frac{x}{n}\right)-\Psi_i\left(\frac{y}{n}\right) &= R_i\left(\frac{x}{n}\right)-R_i\left(\frac{y}{n}\right)+\left(\frac{x}{n}-\frac{y}{n}\right)i\log^{i-1}\frac{x}{n} \\ &\quad +O\left\{\left(\frac{x}{n}-\frac{y}{n}\right)\log^{i-2} x\right\}\quad \text{for } i\ge 1,\end{aligned}$$

and the $O$ depends on $i$.

Finally, using the last formula and (24), we obtain, when $y\ge (x/2)$, $x-y\ge b\log x$, $i\ge 1$ and $x$ is suitably large,

$$\begin{aligned}\sum_{n\le b}\Lambda(n)\left\{\Psi_i\left(\frac{x}{n}\right)-\Psi_i\left(\frac{y}{n}\right)\right\} &= \sum_{n\le b}\Lambda(n)\left\{R_i\left(\frac{x}{n}\right)-R_i\left(\frac{y}{n}\right)\right\} \\ &\quad +\sum_{n\le b}\Lambda(n)\frac{x-y}{n}i\log^{i-1}\frac{x}{n} \\ &\quad +O\{(x-y)\log^{i-1} x\} \\ &= \sum_{n\le b}\Lambda(n)\left\{R_i\left(\frac{x}{n}\right)-R_i\left(\frac{y}{n}\right)\right\} \\ &\quad +(x-y)\log^{i} x \\ &\quad +O\left\{(x-y)\frac{\log^{i} x}{\log\log x}\right\},\end{aligned} \tag{26}$$

and the $O$ depends on $i$.

**Lemma 1.** *If* $y \geq (x/2)$ *and* $x - y \geq x\exp(-\log^{1-\sigma} x)$, *where* $\sigma$ *is a constant,* $0 < \sigma < 1$, *then, when* $x$ *is suitably large, we have*

$$\begin{aligned} R_i(x) - R_i(y) = {} & \{R_{i-1}(x) - R_{i-1}(y)\}\log x \\ & + \sum_{n \leq b} \Lambda(n)\left\{R_{i-1}\left(\frac{x}{n}\right) - R_{i-1}\left(\frac{y}{n}\right)\right\} \\ & + O\{|R_{i-1}(x) - R_{i-1}(y)| \\ & + (x-y)\frac{\log^{i-1} x}{\log\log x}\right\} \quad \text{for } i \geq 2, \end{aligned}$$

and the $O$ depends on $\sigma$ and $i$.

*Proof.* We first consider the remainder term of (13). From (14) and the definitions of $R_i, R$ and $\Lambda_i$, see (6), (2) and (1), we have

$$\begin{aligned} \sum_{y<n\leq x} |\Lambda_{i-1}(n)| &= \sum_{y<n\leq x} \Lambda_{i-1}(n) \\ &= R_{i-1}(x) - R_{i-1}(y) + \sum_{y<n\leq x} (i-1)\log^{i-2} n \\ &= O\{|R_{i-1}(x) - R_{i-1}(y)| + (x-y)\log^{i-2} x\}, \end{aligned}$$

when $y \geq (x/2)$, $x - y \geq x\exp(-\log^{1-\sigma} x)$, $i \geq 2$, and the $O$ depends on $i$.

Now we consider the other terms of (13). Noticing (12), we see that there exists a constant $\sigma_0$ depending only on $\sigma$ such that, for $x \geq \sigma_0$, $x\exp(-\log^{1-\sigma} x) \geq x\exp(-\log x/\log\log x)\log x = b\log x$ and $x\exp(-\log^{1-\sigma} x) \geq x^{5/6}$, and so, if $x$ and $y$ satisfy the given condition $x - y \geq x\exp(-\log^{1-\sigma} x)$ in the lemma, then, when $x \geq \sigma_0$, $x$ and $y$ satisfy $x - y \geq b\log x$ and $x - y \geq x^{5/6}$. That is to say, if $x$ and $y$ satisfy the conditions of the lemma, then, when $x \geq \sigma_0$, $x$ and $y$ also satisfy the conditions of (21), (25) and (26). Hence we can substitute (21), (25) and (26) in (13); doing so and then reducing, the lemma follows.

**4. The estimate of $\Lambda(n) - 1$ in short intervals.** The purpose of this section is to prove

**Lemma 2.** *Let $\delta$ be an arbitrary constant, $0 < \delta < 1$, and let $x$ be suitably large. If, for an arbitrary constant $A$, $A > 1$, the inequality*

$$|R_2(x) - R_2(y)| \leq \frac{1}{A}(x-y)\log x + O\bigg\{(x-y)\frac{\log x}{\log\log x}\bigg\} \tag{27}$$

*holds when*

$$x - y \geq x\exp(-\log^{1-(\delta/2)} x) \quad \text{and} \quad y \geq \frac{x}{2}, \tag{28}$$

*then*

$$|R(x) - R(y)| < \frac{19}{20}(x-y) + O\bigg(\frac{x-y}{\log\log x}\bigg) \tag{29}$$

*holds when*

$$x - y \geq x\exp(-\log^{1-\delta} x), \tag{30}$$

*and the $O$ in (27) depends on $\delta$ and $A$ while the last $O$ depends only on $\delta$.*

In Lemma 2 we obtain an estimate of $\Lambda(n)-1$ in short intervals under the condition (27) which will be proved in the next section.

From (30) we see that the estimate of $R(x) - R(y)$ in Lemma 2 is very refined. Since (30) is determined by (28), the estimated result of $R(x)-R(y)$ in this section is in fact determined by the estimated result of $R_2(x) - R_2(y)$ in the next section.

We write condition (27) in another form. Taking $i = 2$, $\sigma = (\delta/2)$ and substituting (27) in Lemma 1, we get, when $y \geq (x/2)$, $x - y \geq x\exp(-\log^{1-(\delta/2)} x)$ and $x$ is suitably large,

$$\begin{aligned} \bigg|\{R_1(x) - R_1(y)\}\log x + \sum_{n \leq b} \Lambda(n)\bigg\{R_1\bigg(\frac{x}{n}\bigg) - R_1\bigg(\frac{y}{n}\bigg)\bigg\} \\ + O\{|R_1(x) - R_1(y)|\}\bigg| \\ \leq \frac{1}{A}(x-y)\log x + O\bigg\{(x-y)\frac{\log x}{\log\log x}\bigg\}, \end{aligned} \tag{31}$$

where $b$ is defined by (12), and the first $O$ depends on $\delta$ while the latter $O$ depends on $\delta$ and $A$. Since $R_1(x) - R_1(y) = R(x) - R(y)$ and $R_1(x) - R_1(y) \ll x - y$ by the definition of $R_1$, see (6) and (20), we can write (31) in the form

$$\begin{aligned}\{R(x) - R(y)\} \log x = &-\sum_{n \le b} \Lambda(n) \left\{ R\left(\frac{x}{n}\right) - R\left(\frac{y}{n}\right) \right\} \\ &+ \frac{\theta}{A}(x-y)\log x + O\left\{ (x-y)\frac{\log x}{\log\log x} \right\},\end{aligned} \tag{32}$$

where $\theta$ satisfies $|\theta| \le 1$, $x$ and $y$ satisfy the conditions of (31), and the $O$ depends on $\delta$ and $A$.

Thus, Lemma 2 can be proved if we can derive (29) from (32).

We prepare for proving Lemma 2 from Lemma 3–Lemma 11 below.

The following lemma is significant for proving Lemma 2.

**Lemma 3.** *Let $X, Y$ be arbitrarily large positive numbers, $X > 2Y$, and let $f, g$ be real $L$-measurable functions defined on $[0, X]$ satisfying*

$$\frac{1}{x}\int_0^x f^2(y)\,dy \le F, \qquad \frac{1}{x}\int_0^x g^2(y)\,dy \le G, \tag{33}$$

*for $0 < x \le X$, where $F, G$ are positive constants. Define*

$$h(x) = \frac{1}{x}\int_0^x f(x-y)g(y)\,dy, \quad 0 < x \le X. \tag{34}$$

*If $h(x)$ satisfies*

$$\frac{1}{x}\left|\int_0^x h(y)\,dy\right| \le \frac{1}{4}\varepsilon^3\sqrt{FG} \quad \text{for } Y \le x \le X, \tag{35}$$

*where $\varepsilon$ is the reciprocal of an arbitrarily large natural number, then we have*

$$\int_Y^X h^2(y)\,dy \le \left(\frac{1}{2} + K\varepsilon\right) FG(X - Y),$$

*where $K$ is a positive constant independent of $\varepsilon$, $X$, $Y$, $F$ and $G$.*

*Proof.* We prove the lemma only when $F = G = 1$, or we can write $f(x)\sqrt{F}$, $g(x)\sqrt{G}$ and $h(x)\sqrt{FG}$ for $f(x)$, $g(x)$ and $h(x)$, respectively, to transform the formulae into the case $F = G = 1$.

Let $\varepsilon = (1/N)$, $N$ being an arbitrarily large natural number, and let

$$x_n = (1+\varepsilon)^n, \quad n = 1, 2, \dots .$$

If, for arbitrary $n$, when $x_n \geq Y$ and $x_{n+1} \leq X$, the inequality

$$\int_{x_n}^{x_{n+1}} h^2(y)\,dy \leq \left(\frac{1}{2} + K_1\varepsilon\right)(x_{n+1} - x_n) \tag{36}$$

holds, where $K_1$ is a constant independent of $\varepsilon, X, Y, x_n$ and $x_{n+1}$, then we can at once prove the lemma when $F = G = 1$.

Inequality (36) is implicit in the proof of Lemma 1 of [**6**, pp. 2–6]. We cannot obtain (36) from [**6**, Lemma 1] directly. But, in the course of proving Lemma 1 of [**6**], Wirsing has proved the result the same as (36) of our paper by using only the conditions:

$$\int_0^x f^2(y)\,dy \leq x, \qquad \int_0^x g^2(y)\,dy \leq x,$$
$$\text{for } 0 < x \leq x_{n+1},$$

and

$$\left| \int_{x_{n(v-1)}}^{x_{nv}} h(x)\,dx \right| \leq \varepsilon(x_{nv} - x_{n(v-1)}),$$

where

$$x_{nv} = \frac{v}{N}(x_{n+1} - x_n) + x_n, \quad v = 0, 1, \dots, N.$$

When $x_n \geq Y$ and $x_{n+1} \leq X$, these conditions can easily be deduced from (33) and (35) in our paper. And, hence, we can obtain (36) from the proof of Lemma 1 of [**6**]. The lemma is thus proved.

The following argument is how to transform (32) into another form similar to (34), for which Lemma 3 can be applied.

Write

$$a = \exp\left(\frac{\log x}{\log\log x}\right). \tag{37}$$

**Lemma 4.** *If $x$ and $y$ satisfy the conditions of* (31), *then*

$$\sum_{n\le b}\Lambda(n)\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}=\sum_{a<n\le b}\{\Lambda(n)-1\}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}+O\left\{(x-y)\frac{\log x}{\log\log x}\right\},$$

*and the $O$ depends on $\delta$.*

*Proof.* From the definitions of $R$ and $\Psi$, see (2), (10), we have

$$\sum_{n\le x}R\left(\frac{x}{n}\right)=\sum_{n\le x}\Psi\left(\frac{x}{n}\right)-\sum_{n\le x}\left[\frac{x}{n}\right]+2\gamma[x],$$

(symbol $[x]$ denoting the integral part of $x$, see Section 2). Using a familiar formula

$$\sum_{n\le x}\Psi\left(\frac{x}{n}\right)=\sum_{n\le x}\log n=x\log x-x+O(\sqrt{x}),\quad x\ge 1,$$

and the well-known formula of Dirichlet

$$\sum_{n\le x}\left[\frac{x}{n}\right]=x\log x+(2\gamma-1)x+O(\sqrt{x}),\quad x\ge 1,$$

we get

$$\sum_{n\le x}R\left(\frac{x}{n}\right)\ll\sqrt{x}\quad\text{for } x\ge 1. \tag{38}$$

Furthermore, when $x, y$ satisfy the conditions of (31),

$$\begin{aligned}\left|\sum_{b<n\le x}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}\right| &\le\sum_{b<n\le x}\ \sum_{(y/n)<m\le(x/n)}\{\Lambda(m)+1\}\\ &\le\sum_{m\le(x/b)}\{\Lambda(m)+1\}\sum_{(y/m)<n\le(x/m)}1\\ &\ll(x-y)\sum_{m\le(x/b)}\frac{\Lambda(m)+1}{m} \qquad (39)\\ &\ll(x-y)\log(x/b)\\ &=(x-y)\frac{\log x}{\log\log x},\end{aligned}$$

the last step following from (12) and the $\ll$'s depending on $\delta$.

Using (39) and (38), we have, when $x, y$ satisfy the conditions of (31),

$$\begin{aligned}\sum_{n\le b}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\} &= \sum_{n\le x}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}\\ &\quad+O\left\{(x-y)\frac{\log x}{\log\log x}\right\}\\ &=\sum_{n\le x}R\left(\frac{x}{n}\right)-\sum_{n\le y}R\left(\frac{y}{n}\right)\\ &\quad+O\left\{(x-y)\frac{\log x}{\log\log x}\right\}\\ &=O\left\{(x-y)\frac{\log x}{\log\log x}\right\},\end{aligned}$$

this gives

$$\begin{aligned}\sum_{n\le b}\Lambda(n)\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\} &= \sum_{n\le b}\{\Lambda(n)-1\}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}\\ &\quad+O\left\{(x-y)\frac{\log x}{\log\log x}\right\},\end{aligned}\tag{40}$$

and the $O$ depends on $\delta$.

Applying (20) and noticing (37), we have, when $x, y$ satisfy the conditions of (31),

$$\begin{aligned}\sum_{n\le a}\{\Lambda(n)-1\}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\} &\ll (x-y)\sum_{n\le a}\frac{\Lambda(n)+1}{n}\\ &\ll (x-y)\log a\\ &=(x-y)\frac{\log x}{\log\log x},\end{aligned}$$

and the $\ll$'s depend on $\delta$. On substitution in (40), we get the lemma.

Now we smooth the coefficient $\Lambda(n)-1$ in the expression $\sum_{a<n\le b}\{\Lambda(n)-1\}\{R(x/n)-R(y/n)\}$.

First we give a sequence $a_n$ defined by

$$a_0 = 3; \quad a_{n+1} = a_n + [a_n \exp(-\log^{1-(\delta/3)} a_n)], \quad n = 0, 1, 2, \dots . \tag{41}$$

Next we define a function $S$ by

$$\begin{aligned} S(a_n) &= R(a_n), \quad n = 0, 1, 2, \dots , \\ S(x) &= k_n(x - a_n) + R(a_n) \quad \text{for } x \in [a_n, a_{n+1}], \\ & \qquad n = 0, 1, 2, \dots , \end{aligned} \tag{42}$$

where

$$k_n = \frac{R(a_{n+1}) - R(a_n)}{a_{n+1} - a_n}. \tag{43}$$

By (42), the derivative of $S(x)$ is $k_n$ when $a_n < x < a_{n+1}$, i.e., $S'(x) = k_n$ for $x \in (a_n, a_{n+1})$. Now we define $S'(x)$ at the points $a_1, a_2, \dots$ by $S'(a_{n+1}) = k_n$, $n = 0, 1, 2, \dots$ . Thus we have $S'(x) = k_n$ for $x \in (a_n, a_{n+1}]$. Hence, by (43),

$$S'(x) = k_n = \frac{R(a_{n+1}) - R(a_n)}{a_{n+1} - a_n}, \tag{44}$$

for $x \in (a_n, a_{n+1}]$, $n = 0, 1, 2, \dots$ .

**Lemma 5.** *When $x$ is suitably large, we have*

$$|S'(x)| \le 1 + O(\log^{-\delta/4} x),$$

*and the $O$ depends on $\delta$.*

*Proof.* Given any $x$, $x$ being suitably large, we can find a corresponding integer $n$ such that $a_n < x \le a_{n+1}$. Therefore, by (44),

$$|S'(x)| = \frac{|R(a_{n+1}) - R(a_n)|}{a_{n+1} - a_n}, \tag{45}$$

for given $x$. Further, from (41), we see that there exists a constant $\delta_0$ depending only on $\delta$ such that, when $a_n > \delta_0$,

$$a_{n+1} - a_n = [a_n \exp(-\log^{1-(\delta/3)} a_n)] \ge a_{n+1} \exp(-\log^{1-(\delta/4)} a_{n+1});$$

thus, when $a_n > \delta_0$, we can apply (20) to (45) and get, for given $x$,

$$|S'(x)| \leq 1 + O(\log^{-\delta/4} a_{n+1}) = 1 + O(\log^{-\delta/4} x),$$

(for the latter step noticing that $x \leq a_{n+1}$ for given $x$) and the $O$ depends on $\delta$. Because $x$ is arbitrary when $x$ is suitably large, we establish the lemma.

**Lemma 6.**

$$S'(m) = S(m) - S(m-1), \quad m = 4, 5, 6, \dots .$$

*Proof.* Noting that symbol $[x]$, given in Section 2, denotes the integral part of $x$, we see from (41) that $a_0, a_1, a_2, \dots$ are integers. Therefore, given any integer $m$, $m \geq 4$, we can find a corresponding integer $n$ such that $a_n + 1 \leq m \leq a_{n+1}$, that is, $m \in [a_n, a_{n+1}]$ and $m - 1 \in [a_n, a_{n+1}]$; thus, taking $x = m$ and $x = m - 1$ in (42), respectively, we can get, for given $m$,

$$S(m) = k_n(m - a_n) + R(a_n), \tag{46}$$

and

$$S(m-1) = k_n(m - 1 - a_n) + R(a_n). \tag{47}$$

Subtracting (47) from (46) we get, for given $m$,

$$S(m) - S(m-1) = k_n. \tag{48}$$

On the other hand, since $a_n + 1 \leq m \leq a_{n+1}$ for given $m$, we can get from (44)

$$S'(m) = k_n,$$

for given $m$. Combining this with (48), we get

$$S'(m) = S(m) - S(m-1),$$

for given $m$. Because $m$ $(\geq 4)$ is arbitrary, we establish the lemma.

From Lemma 6 we get, for arbitrary positive integers $m, n$, $n < m$,

$$\sum_{i=n+1}^{m} S'(i) = S(m) - S(n). \tag{49}$$

**Lemma 7.** *For arbitrary integer $m$, $m \geq 4$, we have*

$$S'(x) = S'(m) \quad \textit{for } m-1 < x \leq m.$$

*Proof.* In the proof of Lemma 6 we have proved that, corresponding to any given integer $m$, $m \geq 4$, we can find an integer $n$ such that $a_n + 1 \leq m \leq a_{n+1}$. Now $m$ and $n$ are thus fixed. So, if $m - 1 < x \leq m$, we have $x \in (a_n, a_{n+1}]$ so that, from (44), $S'(x) = k_n$ for $m-1 < x \leq m$. Taking $x = m$ in this equality we have $S'(m) = k_n$. Comparing both equalities we have $S'(x) = S'(m)$ for $m - 1 < x \leq m$. Because $m$ ($\geq 4$) is arbitrary, we establish the lemma.

**Lemma 8.** *For $x > 3$,*

$$S(x) - R(x) \ll x \exp\left(-\log^{1-(\delta/3)} \frac{x}{2}\right) \log x,$$

*and the $\ll$ is absolute.*

*Proof.* Given any $x$, $x > 3$, we can find a corresponding integer $n$ such that $a_n < x \leq a_{n+1}$. Now $x$ and $n$ are thus fixed. Plainly,

$$\begin{aligned} |S(x) - R(x)| &= |S(x) - R(a_n) - R(x) + R(a_n)| \\ &\leq |S(x) - R(a_n)| + |R(x) - R(a_n)|. \end{aligned} \tag{50}$$

Since $a_n < x \leq a_{n+1}$, we get, by (42) and (43),

$$S(x) - R(a_n) = k_n(x - a_n) = \frac{R(a_{n+1}) - R(a_n)}{a_{n+1} - a_n}(x - a_n).$$

Since $0 < x - a_n \le a_{n+1} - a_n$ when $a_n < x \le a_{n+1}$, it follows that

$$\begin{aligned} |S(x) - R(a_n)| &\le |R(a_{n+1}) - R(a_n)| \\ &\le \sum_{a_n < m \le a_{n+1}} \{\Lambda(m) + 1\}. \end{aligned} \tag{51}$$

Furthermore, since $a_n < x \le a_{n+1}$, we have

$$\begin{aligned} |R(x) - R(a_n)| &\le \sum_{a_n < m \le x} \{\Lambda(m) + 1\} \\ &\le \sum_{a_n < m \le a_{n+1}} \{\Lambda(m) + 1\}. \end{aligned} \tag{52}$$

Substituting (51) and (52) into (50) we get

$$\begin{aligned} |S(x) - R(x)| &\le 2 \sum_{a_n < m \le a_{n+1}} \{\Lambda(m) + 1\} \\ &\ll (a_{n+1} - a_n) \log a_{n+1}, \end{aligned} \tag{53}$$

and the $\ll$ is absolute.

From (41) it is easily seen that $a_{n+1} - a_n \le a_n$, $n \ge 0$, that is, $a_{n+1} \le 2a_n$, $n \ge 0$. Then, since the $x$ and $n$ given above satisfy $a_n < x \le a_{n+1}$, we have $x \le a_{n+1} \le 2a_n$ and $x > a_n \ge (1/2)a_{n+1}$. And, therefore, from (41) again

$$a_{n+1} - a_n \le a_n \exp(-\log^{1-(\delta/3)} a_n) < x \exp\left(-\log^{1-(\delta/3)} \frac{x}{2}\right);$$

moreover,

$$\log_{a_{n+1}} \le \log(2x) \ll \log x.$$

Substituting in (53) we get

$$S(x) - R(x) \ll x \exp\left(-\log^{1-(\delta/3)} \frac{x}{2}\right) \log x,$$

and the $\ll$ is absolute. Because $x$ is arbitrary, we establish the lemma.

**Lemma 9.** *When $x$ is suitably large, we have*

$$\sum_{a<n\le b}\{\Lambda(n)-1\}\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}$$
$$=\sum_{a<n\le b}S'(n)\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\}$$
$$+O\left\{x\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log^2 x\right\},$$

*and the $O$ is absolute.*

*Proof.* Let

$$H_1=\sum_{a<n\le b}\{\Lambda(n)-1-S'(n)\}R\left(\frac{x}{n}\right).$$

By the definition of $R$, see (2),

$$\begin{aligned}H_1&=\sum_{a<n\le b}\{\Lambda(n)-1-S'(n)\}\left\{\sum_{m\le(x/n)}(\Lambda(m)-1)+2\gamma\right\}\\ &=\sum_{m\le(x/a)}\{\Lambda(m)-1\}\sum_{a<n\le\min((x/m),b)}\{\Lambda(n)-1-S'(n)\}\\ &\quad+2\gamma\sum_{a<n\le b}\{\Lambda(n)-1-S'(n)\}.\end{aligned}\tag{54}$$

Write $d=\min((x/m),b)$. Applying (49) and Lemma 8 we have

$$\begin{aligned}\sum_{a<n\le d}\{\Lambda(n)-1-S'(n)\}&=R([d])-S([d])-\{R([a])-S([a])\}\\ &\ll d\exp\left(-\log^{1-(\delta/3)}\frac{[a]}{2}\right)\log d\\ &\le\frac{x}{m}\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log x,\end{aligned}$$

and the $\ll$ is absolute.

In like manner,

$$\sum_{a<n\le b}\{\Lambda(n)-1-S'(n)\} \ll x\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log x,$$

and the $\ll$ is absolute.

Substituting the last two inequalities in (54), we get

$$\begin{aligned} H_1 &\ll x\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log x\sum_{m\le x}\frac{\Lambda(m)+1}{m} \\ &\ll x\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log^2 x, \end{aligned}$$

and the $\ll$ is absolute. That is, by the definition of $H_1$,

$$\begin{aligned} \sum_{a<n\le b}\{\Lambda(n)-1\}R\left(\frac{x}{n}\right) &= \sum_{a<n\le b}S'(n)R\left(\frac{x}{n}\right) \\ &\quad + O\left\{x\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log^2 x\right\}, \end{aligned}$$

and the $O$ is absolute.

In like manner,

$$\begin{aligned} \sum_{a<n\le b}\{\Lambda(n)-1\}R\left(\frac{y}{n}\right) &= \sum_{a<n\le b}S'(n)R\left(\frac{y}{n}\right) \\ &\quad + O\left\{y\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log^2 y\right\}, \end{aligned}$$

and the $O$ is absolute.

The lemma follows from the last two formulas.

**Lemma 10.** *When $x$ is suitably large, we have*

$$\begin{aligned} \sum_{a<n\le b}S'(n)\left\{R\left(\frac{x}{n}\right)-R\left(\frac{y}{n}\right)\right\} &= \int_a^b S'(t)\left\{R\left(\frac{x}{t}\right)-R\left(\frac{y}{t}\right)\right\}dt \\ &\quad + O\left(\frac{x}{a}\right), \end{aligned}$$

*and the $O$ depends on $\delta$.*

*Proof.* Let

$$H_2 = \sum_{a<n\le b} S'(n)R\left(\frac{x}{n}\right) - \int_a^b S'(t)R\left(\frac{x}{t}\right)dt,$$

then

$$\begin{aligned} H_2 &= \sum_{a<n\le b}\left\{S'(n)R\left(\frac{x}{n}\right) - \int_{n-1}^{n} S'(t)R\left(\frac{x}{t}\right)dt\right\} \\ &\quad - \int_{[b]}^{b} S'(t)R\left(\frac{x}{t}\right)dt + \int_{[a]}^{a} S'(t)R\left(\frac{x}{t}\right)dt. \end{aligned} \tag{55}$$

Using Lemma 5 we can get

$$\begin{aligned} \int_{[b]}^{b} S'(t)R\left(\frac{x}{t}\right)dt &\ll \frac{x}{b}, \\ \int_{[a]}^{a} S'(t)R\left(\frac{x}{t}\right)dt &\ll \frac{x}{a}, \end{aligned} \tag{56}$$

and the $\ll$'s depend on $\delta$. By Lemma 7, we have

$$\int_{n-1}^{n} S'(t)R\left(\frac{x}{t}\right)dt = S'(n)\int_{n-1}^{n} R\left(\frac{x}{t}\right)dt. \tag{57}$$

Substituting (56) and (57) in (55) we get

$$H_2 = \sum_{a<n\le b} S'(n)\int_{n-1}^{n}\left\{R\left(\frac{x}{n}\right) - R\left(\frac{x}{t}\right)\right\}dt + O\left(\frac{x}{a}\right),$$

and, using Lemma 5 again, we have

$$\begin{aligned} H_2 &\ll \sum_{a<n\le b}\int_{n-1}^{n}\sum_{(x/n)<m\le(x/t)}\{\Lambda(m)+1\}\,dt + \frac{x}{a} \\ &\le \sum_{a<n\le b}\ \sum_{(x/n)<m\le x/(n-1)}\{\Lambda(m)+1\} + \frac{x}{a} \\ &\le \sum_{m\le x/(a-1)}\{\Lambda(m)+1\} + \frac{x}{a} \ll \frac{x}{a}, \end{aligned}$$

and the $\ll$ depends on $\delta$.

In like manner,

$$\sum_{a<n\le b} S'(n)R\left(\frac{y}{n}\right) - \int_a^b S'(t)R\left(\frac{y}{t}\right)dt \ll \frac{x}{a},$$

and the $\ll$ depends on $\delta$.

The lemma follows from the last two inequalities.

**Lemma 11.** *If $y \ge (x/2)$, $x - y \ge x\exp(-\log^{1-(\delta/2)} x)$, then when $x$ is suitably large, we have*

$$\{R(x) - R(y)\}\log x = -\int_a^b S'(t)\left\{R\left(\frac{x}{t}\right) - R\left(\frac{y}{t}\right)\right\}dt + \frac{\theta}{A}(x-y)\log x + O\left\{(x-y)\frac{\log x}{\log\log x}\right\},$$

*and the $O$ depends on $\delta$ and $A$.*

*Proof.* Combining Lemma 9 with Lemma 10, we get

$$\text{(58)}\quad \sum_{a<n\le b}\{\Lambda(n)-1\}\left\{R\left(\frac{x}{n}\right) - R\left(\frac{y}{n}\right)\right\} = \int_a^b S'(t)\left\{R\left(\frac{x}{t}\right) - R\left(\frac{y}{t}\right)\right\}dt + O\left\{x\exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log^2 x\right\},$$

and the $O$ depends on $\delta$.

By (37), when $x$ is suitably large, (then $a$ is suitably large),

$$\log\frac{a}{4} \ge \log\sqrt{a} = \frac{1}{2}\log a = \frac{\log x}{2\log\log x}.$$

Using this inequality and the condition of the lemma, we can get, when $x$ is suitably large,

$$
\begin{aligned}
x - y &\geq x \exp(-\log^{1-(\delta/2)} x) \\
&\gg x \exp\left(-\left(\frac{\log x}{2\log\log x}\right)^{1-(\delta/3)}\right)\log^2 x \\
&\geq x \exp\left(-\log^{1-(\delta/3)}\frac{a}{4}\right)\log^2 x,
\end{aligned} \tag{59}
$$

and the $\gg$ depends on $\delta$.

Combining (59), (58), (32) and Lemma 4, the lemma follows immediately.

*Proof of Lemma* 2. Let $u$ be an arbitrary positive number, and let $u$ be suitably large. Put

$$v = u^{1-(\delta/(2-\delta))}; \tag{60}$$

then

$$v^{1-(\delta/2)} = v^{(2-\delta)/2} = u^{(1-(\delta/(2-\delta)))((2-\delta)/2)} = u^{1-\delta}. \tag{61}$$

Also let $g$ be an arbitrary positive number satisfying

$$1 - g \geq \exp(-u^{1-\delta}), \quad g \geq (1/2). \tag{62}$$

Then from (62) we have

$$xg \geq (x/2) \quad \text{for } x > 0. \tag{63}$$

And, from (62) and (61) we get, when $v \leq \log x \leq u$,

$$
\begin{aligned}
x - xg = x(1-g) &\geq x\exp(-u^{1-\delta}) \\
&= x\exp(-v^{1-(\delta/2)}) \\
&\geq x\exp(-\log^{1-(\delta/2)} x).
\end{aligned} \tag{64}
$$

If $y = xg$, then from (63) and (64) we see that, when $v \le \log x \le u$, $x$ and $y$ satisfy the conditions of Lemma 11. Hence, when $v \le \log x \le u$, we can take $y = xg$ in Lemma 11 and get

$$\{R(x) - R(xg)\} \log x = -\int_a^b S'(t)\left\{R\left(\frac{x}{t}\right) - R\left(\frac{xg}{t}\right)\right\} dt + \frac{\theta}{A}(x - xg)\log x + O\left\{(x - xg)\frac{\log x}{\log\log x}\right\},$$

and the $O$ depends on $\delta$ and $A$. And then, on writing $\xi = \log x$, $\eta = \log t$, $\alpha = \log a$ and $\beta = \log b$, we have

$$\{R(e^\xi) - R(e^\xi g)\}\xi = -\int_\alpha^\beta S'(e^\eta)\{R(e^{\xi-\eta}) - R(e^{\xi-\eta}g)\}e^\eta \, d\eta + \frac{\theta}{A}e^\xi(1-g)\xi + O\left\{e^\xi(1-g)\frac{\xi}{\log \xi}\right\}, \tag{65}$$

for $v \le \xi \le u$, and the $O$ depends on $\delta$ and $A$.

Furthermore, from (37) and (12), we have

$$\alpha = \log a = \frac{\log x}{\log\log x} = \frac{\xi}{\log \xi}, \tag{66}$$

$$\beta = \log b = \log x - \frac{\log x}{\log\log x} = \xi - \frac{\xi}{\log \xi}. \tag{67}$$

Define

$$f(\xi) = \begin{cases} \{R(e^\xi) - R(e^\xi g)\}e^{-\xi}(1-g)^{-1} & \text{if } v < \xi \le u, \\ 0 & \text{if } 0 \le \xi \le v, \end{cases} \tag{68}$$

$$g(\xi) = \begin{cases} S'(e^\xi) & \text{if } \xi > v, \\ 0 & \text{if } 0 \le \xi \le v. \end{cases} \tag{69}$$

Noting (64) and $\xi = \log x$, i.e., $x = e^\xi$, we can apply (20) with $\sigma = 1 - (\delta/2)$, $x = e^\xi$ and $y = e^\xi g$ to (68) and get

$$|f(\xi)| \le 1 + O(\xi^{-\delta/2}) = 1 + O(v^{-\delta/2}),$$

for $v < \xi \leq u$, and the $O$ depends on $\delta$. Furthermore, applying Lemma 5 with $x = e^{\xi}$ to (69), we get

$$|g(\xi)| \leq 1 + O(\xi^{-\delta/4}) = 1 + O(v^{-\delta/4}),$$

for $\xi > v$, and the $O$ depends on $\delta$.

Since, from the definitions of $f$ and $g$, $f(\xi) = g(\xi) = 0$ when $0 \leq \xi \leq v$, we get from the last two inequalities

$$\begin{aligned} &|f(\xi)| \leq 1 + O(v^{-\delta/2}), \\ &|g(\xi)| \leq 1 + O(v^{-\delta/4}), \end{aligned} \tag{70}$$

for $0 \leq \xi \leq u$, and both $O$'s depend on $\delta$.

Let

$$w = u^{1-(\delta/(2(2-\delta)))}. \tag{71}$$

From (66) and (67) we see that there exists a constant $\delta_1$ depending only on $\delta$ such that, for $u \geq \delta_1$ and $w \leq \xi \leq u$,

$$\alpha = \frac{\xi}{\log \xi} \geq \frac{w}{\log u} = \frac{1}{\log u} u^{1-(\delta/(2(2-\delta)))} > u^{1-(\delta/(2-\delta))} = v,$$

and similarly

$$\xi - \beta = \frac{\xi}{\log \xi} > v.$$

Hence, if $\alpha \leq \eta \leq \beta$, then $\eta \geq \alpha > v$ and $\xi - \eta \geq \xi - \beta > v$, for $u \geq \delta_1$ and $w \leq \xi \leq u$. Therefore, if $\alpha \leq \eta \leq \beta$, $u \geq \delta_1$ and $w \leq \xi \leq u$, we can get $R(e^{\xi}) - R(e^{\xi} g) = f(\xi) e^{\xi}(1 - g)$, $R(e^{\xi-\eta}) - R(e^{\xi-\eta} g) = f(\xi - \eta) e^{\xi-\eta}(1 - g)$ and $S'(e^{\eta}) = g(\eta)$ from (68) and (69). Substituting in (65) and multiplying the equality by $e^{-\xi}(1 - g)^{-1}$, we get

$$f(\xi)\xi = -\int_{\alpha}^{\beta} g(\eta) f(\xi - \eta)\, d\eta + \frac{\theta}{A}\xi + O\left(\frac{\xi}{\log \xi}\right), \tag{72}$$

for $w \leq \xi \leq u$, and the $O$ depends on $\delta$ and $A$.

Define

$$h(0) = 0; \qquad h(\xi) = \frac{1}{\xi}\int_0^\xi g(\eta)f(\xi-\eta)\,d\eta \quad 0 < \xi \le u. \tag{73}$$

By (70), we have

$$|h(\xi)| \le 1 + O(v^{-\delta/4}) \quad \text{for } 0 \le \xi \le u, \tag{74}$$

and the $O$ depends on $\delta$.

From the definition of $h$, we have

$$\begin{aligned} h(\xi)\xi - \int_\alpha^\beta g(\eta)f(\xi-\eta)\,d\eta &= \int_0^\xi g(\eta)f(\xi-\eta)\,d\eta - \int_\alpha^\beta g(\eta)f(\xi-\eta)\,d\eta \\ &= \int_\beta^\xi g(\eta)f(\xi-\eta)\,d\eta + \int_0^\alpha g(\eta)f(\xi-\eta)\,d\eta, \end{aligned} \tag{75}$$

for $w \le \xi \le u$. From (70) we see that $g(\eta)f(\xi-\eta) \ll 1$ for $0 \le \eta \le \xi$, $w \le \xi \le u$. Substituting in (75) and using (66) and (67), we get

$$h(\xi)\xi - \int_\alpha^\beta g(\eta)f(\xi-\eta)\,d\eta \ll \xi - \beta + \alpha \ll \frac{\xi}{\log \xi},$$

for $w \le \xi \le u$, and the $\ll$ depends on $\delta$. Combining this with (72), we obtain

$$h(\xi)\xi + f(\xi)\xi = \frac{\theta}{A}\xi + O\left(\frac{\xi}{\log \xi}\right) \quad \text{for } w \le \xi \le u,$$

namely,

$$h(\xi) + f(\xi) = \frac{\theta}{A} + O\left(\frac{1}{\log \xi}\right) \quad \text{for } w \le \xi \le u, \tag{76}$$

and the $O$ depends on $\delta$ and $A$.

Trivially,

$$\begin{aligned}
&\int_0^\xi \frac{R(e^\eta) - R(e^\eta g)}{e^\eta(1-g)}\,d\eta \\
&\qquad = \int_0^\xi \frac{1}{e^\eta(1-g)} \sum_{e^\eta g < n \le e^\eta} \{\Lambda(n) - 1\}\,d\eta \\
&\qquad = \sum_{n \le e^\xi} \frac{\Lambda(n) - 1}{1-g} \int_{\log n}^{\min(\log(n/g),\xi)} e^{-\eta}\,d\eta \\
&\qquad = \sum_{n \le e^\xi g} \frac{\Lambda(n) - 1}{1-g} \int_{\log n}^{\log(n/g)} e^{-\eta}\,d\eta \\
&\qquad\quad + O\biggl\{ \sum_{e^\xi g < n \le e^\xi} \frac{\Lambda(n) + 1}{1-g} \int_{\log n}^{\log(n/g)} e^{-\eta}\,d\eta \biggr\} \\
&\qquad = \sum_{n \le e^\xi g} \frac{\Lambda(n) - 1}{n} + O\biggl( \sum_{e^\xi g < n \le e^\xi} \frac{\Lambda(n) + 1}{n} \biggr) \\
&\qquad = O(1),
\end{aligned}$$

for $w \le \xi \le u$, and the $O$ is absolute. Using this, we get from (68)

$$\begin{aligned}
\int_w^\xi f(\eta)\,d\eta &= \int_w^\xi \frac{R(e^\eta) - R(e^\eta g)}{e^\eta(1-g)}\,d\eta \\
&= \int_0^\xi \frac{R(e^\eta) - R(e^\eta g)}{e^\eta(1-g)}\,d\eta - \int_0^w \frac{R(e^\eta) - R(e^\eta g)}{e^\eta(1-g)}\,d\eta \\
&= O(1),
\end{aligned}$$

for $w < \xi \le u$, and the $O$ is absolute.

Writing $\eta$ for $\xi$ in (76) and integrating (76) between the limits $w$ and $\xi$, we get

$$\int_w^\xi h(\eta)\,d\eta = -\int_w^\xi f(\eta)\,d\eta + \frac{\theta_1}{A}(\xi - w) + O\biggl(\frac{\xi}{\log w}\biggr) \quad \text{for } w < \xi \le u,$$

where $|\theta_1| \le 1$, and the $O$ depends on $\delta$ and $A$.

Using the last two formulas, noticing (74), (71) and $w \le (\xi/\log u) \le (\xi/\log \xi)$ for $w \log u \le \xi \le u$, we obtain

$$\begin{aligned}\int_0^\xi h(\eta)\,d\eta &= \int_w^\xi h(\eta)\,d\eta + O(w)\\ &= -\int_w^\xi f(\eta)\,d\eta + \frac{\theta_1}{A}\xi + O\bigg(\frac{\xi}{\log \xi}\bigg)\\ &= \frac{\theta_1}{A}\xi + O\bigg(\frac{\xi}{\log \xi}\bigg),\end{aligned} \tag{77}$$

for $w \log u \le \xi \le u$, and the $O$ depends on $\delta$ and $A$.

Let $N$ be an arbitrarily large natural number, and let $\varepsilon = (1/N)$. Taking $A = 8\varepsilon^{-3}$ in (77), we see from (70) and (77) that there exists a constant $v_0$ depending only on $\delta$ and $\varepsilon$ such that, when $v \ge v_0$,

$$\int_0^\xi f^2(\eta)\,d\eta \le (1+\varepsilon)\xi, \qquad \int_0^\xi g^2(\eta)\,d\eta \le (1+\varepsilon)\xi, \quad \text{for } 0 \le \xi \le u,$$

and

$$\bigg|\int_0^\xi h(\eta)\,d\eta\bigg| \le \frac{1}{4}\varepsilon^3\xi \quad \text{for } w\log u \le \xi \le u.$$

Thus, using Lemma 3 with $F = G = 1+\varepsilon$, $X = u$ and $Y = w\log u$, we get

$$\int_{w\log u}^u h^2(\eta)\,d\eta \le \bigg(\frac{1}{2} + K\varepsilon\bigg)(1+\varepsilon)^2 u,$$

where $K$ is independent of $\varepsilon$, $w$ and $u$.

We choose $\varepsilon$ suitably small such that $(1+\varepsilon)^2 < (11/10)$ and $(1/2) + K\varepsilon < (5/8)$, so that, when $v \ge v_0$,

$$\int_{w\log u}^u h^2(\eta)\,d\eta \le \frac{11}{16}u. \tag{78}$$

Using (74) and noting $|\theta| \le 1$, we get from (76)

$$\begin{aligned}f^2(\xi) &= \bigg\{ -h(\xi) + \frac{\theta}{A} + O\bigg(\frac{1}{\log \xi}\bigg)\bigg\}^2\\ &= h^2(\xi) - 2\frac{\theta}{A}h(\xi) + \frac{\theta^2}{A^2} + O\bigg(\frac{1}{\log\xi}\bigg)\\ &\le h^2(\xi) + \frac{2}{A} + \frac{1}{A^2} + O\bigg(\frac{1}{\log\xi}\bigg),\end{aligned}$$

for $w \le \xi \le u$, and the $O$ depends on $\delta$ and $A$. Integrating this between the limits $w \log u$ and $u$ and taking $A = 48$, we have

$$\int_{w \log u}^{u} f^2(\xi)\, d\xi \le \int_{w \log u}^{u} h^2(\xi)\, d\xi + \frac{u}{16} + O\left(\frac{u}{\log w}\right);$$

using this and (70) and (78), noting (71), we obtain

$$\begin{aligned} \int_0^u f^2(\xi)\, d\xi &= \int_{w \log u}^{u} f^2(\xi)\, d\xi + O(w \log u) \\ &\le \int_{w \log u}^{u} h^2(\xi)\, d\xi + \frac{u}{16} + O\left(\frac{u}{\log u}\right) \\ &\le \frac{3}{4}u + O\left(\frac{u}{\log u}\right), \end{aligned} \tag{79}$$

and the $O$ depends on $\delta$.

Using (72) with $\xi = u$, $A = 20$,

$$\begin{aligned} |f(u)|u \le &\left\{ \int_0^u g^2(\eta)\, d\eta \int_0^u f^2(u - \eta)\, d\eta \right\}^{1/2} \\ &+ \frac{u}{20} + O\left(\frac{u}{\log u}\right). \end{aligned}$$

Since $\int_0^u f^2(u-\eta)\, d\eta = \int_0^u f^2(\xi)\, d\xi$ by putting $\xi = u - \eta$, and since $\int_0^u g^2(\eta)\, d\eta \le u\{1 + O(v^{-\delta/4})\} = u\{1 + O(1/\log u)\}$ by using (70) and (60), it follows, using (79), that

$$\begin{aligned} |f(u)|u &\le \left\{ \frac{3}{4} + O\left(\frac{1}{\log u}\right) \right\}^{1/2} u + \frac{u}{20} + O\left(\frac{u}{\log u}\right) \\ &= \frac{1}{2}\sqrt{3}u + \frac{u}{20} + O\left(\frac{u}{\log u}\right) \\ &< \frac{19}{20}u + O\left(\frac{u}{\log u}\right), \end{aligned}$$

and the $O$ depends on $\delta$. Using (68), this inequality can be written in the form

$$|R(e^u) - R(e^u g)| < \frac{19}{20}(e^u - e^u g) + O\left(\frac{e^u - e^u g}{\log u}\right), \tag{80}$$

and the $O$ depends on $\delta$.

If $x$ and $y$ satisfy the condition (30) and $y \geq (x/2)$, then we can use (80) with $u = \log x$ and $g = (y/x)$ (then $x = e^u$ and $y = e^u g$), it is easy to verify that in this case $g$ satisfies (62) because $1 - g = 1 - (y/x) = ((x-y)/x) \geq \exp(-\log^{1-\delta} x) = \exp(-u^{1-\delta})$ and $g = (y/x) \geq (1/2)$, and get (29) when $y \geq (x/2)$. If $y < (x/2)$, we can get (29) from the prime number theorem too. Thus, we complete the proof of Lemma 2.

**5. The estimate of $\Lambda_i(n) - i\log^{i-1} n$ in short intervals.** In the preceding section we proved (29) under the hypothesis of (27). In this section we shall estimate $R_i(x) - R_i(y)$ for $i \geq 2$ which includes (27).

In the preceding section we use Lemma 1 with $i = 2$ to estimate $R(x) - R(y)$ under the hypothesis of the estimate of $R_2(x) - R_2(y)$. In this section we shall also use Lemma 1, on writing $i+1$ for $i$, to estimate $R_i(x) - R_i(y)$, $i \geq 2$, under the hypothesis of the estimate of $R_{i+1}(x) - R_{i+1}(y)$. Thus we can use the same method as that used in the last section to estimate $R_i(x) - R_i(y)$ for $i \geq 2$. But to estimate $R_i(x) - R_i(y)$ for $i \geq 2$ by using Lemma 1 is simpler than to estimate $R(x) - R(y)$ by using Lemma 1 because it is not effective if we straightforwardly apply Lemma 1 to the estimate of $R(x) - R(y)$ and, in this case, we must transform Lemma 1 into another form for which Wirsing's lemma can be applied, but it is effective if we straightforwardly apply Lemma 1 to the estimate of $R_i(x) - R_i(y)$, $i \geq 2$, and in this case we need not transform Lemma 1 into another form in order to use Wirsing's lemma. Thus we can get the estimate of $R_i(x) - R_i(y)$ when $i \geq 2$ as long as we can get an estimate of $R_i(x) - R_i(y)$ when $i$ is large.

Let $\rho$ be an arbitrary constant, $0 < \rho < (1/2)$, and define

$$w_t(x) = x\exp(-(\log x)^{1-\rho}(\log\log x)^{-t}), \quad t \geq 0. \tag{81}$$

**Lemma 12.** *When $x$ is suitably large, we have*

(i) *If $t < t'$, then $w_t(x) < w_{t'}(x)$.*

(ii) *If $m$ and $n$ are arbitrary integers satisfying $m \geq 1$ and $1 \leq n \leq b$,*

*$b$ being defined by* (12), *then*

$$w_m(x) \geq n w_{m-1}\left(\frac{x}{n}\right).$$

*Proof.* When $x$ is suitably large, $(\log\log x)^t$ is an increasing function of $t$ for every fixed $x$ and then by (81) $w_t(x)$ is also an increasing function of $t$ for every fixed $x$. This establishes (i).

We proceed to prove (ii). From (12),

$$\log\frac{x}{b} = \frac{\log x}{\log\log x},$$

then, when $m \geq 1$, $n \leq b$ and $x$ is suitably large, we have

$$\begin{aligned}(\log x)^{1-\rho}(\log\log x)^{-m} &\leq \left(\frac{\log x}{\log\log x}\right)^{1-\rho}(\log\log x)^{-(m-1)}\\ &= \left(\log\frac{x}{b}\right)^{1-\rho}(\log\log x)^{-(m-1)}\\ &\leq \left(\log\frac{x}{n}\right)^{1-\rho}\left(\log\log\frac{x}{n}\right)^{-(m-1)},\end{aligned}$$

and it follows from (81) that

$$\begin{aligned}\frac{1}{n}w_m(x) &= \frac{x}{n}\exp(-(\log x)^{1-\rho}(\log\log x)^{-m})\\ &\geq \frac{x}{n}\exp\left(-\left(\log\frac{x}{n}\right)^{1-\rho}\left(\log\log\frac{x}{n}\right)^{-(m-1)}\right)\\ &= w_{m-1}\left(\frac{x}{n}\right).\end{aligned}$$

This establishes (ii).

**Lemma 13.** *If $x$ and $y$ are suitably large, $y \geq (x/2)$ and $x - y \geq w_{i-1}(x)$, then*

$$|R_i(x) - R_i(y)| \leq (x-y)i\log^{i-1}x + O\left\{(x-y)\frac{\log^{i-1}x}{\log\log x}\right\},$$

*for $i \geq 1$, and the $O$ depends on $\rho$ and $i$.*

*Proof.* Let us apply inductive method. When $i = 1$, putting $\sigma = 1-\rho$ in (20) and noting (6), we have

$$\begin{aligned} |R_1(x) - R_1(y)| &= |R(x) - R(y)| \\ &\leq (x-y)\{1 + O(\log^{-\rho} x)\} \\ &= x - y + O\bigg\{(x-y)\frac{1}{\log\log x}\bigg\}, \end{aligned}$$

for $y \geq (x/2)$, $x - y \geq x\exp(-\log^{1-\rho} x) = w_0(x)$ and the $O$ depends on $\rho$.

Therefore the lemma is true when $i = 1$.

Suppose that the lemma is true when $i = m$, $m \geq 1$; that is to say, when $y \geq (x/2)$ and $x - y \geq w_{m-1}(x)$,

$$(82)\quad |R_m(x) - R_m(y)| \leq (x-y)m\log^{m-1} x + O\bigg\{(x-y)\frac{\log^{m-1} x}{\log\log x}\bigg\},$$

the $O$ depending on $\rho$ and $m$.

Then writing $(x/n)$ for $x$, $(y/n)$ for $y$ in (82), we have, when $(y/n) \geq x/(2n)$, $(x/n) - (y/n) \geq w_{m-1}(x/n)$ and $n \leq b$, $b$ being given by (12),

$$(83)\quad \begin{aligned} \left|R_m\left(\frac{x}{n}\right) - R_m\left(\frac{y}{n}\right)\right| &\leq \left(\frac{x}{n} - \frac{y}{n}\right) m\log^{m-1}\frac{x}{n} \\ &\quad + O\bigg\{\left(\frac{x}{n} - \frac{y}{n}\right)\frac{\log^{m-1} x}{\log\log x}\bigg\}, \end{aligned}$$

and the $O$ depends on $\rho$ and $m$.

Now we shall use Lemma 1 and (83) to prove that the lemma is true when $i = m+1$. If $y \geq (x/2)$ and $x - y \geq w_m(x)$, then $(y/n) \geq (x/(2n))$ for $n \geq 1$, and by using (ii) of Lemma 12, $(x/n) - (y/n) \geq (1/n)w_m(x) \geq w_{m-1}(x/n)$ for $n \leq b$. That is to say, in this case $x$ and $y$ satisfy the conditions of (83); thus, we can substitute (83) in Lemma 1 as long as we take $i = m+1$ and $\sigma = \rho$ in Lemma 1, noting that, when $x - y \geq w_m(x)$, $x - y \geq w_0(x) = x\exp(-\log^{1-\rho} x)$ by (i) of

Lemma 12 and (81); doing so, and then substituting (82) in it, noting that, when $x - y \geq w_m(x)$, $x - y \geq w_{m-1}(x)$ by (i) of Lemma 12, we get, when $y \geq (x/2)$ and $x - y \geq w_m(x)$,

$$\begin{aligned}|R_{m+1}(x) - R_{m+1}(y)| \leq (x-y)m\log^m x \\ + \sum_{n\leq b}\Lambda(n)\frac{x-y}{n}m\log^{m-1}\frac{x}{n} \\ + O\left\{(x-y)\frac{\log^m x}{\log\log x}\right\},\end{aligned}$$

and, by using (24), the right side of this inequality is

$$(x-y)(m+1)\log^m x + O\left\{(x-y)\frac{\log^m x}{\log\log x}\right\},$$

and the $O$ depends on $\rho$ and $m+1$. Therefore, the lemma is true when $i = m+1$, and this proves the lemma.

**Lemma 14.** *Let $N$ be an arbitrary constant, and let $M$ be an arbitrary and suitably large integer, $M > N \geq 1$. Also, let $i$ be a fixed integer, $i \geq 2$. If*

$$\begin{aligned}|R_{i+1}(x) - R_{i+1}(y)| \leq \frac{N}{M}(x-y)\log^i x \\ + O\left\{(x-y)\frac{\log^i x}{\log\log x}\right\}\end{aligned} \tag{84}$$

*holds when $w_h(x) \leq x - y \leq x\exp(-\sqrt{\log x})$, then*

$$\begin{aligned}|R_i(x) - R_i(y)| \leq \frac{3N}{M}(x-y)\log^{i-1} x \\ + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\}\end{aligned} \tag{85}$$

*holds when $w_{h+M}(x) \leq x - y \leq x\exp(-\sqrt{\log x})$, where $h$ is an integer, $h \geq i$, $x$ and $y$ are suitably large, both $O$'s depend on $\rho$, $i$ and $M$.*

*Proof.* Writing $i+1$ for $i$, putting $\sigma = \rho$ and transposing the terms in Lemma 1 and noting (81), we get

$$\begin{aligned}\{R_i(x) - R_i(y)\} \log x = R_{i+1}(x) - R_{i+1}(y) \\ - \sum_{n \le b} \Lambda(n) \left\{ R_i\left(\frac{x}{n}\right) - R_i\left(\frac{y}{n}\right) \right\} \\ + O\left\{ |R_i(x) - R_i(y)| + (x-y) \frac{\log^i x}{\log\log x} \right\},\end{aligned}$$

for $y \ge (x/2)$, $x - y \ge x \exp(-\log^{1-\rho} x) = w_0(x)$, and the $O$ depends on $\rho$ and $i$. By (i) of Lemma 12 and the given condition $h \ge i \ge 2$ in the lemma, we see that, when $x - y \ge w_h(x)$, $x$ and $y$ satisfy the condition $x - y \ge w_0(x)$ of this equality and the condition $x - y \ge w_{i-1}(x)$ of Lemma 13, and then we can substitute (84) in this equality and substitute Lemma 13 in the remainder term of this equality; finally we get

$$\begin{aligned}|R_i(x) - R_i(y)| \log x \le \frac{N}{M}(x-y) \log^i x \\ + \sum_{n \le b} \Lambda(n) \left| R_i\left(\frac{x}{n}\right) - R_i\left(\frac{y}{n}\right) \right| \\ + O\left\{ (x-y) \frac{\log^i x}{\log\log x} \right\},\end{aligned} \tag{86}$$

for $w_h(x) \le x - y \le x \exp(-\sqrt{\log x})$, and the $O$ depends on $\rho$, $i$ and $M$.

Now we shall use (86) to prove the lemma. First we give a sequence defined by

$$A_0 = i; \qquad A_n = \frac{N}{M} + \frac{1}{i} A_{n-1}, \quad n \ge 1. \tag{87}$$

Secondly, we use deductive method to prove that

$$A_n = \frac{1}{i^{n-1}} + \frac{N}{M} \sum_{j=0}^{n-1} \frac{1}{i^j}, \quad n \ge 1. \tag{88}$$

When $n = 1$, (88) holds obviously. Suppose (88) holds when $n = m$, $m \geq 1$. Then, from (87),

$$A_{m+1} = \frac{N}{M} + \frac{1}{i}A_m = \frac{N}{M} + \frac{1}{i}\left(\frac{1}{i^{m-1}} + \frac{N}{M}\sum_{j=0}^{m-1}\frac{1}{i^j}\right)$$
$$= \frac{1}{i^m} + \frac{N}{M}\sum_{j=0}^{m}\frac{1}{i^j},$$

hence (88) holds when $n = m + 1$, and this proves (88).

Further, we shall use (86) and the deductive method to prove that

$$(89) \qquad |R_i(x) - R_i(y)| \leq A_j(x-y)\log^{i-1} x + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\},$$

for $0 \leq j \leq M + 1$ and $w_{h+j-1}(x) \leq x - y \leq x\exp(-\sqrt{\log x})$, $x$ and $y$ being suitably large, and the $O$ depending on $\rho$, $i$ and $M$.

When $j = 0$, noting $A_0 = i$, we can get (89) from Lemma 13 as long as we can verify that, if $x$ and $y$ satisfy the conditions of (89), when $j = 0$, then $x$ and $y$ satisfy the conditions of Lemma 13. This can be proved as follows. By the given condition $h \geq i$ in this lemma and (i) of Lemma 12, we can get $x - y \geq w_{i-1}(x)$ from the condition $x - y \geq w_{h-1}(x)$ of (89). Furthermore, when $x$ and $y$ are suitably large, we can derive $y \geq x - x\exp(-\sqrt{\log x}) \geq (x/2)$ from the condition $x - y \leq x\exp(-\sqrt{\log x})$ of (89). Hence, (89) is true when $j = 0$.

Suppose that (89) is true when $j = m$, $0 \leq m \leq M$; that is to say,

$$|R_i(x) - R_i(y)| \leq A_m(x-y)\log^{i-1} x + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\},$$

for $w_{h+m-1}(x) \leq x - y \leq x\exp(-\sqrt{\log x})$, $x$ and $y$ being suitably large, and the $O$ depending on $\rho$, $i$ and $M$. Writing $(x/n)$ for $x$, $(y/n)$ for $y$, we have

$$(90) \qquad \left|R_i\left(\frac{x}{n}\right) - R_i\left(\frac{y}{n}\right)\right| \leq A_m\left(\frac{x}{n} - \frac{y}{n}\right)\log^{i-1}\frac{x}{n} + O\left\{\left(\frac{x}{n} - \frac{y}{n}\right)\frac{\log^{i-1} x}{\log\log x}\right\},$$

for $w_{h+m-1}(x/n) \le (x/n) - (y/n) \le (x/n)\exp(-\sqrt{\log(x/n)})$ and $n \le b$, $b$ being defined by (12), $x$ and $y$ being suitably large, and the $O$ depending on $\rho$, $i$ and $M$.

Now we shall use (86) and (90) to prove that (89) is also true when $j = m+1$. We first prove that, if $x$ and $y$ satisfy the conditions of (89) when $j = m+1$, then $x$ and $y$ satisfy the conditions of (90). From the condition $x - y \le x\exp(-\sqrt{\log x})$ of (89), we have $(x/n) - (y/n) \le (x/n)\exp(-\sqrt{\log x}) \le (x/n)\exp(-\sqrt{\log(x/n)})$ when $1 \le n \le x$, and from the condition $x - y \ge w_{h+m}(x)$ of (89) we have, by (ii) of Lemma 12, $(x/n) - (y/n) \ge (1/n)w_{h+m}(x) \ge w_{h+m-1}(x/n)$ when $1 \le n \le b$ and $x$ is suitably large. Moreover, if $x$ and $y$ satisfy the conditions of (89) when $j = m+1$, then, by (i) of Lemma 12, $x$ and $y$ satisfy the conditions of (86). Thus, if $x$ and $y$ satisfy the conditions of (89) when $j = m+1$, we can substitute (90) in (86) and get

$$\begin{aligned}|R_i(x) - R_i(y)|\log x \le \frac{N}{M}(x-y)\log^i x \\ + \sum_{n\le b}\Lambda(n)A_m\frac{x-y}{n}\log^{i-1}\frac{x}{n} \\ + O\bigg\{(x-y)\frac{\log^i x}{\log\log x}\bigg\},\end{aligned}$$

then, using (24) and (87), the righthand side is

$$\begin{aligned}\bigg(\frac{N}{M} + \frac{A_m}{i}\bigg)(x-y)\log^i x + O\bigg\{(x-y)\frac{\log^i x}{\log\log x}\bigg\} \\ = A_{m+1}(x-y)\log^i x + O\bigg\{(x-y)\frac{\log^i x}{\log\log x}\bigg\},\end{aligned}$$

the $O$'s depending on $\rho$, $i$ and $M$. Hence, (89) is also true when $j = m+1$, and this proves (89).

Finally, by the given conditions $i \ge 2$ and $N \ge 1$ in the lemma, we have from (88),

$$\begin{aligned} A_{M+1} &= \frac{1}{i^M} + \frac{N}{M}\sum_{j=0}^{M}\frac{1}{i^j} \\ &\le \frac{1}{2^M} + \frac{N}{M}\sum_{j=0}^{\infty}\frac{1}{2^j} \\ &\le \frac{1+2N}{M} \le \frac{3N}{M}. \end{aligned}$$

From this, (85) follows, on using (89) with $j = M+1$. We thus complete the proof of the lemma.

**Corollary of Lemma 14.** *Let $T, M$ be arbitrary integers, $T > 2$, $M > 3^{T-2}$. If*

$$|R_T(x) - R_T(y)| \le \frac{1}{M}(x-y)\log^{T-1} x + O\left\{(x-y)\frac{\log^{T-1} x}{\log\log x}\right\}$$

*holds when $w_T(x) \le x - y \le x\exp(-\sqrt{\log x})$, then*

$$|R_i(x) - R_i(y)| \le \frac{3^{T-i}}{M}(x-y)\log^{i-1} x + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\}$$

*holds when $2 \le i \le T-1$ and $w_{T+(T-i)M}(x) \le x-y \le x\exp(-\sqrt{\log x})$, both $O$'s depend on $\rho$ and $M$.*

*Proof.* Applying Lemma 14 and inductive method, we can get the corollary at once.

**Lemma 15.** *Let $\sigma$ and $\sigma'$ be arbitrary constants satisfying $0 < \sigma' \le \sigma < 1$, and let $x, y$ be suitably large satisfying $(y/2)\exp(-\log^{\sigma} y) \le x - y \le y\exp(-\log^{\sigma'} y)$. Then we have*

$$\Psi_i(x) - \Psi_i(y) = (x-y)m_i(x) + O\{3^i(x-y)(\log y)^{\sigma(i+1)}\},$$

*for $2 \le i \le (\log\log x)^{-1}\log^{\sigma'} y$, where $m_i$ is defined by (4), and the $O$ depends on $\sigma$ and $\sigma'$.*

*Proof.* Write $q = (x-y)\log^{-i} x$. From the definitions of $\Psi_i$ and $\Lambda_i$, see (9) and (1), we have

$$(91)\quad \begin{aligned}\Psi_i(x) - \Psi_i(y) &= \sum_{y<n\le x} \Lambda_i(n) = \sum_{y<nm\le x} \mu(n)\log^i m \\ &= \sum_{n\le q} \mu(n) \sum_{(y/n)<m\le(x/n)} \log^i m \\ &\quad + \sum_{m\le(x/q)} \log^i m \sum_{\max(q,(y/m))<n\le(x/m)} \mu(n) \\ &= \sum_1 + \sum_2, \text{ say.}\end{aligned}$$

Applying (22), we have, when $n \le q$ and $(y/n) < m \le (x/n)$,

$$\log^i(x/n) - \log^i m < \log^i(x/n) - \log^i(y/n) \le i\frac{x-y}{y}\log^{i-1} x.$$

Furthermore, by the conditions of the lemma, we have

$$\frac{x-y}{y} \le \exp(-\log^{\sigma'} y),$$

$$\begin{aligned}\log^{i-1} x &= \exp(i\log\log x)\log^{-1} x \\ &\le \exp(\log^{\sigma'} y)\log^{-1} x.\end{aligned}$$

Combining these inequalities, we get, when $n \le q$ and $(y/n) < m \le (x/n)$,

$$\log^i(x/n) - \log^i m < i\log^{-1} x.$$

Substituting in $\sum_1$, we get

$$\begin{aligned}\sum_1 &= \sum_{n\le q}\mu(n)\sum_{(y/n)<m\le(x/n)}\left\{\log^i \frac{x}{n} + O(i\log^{-1} x)\right\} \\ &= \sum_{n\le q}\mu(n)\left\{\frac{x}{n} - \frac{y}{n} + O(1)\right\}\left\{\log^i \frac{x}{n} + O(i\log^{-1} x)\right\} \\ &= (x-y)\sum_{n\le q}\frac{\mu(n)}{n}\log^i\frac{x}{n} + O\left\{\sum_{n\le q}\log^i x + (x-y)\sum_{n\le q}\frac{1}{n} i\log^{-1} x\right\} \\ &= (x-y)m_i(x) - \sum_3 + O\{q\log^i x + (x-y)i\},\end{aligned}$$

where

$$\sum_3 = (x-y) \sum_{q<n\le x} \frac{\mu(n)}{n} \log^i \frac{x}{n}.$$

It follows from the definition of $q$ that

$$\sum_1 = (x-y)m_i(x) - \sum_3 + O\{(x-y)i\}, \tag{92}$$

and the $O$ depends on $\sigma$ and $\sigma'$.

Obviously,

$$\left|\sum_3\right| \le (x-y) \sum_{q<n\le x} \frac{1}{n} \log^i \frac{x}{q} \ll (x-y)\log^{i+1}\frac{x}{q}, \tag{93}$$

$$\begin{aligned}\left|\sum_2\right| &\le \sum_{m\le(x/q)} \log^i \frac{x}{q} \sum_{(y/m)<n\le(x/m)} 1 \ll \log^i \frac{x}{q} \sum_{m\le(x/q)} \frac{x-y}{m} \\ &\ll (x-y)\log^{i+1}\frac{x}{q},\end{aligned} \tag{94}$$

and the $\ll$'s depend on $\sigma$ and $\sigma'$.

By the definition of $q$, we have

$$\log\frac{x}{q} = \log\left(\frac{x}{x-y}\log^i x\right) = \log\frac{x}{x-y} + i\log\log x.$$

By the given condition $x-y \ge (y/2)\exp(-\log^\sigma y)$ in the lemma, we have

$$\log\frac{x}{x-y} \le \log\left\{\frac{2x}{y}\exp(\log^\sigma y)\right\} = \log\frac{2x}{y} + \log^\sigma y.$$

Further, by the given condition $x-y \le y\exp(-\log^{\sigma'} y)$ in the lemma, we have $(x/y)-1 = (x-y)/y \le \exp(-\log^{\sigma'} y) \le 1$, namely, $(x/y) \le 2$, and consequently, $\log(2x/y) \le \log 4 < \log\log x$ when $x$ is suitably large.

Combining these results, we get, when $x$ is suitably large,

$$\log\frac{x}{q} < \log^\sigma y + 2i\log\log x.$$

Since, by the given conditions $i \le (\log\log x)^{-1}\log^{\sigma'} y$ and $\sigma' \le \sigma$ in the lemma, $i\log\log x \le \log^{\sigma'} y \le \log^{\sigma} y$, it follows that, when $x$ is suitably large,

$$\log\frac{x}{q} < 3\log^{\sigma} y.$$

Substituting in (93) and (94), we have

$$\sum_3 \ll (x-y)(3\log^{\sigma} y)^{i+1}, \tag{95}$$

$$\sum_2 \ll (x-y)(3\log^{\sigma} y)^{i+1}, \tag{96}$$

and both $\ll$'s depend on $\sigma$ and $\sigma'$.

Substitute (95) into (92), and then substitute (92) and (96) in (91). The lemma follows immediately.

**Lemma 16.**

$$m_i(x) = i\log^{i-1} x + O(\log^{i-2} x),$$

*for $i \ge 2$, and the $O$ depends on $i$.*

Balog has proved in [**1**, p. 290] that $M_i(x) = ix\log^{i-1} x + O(i(i-1)x\log^{i-2} x)$ for $2 \le i \le \log x$, where $M_i(x) = \sum_{n\le x}\mu(n)(x/n)\log^i(x/n)$, and the $O$ is $i$-uniform. By the definition of $m_i$, see (4), we see that $m_i(x) = (1/x)M_i(x)$, and Lemma 16 follows immediately.

**Lemma 17.** *Let $\sigma, \sigma'$ be defined as Lemma 15, and let $x, y$ be suitably large satisfying $y\exp(-\log^{\sigma} y) \le x - y \le y\exp(-\log^{\sigma'} y)$. Then we have*

$$R_i(x) - R_i(y) \ll (x-y)(\log^{\sigma(i+1)} x + \log^{i-2} x),$$

*for $i \ge 2$ and the $\ll$ depends on $\sigma$, $\sigma'$ and $i$.*

*Proof.* From the definitions of $R_i$ and $\Psi_i$, see (6) and (9), we have

$$\begin{aligned} R_i(x) - R_i(y) &= \sum_{y<n\le x}\{\Lambda_i(n) - i\log^{i-1} n\} \\ &= \Psi_i(x) - \Psi_i(y) - i\sum_{y<n\le x}\log^{i-1} n, \end{aligned} \tag{97}$$

for $i \geq 2$. Using (22), we have, for $y < n \leq x$, $i \geq 2$,

$$\log^{i-1} x - \log^{i-1} n \leq \log^{i-1} x - \log^{i-1} y \leq (i-1)\frac{x-y}{y}\log^{i-2} x.$$

Since $((x-y)/y) \leq \exp(-\log^{\sigma'} y) \leq 1$ by the given condition $x - y \leq y\exp(-\log^{\sigma'} y)$ in the lemma, we get, for $y < n \leq x$, $i \geq 2$,

$$\log^{i-1} x - \log^{i-1} n \ll \log^{i-2} x,$$

and the $\ll$ depends on $i$. It follows that

$$\begin{aligned}\sum_{y<n\leq x}\log^{i-1} n &= \sum_{y<n\leq x}\{\log^{i-1} x + O(\log^{i-2} x)\}\\ &= (x-y)\log^{i-1} x + O\{\log^{i-1} x + (x-y)\log^{i-2} x\}, \qquad (98)\end{aligned}$$

and the $O$ depends on $i$.

From the conditions of the lemma, we see that $\log^{i-1} x \ll (x-y)\log^{i-2} x$ for $i \geq 2$, and the $\ll$ depends on $\sigma$. And, therefore, the remainder term of (98) is $O\{(x-y)\log^{i-2} x\}$, the $O$ depending on $\sigma$ and $i$. Substituting (98) into (97), we have

$$\begin{aligned}R_i(x) - R_i(y) = \Psi_i(x) - \Psi_i(y) - i(x-y)\log^{i-1} x\\ + O\{(x-y)\log^{i-2} x\}, \qquad (99)\end{aligned}$$

for $i \geq 2$, and the $O$ depends on $\sigma$ and $i$.

Combining Lemma 15 and Lemma 16 with (99), the lemma follows.

**Lemma 18.** *Let $\rho$ be as before, and let $A$ be an arbitrary constant, $A > 1$. Then, when $x\exp(-\log^{1-2\rho} x) \leq x - y \leq x\exp(-\sqrt{\log x})$ and $x, y$ are suitably large, we have*

$$|R_i(x) - R_i(y)| \leq \frac{1}{A}(x-y)\log^{i-1} x + O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\},$$

*for $i \geq 2$, and the $O$ depends on $\rho$, $i$ and $A$.*

*Proof.* Taking $\sigma = 1 - \rho$ and $\sigma' = (1/3)$ in Lemma 17, we get, when $y\exp(-\log^{1-\rho} y) \leq x - y \leq y\exp(-\log^{1/3} y)$,

$$R_i(x) - R_i(y) \ll (x-y)\{(\log x)^{(1-\rho)(i+1)} + \log^{i-2} x\}, \qquad (100)$$

for $i \geq 2$, and the $\ll$ depends on $\rho$ and $i$.

Let $T = [3/\rho]$; then, if $i \geq T$,

$$\begin{aligned}(1-\rho)(i+1) &= i+1-\rho(i+1) \leq i+1-\rho(T+1) \\ &= i+1-\rho\left(\left[\frac{3}{\rho}\right]+1\right) \leq i-2.\end{aligned}$$

Substituting in (100) we obtain, when $i \geq T$, $y\exp(-\log^{1-\rho} y) \leq x-y \leq y\exp(-\log^{1/3} y)$,

$$R_i(x) - R_i(y) \ll (x-y)\log^{i-2} x, \tag{101}$$

and the $\ll$ depends on $\rho$ and $i$. Hence, the lemma is true when $i \geq T$.

When $2 \leq i \leq T-1$, we apply the corollary of Lemma 14. By (i) of Lemma 12 and (81), we have, when $y < x$ and $x, y$ are suitably large,

$$w_T(x) > w_0(x) = x\exp(-\log^{1-\rho} x) \geq y\exp(-\log^{1-\rho} y).$$

Hence, when $w_T(x) \leq x-y \leq x\exp(-\sqrt{\log x})$, we can use (101) with $i = T$ and get

$$\begin{aligned}|R_T(x) - R_T(y)| \leq{}& \frac{1}{3^T[A+1]}(x-y)\log^{T-1} x \\ &+ O\left\{(x-y)\frac{\log^{T-1} x}{\log\log x}\right\},\end{aligned}$$

and the $O$ depends on $\rho$ and $T$.

Consequently, using the corollary of Lemma 14 with $M = 3^T[A+1]$ and noting (i) of Lemma 12, we obtain, when $2 \leq i \leq T-1$, $w_L(x) \leq x-y \leq x\exp(-\sqrt{\log x})$, where $L = T + 3^T[A+1]T$,

$$\begin{aligned}|R_i(x) - R_i(y)| \leq{}& \frac{1}{A}(x-y)\log^{i-1} x \\ &+ O\left\{(x-y)\frac{\log^{i-1} x}{\log\log x}\right\},\end{aligned} \tag{102}$$

and the $O$ depends on $\rho$, $T$ and $A$.

From (81) we see that there exists a constant $x_0$ depending only on $\rho$, $T$ and $A$ such that, when $x \geq x_0$,

$$w_L(x) = x\exp(-(\log x)^{1-\rho}(\log\log x)^{-L}) \leq x\exp(-\log^{1-2\rho} x).$$

Therefore (102) holds when $x$ and $y$ satisfy the conditions of the lemma. In addition, by the definition of $T$, $T$ only depends on $\rho$. Thus we establish the lemma when $2 \leq i \leq T-1$.

**Lemma 19.** *Let $\delta$ be an arbitrary constant, $0 < \delta < 1$. If $x - y \geq x\exp(-\log^{1-\delta} x)$ and $x$ is suitably large, then*

$$|R(x) - R(y)| < \frac{19}{20}(x-y) + O\left(\frac{x-y}{\log\log x}\right),$$

*and the $O$ depends on $\delta$.*

*Proof.* Taking $\rho = (\delta/4)$ and $i = 2$ in Lemma 18, we get, when $x\exp(-\log^{1-(\delta/2)} x) \leq x - y \leq x\exp(-\sqrt{\log x})$,

$$|R_2(x) - R_2(y)| \leq \frac{1}{A}(x-y)\log x + O\left\{(x-y)\frac{\log x}{\log\log x}\right\}, \tag{103}$$

where $A$ is an arbitrary constant, $A > 1$ and the $O$ depends on $\delta$ and $A$.

If $x - y > x\exp(-\sqrt{\log x})$ and $y \geq (x/2)$, then give arbitrarily a sequence $z_0, z_1, \ldots, z_m$ satisfying $y = z_0 < z_1 < \cdots < z_{m-1} < z_m = x$ and $z_n\exp(-\log^{1-(\delta/2)} z_n) \leq z_n - z_{n-1} \leq z_n\exp(-\sqrt{\log z_n})$, $n = 1, 2, \ldots, m$, so that $R_2(x) - R_2(y) = R_2(z_m) - R_2(z_{m-1}) + R_2(z_{m-1}) - R_2(z_{m-2}) + \cdots + R_2(z_1) - R_2(z_0)$, and from this we see, by applying (103) to every $R_2(z_n) - R_2(z_{n-1})$, $1 \leq n \leq m$, that (103) still holds when $x - y > x\exp(-\sqrt{\log x})$ and $y \geq (x/2)$. Thus, applying Lemma 2 we can get the lemma at once.

## 6. Balog's identity.

**Lemma 20.**

$$R(x)\log^k x+\sum_{i=1}^{k}\binom{k}{i}\sum_{n\le x}\Lambda_i(n)\log^{k-i}\frac{x}{n}R\left(\frac{x}{n}\right) =\sum_{n\le x}\mu(n)\log^k\frac{x}{n}\tilde{R}\left(\frac{x}{n}\right),$$

*where*

$$\tilde{R}(x)=\sum_{n\le x}R\left(\frac{x}{n}\right).$$

This lemma is given by Balog [**1**, p. 288]. It is the base of the proof of the main theorem in the paper.

From (38),

$$\tilde{R}(x)\ll\sqrt{x}\quad\text{for } x\ge 1.$$

Using this and writing $r=x\log^{-2k}x$, we get, when $k\le\sqrt{\log x}$ and $x$ is suitably large,

$$\begin{aligned}\sum_{n\le x}\mu(n)\log^k\frac{x}{n}\tilde{R}\left(\frac{x}{n}\right)&\ll\sum_{n\le x}\log^k\frac{x}{n}\sqrt{\frac{x}{n}}\\ &\le\sum_{n\le r}\log^k x\sqrt{\frac{x}{n}}+\sum_{r<n\le x}\log^k\frac{x}{r}\sqrt{\frac{x}{n}}\\ &\ll\sqrt{xr}\log^k x+x\log^k\frac{x}{r}\\ &\ll x(2k\log\log x)^k,\end{aligned}\tag{104}$$

and the $\ll$ is $k$-uniform.

Write

$$c=\exp(\sqrt{\log x}),\tag{105}$$

$$I = \sum_{i=1}^{k} \binom{k}{i} \sum_{n \le c} \Lambda_i(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right), \tag{106}$$

$$J = \sum_{i=1}^{k} \binom{k}{i} \sum_{c < n \le x} \Lambda_i(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right). \tag{107}$$

Substituting (104) in Lemma 20 and using the notations (106) and (107), we have, when $k \le \sqrt{\log x}$ and $x$ is suitably large,

$$R(x) \log^k x + I + J = O\{x(2k \log\log x)^k\}, \tag{108}$$

and the $O$ is $k$-uniform.

In (108), $I$ can be straightforwardly calculated, so the key of (108) is how to deal with $J$.

Let

$$u_i(n) = \Lambda_i(n) - m_i(n), \quad n \ge 1, \quad i \ge 1, \tag{109}$$

where $m_i$ is defined by (4). Then

$$\Lambda_i(n) = m_i(n) + u_i(n), \quad n \ge 1, \quad i \ge 1.$$

Substituting this into (107),

$$J = \sum_{i=1}^{k} \binom{k}{i} \sum_{c < n \le x} \{m_i(n) + u_i(n)\} \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right) = J_1 + J_2, \tag{110}$$

where

$$J_1 = \sum_{i=1}^{k} \binom{k}{i} \sum_{c < n \le x} m_i(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right), \tag{111}$$

$$J_2 = \sum_{i=1}^{k} \binom{k}{i} \sum_{c < n \le x} u_i(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right). \tag{112}$$

We shall now deal with $J_1$, while $J_2$ will be specially discussed in Section 8.

Define

$$m(x) = m_0(x) = \sum_{n \le x} \frac{\mu(n)}{n}. \tag{113}$$

**Lemma 21.** *Let $A$ be an arbitrary positive number. Then*

(i) $m(x) = O(\log^{-A} x)$,

(ii) $m_1(x) = 1 + O(\log^{-A} x)$,

*and both $O$'s depend on $A$.*

*Proof.* On the one hand, Wirsing [**6**, pp. 7–8] proves that, if

$$\sum_{n \le x} \left( \frac{1}{n} - \frac{\Lambda(n)}{n} \right) - 2\gamma = O(\log^{-A'} x) \tag{114}$$

holds, then both $m(x) = O(\log^{-A'} x)$ and $m_1(x) = 1 + O(\log^{-A'} x \log\log x)$ hold, where $A'$ is an arbitrary positive number, and all the $O$'s depend on $A'$. On the other hand, we can obtain (114) from any one of [**2, 6**] and [**3**]. Putting $A = A' - 1$, the lemma follows immediately.

Part (ii) of Lemma 21 will be applied in Section 8.

**Lemma 22.**

$$m_i(x) \ll i \log^{i-1} x \quad \textit{for } i \ge 1,$$

*and the $\ll$ is $i$-uniform.*

*Proof.* By (4) and partial summation

$$\begin{aligned} m_i(x) &= \sum_{n \le x} \frac{\mu(n)}{n} \log^i \frac{x}{n} \\ &= m([x]) \log^i \frac{x}{[x]} \\ &\quad + \sum_{n \le x-1} m(n) \left( \log^i \frac{x}{n} - \log^i \frac{x}{n+1} \right) \quad \text{for } i \ge 1. \end{aligned} \tag{115}$$

Taking $A = 4$ in (i) of Lemma 21,

$$m(n) \ll \log^{-4} n \quad \text{for } n \geq 2. \tag{116}$$

Using (22),

$$\log^i \frac{x}{n} - \log^i \frac{x}{n+1} \leq \frac{1}{n} i \log^{i-1} x \quad \text{for } 1 \leq n \leq x-1, \quad i \geq 1. \tag{117}$$

Substituting (116) and (117) in (115), we get

$$m_i(x) \ll i \log^{i-1} x + \sum_{2 \leq n \leq x-1} \frac{1}{n} (\log n)^{-4} i \log^{i-1} x \ll i \log^{i-1} x,$$

and the $\ll$ is $i$-uniform. The lemma is proved.

**Lemma 23.** *If $k \leq \sqrt{\log x}$, then*

$$J_1 \ll k \log^{k-1} x \sum_{n \leq c} \left| R\left(\frac{x}{n}\right) \right| + \log^{-2} x \sum_{c < n \leq x} \log^k \frac{x}{n} \left| R\left(\frac{x}{n}\right) \right| + x,$$

*and the $\ll$ is absolute.*

*Proof.* Substituting (4) into (111), we have

$$\begin{aligned} J_1 &= \sum_{i=1}^{k} \binom{k}{i} \sum_{c < n \leq x} \sum_{m \leq n} \frac{\mu(m)}{m} \log^i \frac{n}{m} \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right) \\ &= \sum_{c < n \leq x} \sum_{m \leq n} \frac{\mu(m)}{m} \sum_{i=1}^{k} \binom{k}{i} \log^i \frac{n}{m} \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right). \end{aligned} \tag{118}$$

By the binomial theorem,

$$\begin{aligned} \sum_{i=1}^{k} \binom{k}{i} \log^i \frac{n}{m} \log^{k-i} \frac{x}{n} &= \sum_{i=0}^{k} \binom{k}{i} \log^i \frac{n}{m} \log^{k-i} \frac{x}{n} - \log^k \frac{x}{n} \\ &= \left( \log \frac{n}{m} + \log \frac{x}{n} \right)^k - \log^k \frac{x}{n} \\ &= \log^k \frac{x}{m} - \log^k \frac{x}{n}, \end{aligned}$$

$$1 \leq m \leq n \leq x.$$

Substituting in (118), we get

$$J_1 = \sum_{c<n\le x} \sum_{m\le n} \frac{\mu(m)}{m}\left(\log^k \frac{x}{m} - \log^k \frac{x}{n}\right) R\left(\frac{x}{n}\right)$$
$$= \sum_{c<n\le x} \left\{ \sum_{m\le n} \frac{\mu(m)}{m} \log^k \frac{x}{m} - \log^k \frac{x}{n} \sum_{m\le n} \frac{\mu(m)}{m} \right\} R\left(\frac{x}{n}\right).$$

Thus, using notations (4) and (113), we have

(119)
$$J_1 = \sum_{c<n\le x} \left\{ m_k(x) - \sum_{n<m\le x} \frac{\mu(m)}{m} \log^k \frac{x}{m} - m(n) \log^k \frac{x}{n} \right\} R\left(\frac{x}{n}\right).$$

Using (116) via partial summation

$$\sum_{n<m\le x} \frac{\mu(m)}{m} \log^k \frac{x}{m} \ll \log^{-4} n \log^k \frac{x}{n},$$

and the $\ll$ is absolute. Substituting this and (116) into (119), we get

$$J_1 = \sum_{c<n\le x} \left\{ m_k(x) + O\left( \log^{-4} n \log^k \frac{x}{n} \right) \right\} R\left(\frac{x}{n}\right)$$

(120)
$$= m_k(x) \sum_{c<n\le x} R\left(\frac{x}{n}\right) + O\left\{ \log^{-4} c \sum_{c<n\le x} \log^k \frac{x}{n} \left| R\left(\frac{x}{n}\right) \right| \right\},$$

and the $O$ is absolute.

From (38),

$$\sum_{c<n\le x} R\left(\frac{x}{n}\right) = -\sum_{n\le c} R\left(\frac{x}{n}\right) + O(\sqrt{x}).$$

Putting $i = k$ in Lemma 22,

$$m_k(x) \ll k \log^{k-1} x,$$

and the $\ll$ is $k$-uniform. Combining both formulae, we have

$$m_k(x) \sum_{c<n\le x} R\left(\frac{x}{n}\right) \ll k \log^{k-1} x \sum_{n\le c} \left| R\left(\frac{x}{n}\right) \right| + \sqrt{x} k \log^{k-1} x,$$

and the $\ll$ is absolute. By the given condition $k \le \sqrt{\log x}$ in the lemma, the last term is $\sqrt{x} k e^{(k-1)\log\log x} \le \sqrt{x \log x} e^{\sqrt{\log x}\log\log x} \ll x$.

On substituting in (120), noting that $\log^{-4} c = \log^{-2} x$ from (105), the lemma is proved.

**7. The estimate of $u_i(n)$ in short intervals.** Define

$$U_i(x) = \sum_{n \le x} u_i(n). \tag{121}$$

**Lemma 24.** *Let $\sigma$ be an arbitrary constant, $0 < \sigma < 1$, and let $x, y$ be suitably large satisfying*

$$\frac{y}{2}\exp(-\log^\sigma y) \le x - y \le y \exp(-\log^\sigma y);$$

*then we have*

(i) $U_i(x) - U_i(y) \ll 3^i (x-y) \log^{\sigma(i+1)} y$, *for* $2 \le i \le (\log\log x)^{-1} \log^\sigma y$, *and the $\ll$ depends on $\sigma$.*

(ii) $\sum_{y<n\le x} |u_i(n)| \ll (x-y)\log^{i+1} x$, *for $i \ge 1$, and the $\ll$ depends on $\sigma$.*

*Proof.* From the definitions of $U_i$, $u_i$ and $\Psi_i$, see (121), (109) and (9), we have

$$\begin{aligned} U_i(x) - U_i(y) &= \sum_{y<n\le x} u_i(n) = \sum_{y<n\le x} \{\Lambda_i(n) - m_i(n)\} \\ &= \Psi_i(x) - \Psi_i(y) - \sum_{y<n\le x} m_i(n). \end{aligned} \tag{122}$$

Taking $\sigma' = \sigma$ in Lemma 15,

$$\Psi_i(x) - \Psi_i(y) = (x-y) m_i(x) + O\{3^i (x-y)\log^{\sigma(i+1)} y\}, \tag{123}$$

for $2 \le i \le (\log\log x)^{-1} \log^\sigma y$, and the $O$ depends on $\sigma$.

From the definition of $m_i$, see (4), we have

$$\begin{aligned}\sum_{y<n\le x} m_i(n) &= \sum_{y<n\le x}\sum_{m\le n}\frac{\mu(m)}{m}\log^i\frac{n}{m}\\ &= \sum_{m\le s}\frac{\mu(m)}{m}\sum_{y<n\le x}\log^i\frac{n}{m} + \sum_{y<n\le x}\sum_{s<m\le n}\frac{\mu(m)}{m}\log^i\frac{n}{m} \qquad (124)\\ &= \Delta_1+\Delta_2,\end{aligned}$$

say, where $s = x\log^{-1} x$.

Using (22), we have, when $m\le s$, $y<n\le x$,

$$\log^i\frac{x}{m}-\log^i\frac{n}{m}\le\log^i\frac{x}{m}-\log^i\frac{y}{m}\le i\frac{x-y}{y}\log^{i-1}x;$$

further, by the conditions of the lemma we have

$$\frac{x-y}{y}\le\exp(-\log^\sigma y),$$

$$\log^{i-1}x=\exp(i\log\log x)\log^{-1}x\le\exp(\log^\sigma y)\log^{-1}x,$$

and so

$$\log^i\frac{x}{m}-\log^i\frac{n}{m}\le i\log^{-1}x,$$

when $m\le s$, $y<n\le x$. It follows that

$$\begin{aligned}\Delta_1 &= \sum_{m\le s}\frac{\mu(m)}{m}\sum_{y<n\le x}\left\{\log^i\frac{x}{m}+O(i\log^{-1}x)\right\}\\ &= \sum_{m\le s}\frac{\mu(m)}{m}\{x-y+O(1)\}\left\{\log^i\frac{x}{m}+O(i\log^{-1}x)\right\}\\ &= (x-y)\sum_{m\le s}\frac{\mu(m)}{m}\log^i\frac{x}{m}\\ &\quad + O\left\{\sum_{m\le s}\frac{1}{m}\log^i x+(x-y)\sum_{m\le s}\frac{1}{m}i\log^{-1}x\right\} \qquad (125)\\ &= (x-y)m_i(x)-(x-y)\sum_{s<m\le x}\frac{\mu(m)}{m}\log^i\frac{x}{m}\\ &\quad + O\{\log^{i+1}x+(x-y)i\},\end{aligned}$$

and the $O$ is absolute.

By the definition of $s$, we have

$$\begin{aligned}\left|\sum_{s<m\le x}\frac{\mu(m)}{m}\log^i\frac{x}{m}\right| &\le \sum_{s<m\le x}\frac{1}{m}\log^i\frac{x}{s}\\ &\ll \log^{i+1}\frac{x}{s}\\ &= (\log\log x)^{i+1},\end{aligned}$$

and the $\ll$ is absolute. Furthermore, by the conditions of the lemma, we have

$$\begin{aligned}\log^{i+1} x &= \exp(i\log\log x)\log x\\ &\le \exp(\log^\sigma y)\log x\\ &\ll \frac{y}{2}\exp(-\log^\sigma y)\\ &\le x-y,\end{aligned}$$

and the $\ll$ depends on $\sigma$. Substituting these into (125), we obtain

$$\Delta_1 = (x-y)m_i(x) + O\{(x-y)(\log\log x)^{i+1}\}, \tag{126}$$

and the $O$ depends on $\sigma$.

Now we consider $\Delta_2$. From the definition of $s$, we have

$$\begin{aligned}|\Delta_2| &\le \sum_{y<n\le x}\sum_{s<m\le n}\frac{1}{m}\log^i\frac{x}{s}\\ &\ll (x-y)\log^{i+1}\frac{x}{s}\\ &= (x-y)(\log\log x)^{i+1},\end{aligned}\tag{127}$$

and the $\ll$ is absolute.

Substituting (126) and (127) into (124), we have

$$\sum_{y<n\le x} m_i(n) = (x-y)m_i(x) + O\{(x-y)(\log\log x)^{i+1}\},$$

and the $O$ depends on $\sigma$. Since $\log\log x \le i\log\log x \le \log^\sigma y$ by the condition of the lemma, the remainder term is $O\{(x-y)\log^{\sigma(i+1)} y\}$. Substituting the last equality and (123) in (122), we establish (i).

We proceed to prove (ii). From the definitions of $u_i$, $\Lambda_i$ and $m_i$, see (109), (1) and (4), we have, when $i \geq 1$ and $n$ is suitably large,

$$\begin{aligned}|u_i(n)| &\leq |\Lambda_i(n)| + |m_i(n)| \\ &= \left|\sum_{m|n} \mu(m) \log^i \frac{n}{m}\right| + \left|\sum_{m\leq n} \frac{\mu(m)}{m} \log^i \frac{n}{m}\right| \\ &\leq \log^i n \sum_{m|n} 1 + \log^i n \sum_{m\leq n} \frac{1}{m} \\ &\ll d(n) \log^i n + \log^{i+1} n,\end{aligned}$$

where $d$ is a divisor function, and the $\ll$ is $i$-uniform. Hence, when $i \geq 1$,

$$\sum_{y<n\leq x} |u_i(n)| \ll \log^i x \sum_{y<n\leq x} d(n) + (x-y) \log^{i+1} x, \tag{128}$$

and the $\ll$ is absolute.

Using the formula of Dirichlet

$$\sum_{n\leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$$

and (22), we have

$$\begin{aligned}\sum_{y<n\leq x} d(n) &= x \log x - y \log y + O(x - y + \sqrt{x}) \\ &= (x-y) \log x + y(\log x - \log y) \\ &\quad + O(x - y + \sqrt{x}) \\ &= O\{(x-y) \log x + \sqrt{x}\},\end{aligned} \tag{129}$$

and the $O$ is absolute. Furthermore, from the condition of the lemma, we have

$$\sqrt{x} \ll \frac{y}{2} \exp(-\log^\sigma y) \leq x - y,$$

and the $\ll$ depends on $\sigma$. Substitute this into (129) and then substitute (129) into (128). We establish (ii) at once.

**8. A transformation of $J_2$.** Define

$$\phi_i(x,\sigma,A,B)=\begin{cases}\dfrac{19}{20}+O\Big(\dfrac{1}{\log\log x}\Big) & \text{if } i=1,\\ \dfrac{1}{A}\log^{i-1}x+O\Big(\dfrac{\log^{i-1}x}{\log\log x}\Big) & \text{if } 2\le i\le B,\\ O\{3^i(\log x)^{(1-\sigma)(i+1)}\} & \text{if } i>B,\end{cases} \tag{130}$$

$0<\sigma<1$, $A\ge 1$, $B\ge 2$, where the first and last $O$'s depend only on $\sigma$, while the second $O$ depends on $\sigma$, $A$ and $B$.

In this section we shall prove

**Lemma 25.** *Let $\lambda$ be an arbitrary constant, $0<\lambda<(1/4)$, and let $A,B$ be arbitrarily large constants. If $k\le\log^{(1/2)-\lambda}x$, then, when $x$ is suitably large,*

$$|J_2|\le\sum_{i=1}^{k}\binom{k}{i}\sum_{c<n\le x}\phi_i(n,\lambda,A,B)\log^{k-i}\frac{x}{n}\left|R\Big(\frac{x}{n}\Big)\right| + O(x), \tag{131}$$

*and the $O$ depends on $\lambda$.*

In (131), $\lambda$ will be taken sufficiently small, while $A$ and $B$, being at our disposal and independent of $\lambda$, will be chosen large enough later.

Before proving Lemma 25 we shall first prove several lemmas.

We first give a sequence $b_n$ defined by

$$b_0=3,\quad b_{n+1}=b_n+[b_n\exp(-\log^{1-\lambda}b_n)]\quad\text{for } n\ge 0. \tag{132}$$

We then define the function $V_i$ by

$$\begin{gathered}V_i(b_n)=U_i(b_n)\quad\text{for } i\ge 1,\quad n\ge 0,\\ V_i(x)=k_{n,i}(x-b_n)+U_i(b_n)\quad\text{for } i\ge 1,\\ x\in[b_n,b_{n+1}],\quad n\ge 0,\end{gathered} \tag{133}$$

where

$$k_{n,i}=\frac{U_i(b_{n+1})-U_i(b_n)}{b_{n+1}-b_n}. \tag{134}$$

By (133), the derivative of $V_i(x)$ is $k_{n,i}$ when $b_n < x < b_{n+1}$, that is, $V_i'(x) = k_{n,i}$ for $x \in (b_n, b_{n+1})$. Now we define the derivative of $V_i(x)$ at the points $b_1, b_2, \dots$ by $V_i'(b_{n+1}) = k_{n,i}$ for $i \geq 1$ and $n \geq 0$. Thus, by (134), we have

$$V_i'(x) = k_{n,i} = \frac{U_i(b_{n+1}) - U_i(b_n)}{b_{n+1} - b_n}, \tag{135}$$

for $i \geq 1$, $x \in (b_n, b_{n+1}]$ and $n \geq 0$.

**Lemma 26.** *If $k$ and $\lambda$ satisfy the conditions of Lemma 25, then there exists a constant $\lambda_0$ depending only on $\lambda$ such that, when $x > \lambda_0$,*

$$k \leq \frac{1}{4\log\log(2x)} \log^{1-\lambda} \frac{c}{4}.$$

*Proof.* By the definition of $c$, see (105), we have, when $x$ is suitably large,

$$\begin{aligned} \log^{1-\lambda} \frac{c}{4} &\geq \log^{1-\lambda} \sqrt{c} = \left(\frac{1}{2}\right)^{1-\lambda} \log^{1-\lambda} c \\ &\geq \frac{1}{2} \log^{1-\lambda} c = \frac{1}{2} \log^{(1-\lambda)/2} x. \end{aligned}$$

Furthermore, it is evident that there exists a constant $\lambda_1$ depending only on $\lambda$ such that, when $x > \lambda_1$,

$$8 \log\log(2x) \leq \log^{\lambda/2} x.$$

Combining the above results, we have that there exists a constant $\lambda_0$ $(\geq \lambda_1)$ depending only on $\lambda$ such that, when $x > \lambda_0$,

$$\frac{1}{4\log\log(2x)} \log^{1-\lambda} \frac{c}{4} \geq \log^{(1/2)-\lambda} x.$$

Since $k$ satisfies the condition $k \leq \log^{(1/2)-\lambda} x$ in Lemma 25, the lemma is proved.

**Lemma 27.** *Let $\lambda$, $A$ and $B$ be defined in Lemma 25, and let $k$ satisfy the condition of Lemma 25. Then, when $x$ is suitably large, we have*

$$|V_i'(t)| \leq \phi_i(t, \lambda, A, B) \quad \textit{for } c \leq t \leq x, \quad 1 \leq i \leq k.$$

*Proof.* Given any number $t$, $c \leq t \leq x$, we can find a corresponding integer $n$, such that $b_n < t \leq b_{n+1}$. Now $t$ and $n$ are thus fixed, so that, from (135),

$$V_i'(t) = \frac{U_i(b_{n+1}) - U_i(b_n)}{b_{n+1} - b_n} \quad \text{for } i \geq 1. \tag{136}$$

If $1 \leq i \leq B$, we have from the definitions of $U_i$ and $u_i$, see (121) and (109),

$$\begin{aligned} U_i(b_{n+1}) - U_i(b_n) &= \sum_{b_n < m \leq b_{n+1}} u_i(m) \\ &= \sum_{b_n < m \leq b_{n+1}} \{\Lambda_i(m) - m_i(m)\}. \end{aligned}$$

Taking $A = 1$ in (ii) of Lemma 21 we have $m_1(m) = 1 + O(\log^{-1} m)$. Further, using Lemma 16 we have $m_i(m) = i \log^{i-1} m + O(\log^{i-2} m)$ for $2 \leq i \leq B$, and the $O$ depends on $i$. Substituting in the above equality, we get

$$\begin{aligned} U_i(b_{n+1}) - U_i(b_n) &= \sum_{b_n < m \leq b_{n+1}} \{\Lambda_i(m) - i \log^{i-1} m + O(\log^{i-2} m)\} \\ &= R_i(b_{n+1}) - R_i(b_n) \qquad (137) \\ &\quad + O\left\{(b_{n+1} - b_n) \frac{\log^{i-1} b_{n+1}}{\log\log b_{n+1}}\right\} \quad \text{for } 1 \leq i \leq B, \end{aligned}$$

the latter step following from (6) and $R_1(b_{n+1}) - R_1(b_n) = R(b_{n+1}) - R(b_n) = \sum_{b_n < m \leq b_{n+1}} \{\Lambda(m) - 1\} = \sum_{b_n < m \leq b_{n+1}} \{\Lambda_1(m) - 1\}$, by (6), (2), (1) and the $O$ depending on $i$.

By (132),

$$b_{n+1} - b_n = [b_n \exp(-\log^{1-\lambda} b_n)] \geq b_{n+1} \exp(-\log^{1-(\lambda/2)} b_{n+1}),$$

when $b_n > \lambda_2$, where $\lambda_2$ is a constant depending only on $\lambda$, hence we can use Lemma 19 with $\delta = (\lambda/2)$, $x = b_{n+1}$ and $y = b_n$, and get

$$|R(b_{n+1}) - R(b_n)| < \frac{19}{20}(b_{n+1} - b_n) + O\biggl(\frac{b_{n+1} - b_n}{\log\log b_{n+1}}\biggr),$$

and the $O$ depends on $\lambda$. Noting that $R_1(x) = R(x)$, see (6), we can substitute this inequality in (137) and get

$$|U_1(b_{n+1}) - U_1(b_n)| < \frac{19}{20}(b_{n+1} - b_n) + O\biggl(\frac{b_{n+1} - b_n}{\log\log b_{n+1}}\biggr),$$

then substituting this into (136), and noticing (130), we obtain

$$\begin{aligned} |V_1'(t)| &< \frac{19}{20} + O\biggl(\frac{1}{\log\log b_{n+1}}\biggr) \\ &= \frac{19}{20} + O\biggl(\frac{1}{\log\log t}\biggr) \\ &= \phi_1(t, \lambda, A, B), \end{aligned} \tag{138}$$

and the $O$ depends on $\lambda$.

Next, by (132) and the given condition $\lambda < (1/4)$ in Lemma 25,

$$\begin{aligned} b_{n+1} - b_n = [b_n \exp(-\log^{1-\lambda} b_n)] \\ \begin{cases} \geq b_{n+1} \exp(-\log^{1-(\lambda/2)} b_{n+1}), \\ \leq b_{n+1} \exp(-\sqrt{\log b_{n+1}}), \end{cases} \end{aligned}$$

when $b_n > \lambda_3$, where $\lambda_3$ ($\geq \lambda_2$) is a constant depending only on $\lambda$; hence, we can use Lemma 18 with $\rho = (\lambda/4)$, $x = b_{n+1}$ and $y = b_n$, and get

$$\begin{aligned} |R_i(b_{n+1}) - R_i(b_n)| \leq \frac{1}{A}(b_{n+1} - b_n) \log^{i-1} b_{n+1} \\ + O\biggl\{(b_{n+1} - b_n)\frac{\log^{i-1} b_{n+1}}{\log\log b_{n+1}}\biggr\}, \end{aligned}$$

for $2 \leq i \leq B$, and the $O$ depends on $\lambda$, $i$ and $A$. Substituting this inequality into (137) and then substituting (137) into (136), we get

$$|V_i'(t)| \leq \frac{1}{A} \log^{i-1} b_{n+1} + O\biggl(\frac{\log^{i-1} b_{n+1}}{\log\log b_{n+1}}\biggr), \tag{139}$$

for $2 \le i \le B$, and the $O$ depends on $\lambda$, $i$ and $A$.

Noting (132), we have $b_{n+1} \le 2b_n < 2t$ when $b_n < t \le b_{n+1}$. Hence, $\log^{i-1} b_{n+1} \le (\log t + \log 2)^{i-1} = (\log t)^{i-1}(1 + (\log 2/\log t))^{i-1} = (\log t)^{i-1}\{1 + O(1/(\log t))\}$ for $b_n < t \le b_{n+1}$, and the $O$ depends on $i$. Substituting this into (139) and noticing (130), we obtain

$$|V_i'(t)| \le \frac{1}{A}\log^{i-1} t + O\left(\frac{\log^{i-1} t}{\log\log t}\right) = \phi_i(t, \lambda, A, B), \tag{140}$$

for $2 \le i \le B$, and the $O$ depends on $\lambda, B$ and $A$.

If $B < i \le k$, we first prove that

$$b_n \ge \frac{c}{2}, \quad b_{n+1} < 2x, \tag{141}$$

$b_n$ and $b_{n+1}$ still satisfying $b_n < t \le b_{n+1}$, $c \le t \le x$, given in the beginning of the proof.

From (132), we see that $b_{n+1} \le 2b_n$. Since $b_n$ and $b_{n+1}$ satisfy $b_n < t \le b_{n+1}$, $t$ satisfying $c \le t \le x$, it follows that $b_n \ge (1/2)b_{n+1} \ge (1/2)t \ge (1/2)c$ and $b_{n+1} \le 2b_n < 2t \le 2x$. This proves (141).

Now $t$ and $n$ being still fixed as above, we apply (i) of Lemma 24 to (136) when $B < i \le k$. Since $i \le k$ and $k$ satisfies the condition of Lemma 25, we have by using Lemma 26 and (141)

$$i \le k \le \frac{1}{4\log\log(2x)}\log^{1-\lambda}\frac{c}{4} < \frac{1}{\log\log b_{n+1}}\log^{1-\lambda} b_n,$$

for $x > \lambda_0$. And so, we can use (i) of Lemma 24 with $\sigma = 1 - \lambda$, $x = b_{n+1}$ and $y = b_n$, (in this case, from the last inequality we see that $i$ satisfies the condition of (i) of Lemma 24 provided $B < i \le k$ and from (132) we see that $b_{n+1} - b_n$ satisfies the condition of Lemma 24), and get

$$U_i(b_{n+1}) - U_i(b_n) \ll 3^i(b_{n+1} - b_n)(\log b_n)^{(1-\lambda)(i+1)},$$

for $B < i \le k$, and the $\ll$ depends on $\lambda$. Substituting this into (136), we obtain

$$V_i'(t) \ll 3^i(\log b_n)^{(1-\lambda)(i+1)} < 3^i(\log t)^{(1-\lambda)(i+1)},$$

for $B < i \le k$, and the $\ll$ depends on $\lambda$. By (130), this inequality can be written in the form

$$|V_i'(t)| \le \phi_i(t, \lambda, A, B), \tag{142}$$

for $B < i \le k$, and the $O$ included in $\phi_i$ depends on $\lambda$.

Combining (138), (140) and (142), noticing that $t$ is arbitrary within the interval $[c, x]$, we get the lemma at once.

**Lemma 28.**

$$V_i'(m) = V_i(m) - V_i(m-1), \quad i \ge 1; \quad m = 4, 5, 6, \dots .$$

The proof is completely the same as Lemma 6.

From Lemma 28, we get, for arbitrary integers $m, n, 3 \le n < m$ and $i \ge 1$,

$$\sum_{j=n+1}^{m} V_i'(j) = V_i(m) - V_i(n). \tag{143}$$

**Lemma 29.**

$$V_i(t) - U_i(t) \ll t \exp\left(-\log^{1-\lambda} \frac{t}{2}\right) \log^{i+1}(2t),$$

*for $t > 3$ and $i \ge 1$, and the $\ll$ depends on $\lambda$.*

*Proof.* We shall prove this lemma in a similar manner as Lemma 8.

Given any number $t$, $t > 3$, we can find a corresponding positive integer $n$ such that $b_n < t \le b_{n+1}$. Now $t$ and $n$ are thus fixed. In the same way as (50)–(52), we can obtain the following corresponding results:

$$\begin{aligned} |V_i(t) - U_i(t)| &\le |V_i(t) - U_i(b_n)| + |U_i(t) - U_i(b_n)|, \\ |V_i(t) - U_i(b_n)| &\le |U_i(b_{n+1}) - U_i(b_n)| \\ &\le \sum_{b_n < m \le b_{n+1}} |u_i(m)|, \\ |U_i(t) - U_i(b_n)| &\le \sum_{b_n < m \le b_{n+1}} |u_i(m)|. \end{aligned}$$

Substituting the last two inequalities in the first inequality, we get

$$|V_i(t) - U_i(t)| \le 2 \sum_{b_n < m \le b_{n+1}} |u_i(m)|.$$

Using (ii) of Lemma 24 with $\sigma = 1-\lambda$, $x = b_{n+1}$ and $y = b_n$, it follows that

$$V_i(t) - U_i(t) \ll (b_{n+1} - b_n) \log^{i+1} b_{n+1} \quad \text{for } i \ge 1,$$

and the $\ll$ depends on $\lambda$. And so, by (132),

$$V_i(t) - U_i(t) \ll b_n \exp(-\log^{1-\lambda} b_n) \log^{i+1} b_{n+1} \quad \text{for } i \ge 1,$$

and the $\ll$ depends on $\lambda$. Since $t \le b_{n+1} \le 2b_n < 2t$ by (132) and the given condition $b_n < t \le b_{n+1}$ in the beginning of the proof, the right side of the inequality is $\le t\exp(-\log^{1-\lambda}(t/2))\log^{i+1}(2t)$.

Because $t$ $(> 3)$ is arbitrary, we establish the lemma.

*Proof of Lemma* 25. Let

$$G = \sum_{c < n \le x} \{u_i(n) - V_i'(n)\} \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right), \quad 1 \le i \le k. \tag{144}$$

From the definition of $R$, see (2),

$$R\left(\frac{x}{n}\right) = \sum_{m \le (x/n)} \{\Lambda(m) - 1\} + 2\gamma \quad \text{for } c < n \le x.$$

Substituting this into (144) we get, for $1 \le i \le k$,

$$\begin{aligned} G &= \sum_{c<n\le x} \{u_i(n) - V_i'(n)\} \log^{k-i} \frac{x}{n} \sum_{m \le (x/n)} \{\Lambda(m) - 1\} \\ &\quad + 2\gamma \sum_{c<n\le x} \{u_i(n) - V_i'(n)\} \log^{k-i} \frac{x}{n} \\ &= G_1 + G_2, \end{aligned} \tag{145}$$

say. Plainly, for $1 \le i \le k$,

$$G_1 = \sum_{m \le (x/c)} \{\Lambda(m) - 1\} \sum_{c < n \le (x/m)} \{u_i(n) - V_i'(n)\} \log^{k-i} \frac{x}{n}. \tag{146}$$

By (143) and Lemma 29, we get

$$\begin{aligned}\sum_{c<n\le t}\{u_i(n)-V_i'(n)\} &= \{U_i([t])-V_i([t])\}-\{U_i([c])-V_i([c])\}\\ &\ll t\exp\left(-\log^{1-\lambda}\frac{[c]}{2}\right)\log^{i+1}(2t)\\ &\le t\exp\left(-\log^{1-\lambda}\frac{c}{4}\right)\log^{i+1}(2x),\end{aligned}$$

for $c < t \le x$ and $1 \le i \le k$, and the $\ll$ depends on $\lambda$. Using this via partial summation

$$\sum_{c<n\le(x/m)}\{u_i(n)-V_i'(n)\}\log^{k-i}\frac{x}{n} \ll \frac{x}{m}\exp\left(-\log^{1-\lambda}\frac{c}{4}\right)\log^{k+1}(2x), \tag{147}$$

for $1 \le i \le k$ and $1 \le m \le (x/c)$, and the $\ll$ depends on $\lambda$. Substituting in (146),

$$\begin{aligned}G_1 &\ll x\exp\left(-\log^{1-\lambda}\frac{c}{4}\right)\log^{k+1}(2x)\sum_{m\le x}\frac{\Lambda(m)+1}{m}\\ &\ll x\exp\left(-\log^{1-\lambda}\frac{c}{4}\right)\log^{k+2}(2x),\end{aligned}$$

for $1 \le i \le k$, and the $\ll$ depends on $\lambda$. Furthermore, using (147) with $m = 1$,

$$G_2 \ll x\exp\left(-\log^{1-\lambda}\frac{c}{4}\right)\log^{k+1}(2x) \quad \text{for } 1 \le i \le k,$$

and the $\ll$ depends on $\lambda$.

Substituting the last two inequalities in (145) we get

$$G \ll x\exp\left(-\log^{1-\lambda}\frac{c}{4}\right)\log^{k+2}(2x) \quad \text{for } 1 \le i \le k,$$

and the $\ll$ depends on $\lambda$. By (144) we can write this inequality in the form

$$\sum_{c<n\leq x} u_i(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right) = \sum_{c<n\leq x} V_i'(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right) + O\left\{x \exp\left(-\log^{1-\lambda} \frac{c}{4}\right) \log^{k+2}(2x)\right\},$$

for $1 \leq i \leq k$, and the $\ll$ depends on $\lambda$. Substituting this into (112), we have

$$J_2 = \sum_{i=1}^{k} \binom{k}{i} \sum_{c<n\leq x} V_i'(n) \log^{k-i} \frac{x}{n} R\left(\frac{x}{n}\right) + O\left\{\sum_{i=0}^{k} \binom{k}{i} x \exp\left(-\log^{1-\lambda} \frac{c}{4}\right) \log^{k+2}(2x)\right\},$$

and the $O$ depends on $\lambda$. Applying Lemma 27 to the main term and applying the binomial theorem to the remainder term

$$|J_2| \leq \sum_{i=1}^{k} \binom{k}{i} \sum_{c<n\leq x} \phi_i(n, \lambda, A, B) \log^{k-i} \frac{x}{n} \left|R\left(\frac{x}{n}\right)\right| + O\left\{2^k x \exp\left(-\log^{1-\lambda} \frac{c}{4}\right) \log^{k+2}(2x)\right\}, \tag{148}$$

and the $O$ depends on $\lambda$.

It is evident that $k \log 2 + (k+2) \log\log(2x) \leq 4k \log\log(2x)$ when $x$ is suitably large and, by Lemma 26, $4k \log\log(2x) \leq \log^{1-\lambda}(c/4)$ when $x > \lambda_0$. Hence, when $x > \lambda_4$, where $\lambda_4$ $(\geq \lambda_0)$ is some constant depending only on $\lambda$,

$$\begin{aligned} 2^k \log^{k+2}(2x) &= \exp(k \log 2 + (k+2) \log\log(2x)) \\ &\leq \exp(4k \log\log(2x)) \\ &\leq \exp\left(\log^{1-\lambda} \frac{c}{4}\right). \end{aligned}$$

Substituting in the remainder term of (148), Lemma 25 follows immediately.

**9. The main result of this paper.** Let $\varepsilon$ be an arbitrarily small constant, $\varepsilon > 0$, and define

$$C(x) = \exp(-\log^{(1/2)-\varepsilon} x); \tag{149}$$

$$Q(x) = x^{-1}C^{-1}(x)R(x), \quad x \geq 1; \tag{150}$$

$$A(x) = \sup_{1\leq t\leq x} |Q(t)|, \quad x \geq 1. \tag{151}$$

Then, from (150),

$$R(x) = xC(x)Q(x), \quad x \geq 1. \tag{152}$$

We take

$$k = [\log^{(1-\varepsilon)/2} x]; \tag{153}$$

then there exists a constant $\varepsilon_0$ depending only on $\varepsilon$ such that, when $x > \varepsilon_0$,

$$k > \log^{(1/2)-\varepsilon} x \log\log x. \tag{154}$$

**Lemma 30.** *If* $2 \leq n \leq x$, *then*

(i) $\log^{k-1}(x/n) \leq \log^{k-1} x e^{-(k-1)(\log n/\log x)}$,

(ii) $(\log x - (\log n)/2)^{k-1} \leq \log^{k-1} x e^{-(k-1)\log n/(2\log x)}$,

(iii) $C(x/n) \leq C(x)\exp(\log^{-(1/2)-\varepsilon} x \log n)$.

*Proof.* If $1 < t < x$, then

$$(\log x - \log t)^{k-1} = \log^{k-1} x \left(1 - \frac{\log t}{\log x}\right)^{k-1} = \log^{k-1} x e^{(k-1)\log(1-(\log t/\log x))}.$$

Since

$$\log\left(1 - \frac{\log t}{\log x}\right) = -\frac{\log t}{\log x} - \frac{1}{2}\left(\frac{\log t}{\log x}\right)^2 - \cdots \leq -\frac{\log t}{\log x}$$

for $1 < t < x$, we have

$$(155)\quad (\log x - \log t)^{k-1} \le \log^{k-1} x e^{-(k-1)(\log t/\log x)} \quad \text{for } 1 < t < x.$$

Putting $t = n$ in (155), we have

$$\log^{k-1}\frac{x}{n} = (\log x - \log n)^{k-1} \le \log^{k-1} x e^{-(k-1)(\log n/\log x)}.$$

So (i) holds when $1 < n < x$. And, when $n = x$, (i) holds too obviously. Thus we establish (i).

Putting $t = \sqrt{n}$ in (155), we get (ii) at once.

We proceed to prove (iii). By the definition of $C$, see (149), the proof of (iii) is just to prove that

$$\exp\left(-\log^{(1/2)-\varepsilon}\frac{x}{n}\right) \le \exp(-\log^{(1/2)-\varepsilon} x + \log^{-(1/2)-\varepsilon} x \log n)$$

$$(156)\qquad \text{for } 2 \le n \le x.$$

Since

$$\begin{aligned}\log^{(1/2)-\varepsilon}\frac{x}{n} &= (\log x - \log n)^{(1/2)-\varepsilon} = \log^{(1/2)-\varepsilon} x\left(1 - \frac{\log n}{\log x}\right)^{(1/2)-\varepsilon}\\ &\ge \log^{(1/2)-\varepsilon} x\left(1 - \frac{\log n}{\log x}\right)\\ &= \log^{(1/2)-\varepsilon} x - \log^{-(1/2)-\varepsilon} x \log n,\end{aligned}$$

for $2 \le n \le x$, (156) follows from this.

The proof of the lemma is completed.

Now we estimate $J_2$. Write

(157)
$$Z_1 = k\sum_{c<n\le x}\phi_1\left(n, \frac{\varepsilon}{2}, A, B\right)\log^{k-1}\frac{x}{n}\left|R\left(\frac{x}{n}\right)\right|,$$

(158)
$$Z_2 = \sum_{i=2}^{k}\binom{k}{i}\sum_{c<n\le x}\phi_i\left(n, \frac{\varepsilon}{2}, A, B\right)\log^{k-i}\frac{x}{n}\left|R\left(\frac{x}{n}\right)\right|.$$

Using Lemma 25 with $\lambda = (\varepsilon/2)$, in this case by (153), $k$ satisfies the condition of Lemma 25, we get, when $x$ is suitably large

$$|J_2| \le Z_1 + Z_2 + O(x), \tag{159}$$

and the $O$ depends on $\varepsilon$.

Substituting (130) in (157),

$$Z_1 = k\biggl\{\frac{19}{20} + O\biggl(\frac{1}{\log\log c}\biggr)\biggr\} \sum_{c<n\le x} \log^{k-1}\frac{x}{n}\biggl|R\biggl(\frac{x}{n}\biggr)\biggr|, \tag{160}$$

and the $O$ depends on $\varepsilon$.

We shall next prove, by a suitable choice of $A$ and $B$, that

$$\phi_i\biggl(n, \frac{\varepsilon}{2}, A, B\biggr) \le \biggl\{\frac{1}{80} + O\biggl(\frac{1}{\log\log c}\biggr)\biggr\}\biggl(\frac{\log n}{2}\biggr)^{i-1}, \tag{161}$$

for $c < n \le x$ and $2 \le i \le k$, and the $O$ depends on $\varepsilon$.

We take $\sigma = (\varepsilon/2)$, $B = (6/\varepsilon)$ and $A = 80 \cdot 2^B$ in (130). Then, if $2 \le i \le B$, we can get (161) from (130) at once, and if $B < i \le k$, we can get from (130),

$$\begin{aligned} \phi_i\biggl(n, \frac{\varepsilon}{2}, A, B\biggr) &\ll 3^i (\log n)^{(1-(\varepsilon/2))(i+1)} \\ &\le (3\log^{1-(\varepsilon/2)} n)^{i+1} \\ &= \biggl(\frac{\log n}{2} \cdot 6\log^{-(\varepsilon/2)} n\biggr)^{i+1} \\ &= \biggl(\frac{\log n}{2}\biggr)^{i+1} (6\log^{-(\varepsilon/2)} n)^{i+1}, \end{aligned} \tag{162}$$

for $c < n \le x$, and the $\ll$ depends on $\varepsilon$.

It is easily seen that there exists a constant $\varepsilon_1$ depending only on $\varepsilon$ such that, when $n > c > \varepsilon_1$, $6\log^{-\varepsilon/2} n < 1$. And so, when $B < i \le k$ and $c < n \le x$,

$$(6\log^{-\varepsilon/2} n)^{i+1} \le (6\log^{-\varepsilon/2} n)^B = 6^B \log^{-(\varepsilon/2)B} n \ll \log^{-(\varepsilon/2)B} n,$$

and the $\ll$ depends on $B$; since we have taken $B = (6/\varepsilon)$, the last expression is $\log^{-3} n$. Substituting in (162) we get, when $B < i \leq k$ and $c < n \leq x$,

$$\phi_i\left(n, \frac{\varepsilon}{2}, A, B\right) \ll \left(\frac{\log n}{2}\right)^{i+1} \log^{-3} n \leq \left(\frac{\log n}{2}\right)^{i-1} \log^{-1} c,$$

and the $\ll$ depends on $\varepsilon$. This gives (161) when $B < i \leq k$.

Substituting (161) in (158),

$$Z_2 \leq \left\{\frac{1}{80} + O\left(\frac{1}{\log\log c}\right)\right\} \cdot \sum_{i=2}^{k} \binom{k}{i} \sum_{c<n\leq x} \left(\frac{\log n}{2}\right)^{i-1} \log^{k-i} \frac{x}{n}\left|R\left(\frac{x}{n}\right)\right|, \tag{163}$$

and the $O$ depends on $\varepsilon$.

Since

$$\binom{k}{i} = \frac{k!}{i!(k-i)!} \leq \frac{k!}{(i-1)!(k-i)!} = k\binom{k-1}{i-1},$$

for $1 \leq i \leq k$, we have

$$\sum_{i=2}^{k} \binom{k}{i} \left(\frac{\log n}{2}\right)^{i-1} \log^{k-i} \frac{x}{n} \leq k \sum_{i=2}^{k} \binom{k-1}{i-1} \left(\frac{\log n}{2}\right)^{i-1} \log^{k-i} \frac{x}{n};$$

putting $j = i - 1$ and using the binomial theorem, the right side of this inequality is

$$\begin{aligned}
k\sum_{j=1}^{k-1}\binom{k-1}{j}\left(\frac{\log n}{2}\right)^{j}\log^{k-1-j}\frac{x}{n} \\
&\le k\sum_{j=0}^{k-1}\binom{k-1}{j}\left(\frac{\log n}{2}\right)^{j}\log^{k-1-j}\frac{x}{n} \\
&= k\left(\frac{\log n}{2}+\log\frac{x}{n}\right)^{k-1} \\
&= k\left(\log x-\frac{\log n}{2}\right)^{k-1}.
\end{aligned}$$

Substituting in (163),

$$\begin{aligned}
Z_2 \le \left\{\frac{1}{80}+O\left(\frac{1}{\log\log c}\right)\right\}k \\
\cdot\sum_{c<n\le x}\left(\log x-\frac{\log n}{2}\right)^{k-1}\left|R\left(\frac{x}{n}\right)\right|,
\end{aligned} \tag{164}$$

and the $O$ depends on $\varepsilon$.

**Lemma 31.** *Let*

$$Y_1 = \sum_{c<n\le x}\log^{k-1}\frac{x}{n}\left|R\left(\frac{x}{n}\right)\right|, \tag{165}$$

$$Y_2 = \sum_{c<n\le x}\left(\log x-\frac{\log n}{2}\right)^{k-1}\left|R\left(\frac{x}{n}\right)\right|. \tag{166}$$

*Then*

$$Y_m \le \frac{m}{k}\left\{1+O\left(\frac{1}{\log\log x}\right)\right\}x\log^k xC(x)A\left(\frac{x}{2}\right), \quad m=1,2,$$

*and the* $O$ *depends on* $\varepsilon$.

*Proof.* Substituting (i) and (ii) of Lemma 30 in (165) and (166), respectively, we have

$$Y_m \le \log^{k-1} x \sum_{c<n\le x} \exp\left(-\frac{(k-1)\log n}{m\log x}\right)\left|R\left(\frac{x}{n}\right)\right|, \quad m=1,2. \tag{167}$$

From (152) and (151),

$$\left|R\left(\frac{x}{n}\right)\right| = \frac{x}{n}C\left(\frac{x}{n}\right)\left|Q\left(\frac{x}{n}\right)\right| \le \frac{x}{n}C\left(\frac{x}{n}\right)A\left(\frac{x}{2}\right),$$

for $2 \le n \le x$. Then using (iii) of Lemma 30 we have

$$\left|R\left(\frac{x}{n}\right)\right| \le \frac{x}{n}C(x)\exp(\log^{-(1/2)-\varepsilon} x \log n)A\left(\frac{x}{2}\right), \tag{168}$$

for $2 \le n \le x$.

Substituting this in (167),

$$Y_m \le \log^{k-1} x \sum_{c<n\le x} \frac{x}{n}C(x)\exp(-l_m \log n)A\left(\frac{x}{2}\right), \quad m=1,2, \tag{169}$$

where

$$l_m = \frac{k-1}{m\log x} - \log^{-(1/2)-\varepsilon} x. \tag{170}$$

Trivially,

$$\begin{aligned}\sum_{c<n\le x} \frac{1}{n}\exp(-l_m\log n) = \sum_{c<n\le x} n^{-l_m-1} &= \int_c^x t^{-l_m-1}\,dt + O(c^{-l_m-1})\\ &\le \int_1^\infty t^{-l_m-1}\,dt + O(1)\\ &= \frac{1}{l_m} + O(1);\end{aligned}$$

substituting in (169) we have

$$Y_m \le \left\{\frac{1}{l_m} + O(1)\right\} x \log^{k-1} x C(x) A\left(\frac{x}{2}\right), \quad m = 1, 2, \tag{171}$$

and the $O$ is absolute.

From (170) and (154) we have

$$\begin{aligned} l_m &= \frac{k}{m \log x}\left(1 - \frac{1}{k} - \frac{m}{k}\log^{(1/2)-\varepsilon} x\right) \\ &= \frac{k}{m\log x}\left\{1 + O\left(\frac{1}{\log\log x}\right)\right\}, \quad m = 1, 2; \end{aligned}$$

therefore, noting (153),

$$\begin{aligned} \frac{1}{l_m} + O(1) &= \frac{m\log x}{k}\left\{1 + O\left(\frac{1}{\log\log x}\right)\right\}^{-1} + O(1) \\ &= \frac{m\log x}{k}\left\{1 + O\left(\frac{1}{\log\log x}\right)\right\}, \quad m = 1, 2, \end{aligned}$$

and the $O$ depends on $\varepsilon$. Substituting in (171) the lemma follows.

**Lemma 32.**

$$|J_2| \le \left\{\frac{39}{40} + O\left(\frac{1}{\log\log x}\right)\right\} x \log^k x C(x) A\left(\frac{x}{2}\right) + O(x),$$

*and the $O$'s depend on $\varepsilon$.*

*Proof.* Substituting Lemma 31 in (160) and (164) we have

$$\begin{aligned} Z_1 &\le \left\{\frac{19}{20} + O\left(\frac{1}{\log\log c}\right)\right\} x \log^k x C(x) A\left(\frac{x}{2}\right), \\ Z_2 &\le \left\{\frac{1}{40} + O\left(\frac{1}{\log\log c}\right)\right\} x \log^k x C(x) A\left(\frac{x}{2}\right), \end{aligned}$$

and the $O$'s depend on $\varepsilon$. Noting (105), the remainder terms in $Z_1$ and $Z_2$ are $O(1/(\log\log x))$. On substitution in (159) the lemma follows.

**Lemma 33.**

$$J_1 \ll \log^{k-(1/2)} x|R(x)| + x\log^{k-(\varepsilon/2)} xC(x)A\left(\frac{x}{2}\right) + x,$$

*and the $\ll$ depends on $\varepsilon$.*

*Proof.* If $n \le c$, then by (105), $\log n \le \log c = \sqrt{\log x}$, and therefore,

$$\exp(\log^{-(1/2)-\varepsilon} x \log n) \ll 1 \quad \text{for } n \le c;$$

substituting in (168) we obtain

$$R\left(\frac{x}{n}\right) \ll \frac{x}{n}C(x)A\left(\frac{x}{2}\right) \quad \text{for } 2 \le n \le c, \tag{172}$$

and the $O$ is absolute. It follows that, using (105) again,

$$\sum_{2\le n\le c}\left|R\left(\frac{x}{n}\right)\right| \ll x\log cC(x)A\left(\frac{x}{2}\right) = x\sqrt{\log x}C(x)A\left(\frac{x}{2}\right), \tag{173}$$

and the $\ll$ is absolute.

Using Lemma 31,

$$\begin{aligned}\sum_{c<n\le x}\log^k \frac{x}{n}\left|R\left(\frac{x}{n}\right)\right| &\le \log x\sum_{c<n\le x}\log^{k-1}\frac{x}{n}\left|R\left(\frac{x}{n}\right)\right| \\ &= Y_1\log x \ll x\log^{k+1} xC(x)A\left(\frac{x}{2}\right),\end{aligned} \tag{174}$$

and the $O$ depends on $\varepsilon$.

Substituting (173) and (174) in Lemma 23 and noticing (153), the lemma is proved.

**Lemma 34.**

$$I \ll x\log^{k-(\varepsilon/2)} xC(x)A\left(\frac{x}{2}\right),$$

*and the $\ll$ depends on $\varepsilon$.*

*Proof.* From the definition of $\Lambda_i(n)$, see (1), we see that $\Lambda_i(1) = \mu(1)\log^i 1 = 0$ for $i \geq 1$. Therefore, we get from (106) and (14)

$$|I| \leq \sum_{i=1}^{k} \binom{k}{i} \sum_{2\leq n\leq c} \Lambda_i(n) \log^{k-i} x \left| R\left(\frac{x}{n}\right)\right|.$$

Substituting (172) into this inequality,

$$I \ll x \sum_{i=1}^{k} \binom{k}{i} \sum_{2\leq n\leq c} \frac{\Lambda_i(n)}{n} \log^{k-i} x C(x) A\left(\frac{x}{2}\right), \tag{175}$$

and the $O$ is absolute.

Writing $D = (2/\varepsilon) + 1$ and noting (105) we get from (16)

$$\sum_{n\leq c} \frac{\Lambda_i(n)}{n} \ll \log^i c = \log^{i/2} x \quad \text{for } 1 \leq i \leq D, \tag{176}$$

and the $\ll$ depends on $D$.

If $D < i \leq k$, from the definition of $\Lambda_i$, see (1), we have

$$\Lambda_i(n) = \sum_{m|n} \mu(m) \log^i \frac{n}{m} \leq \log^i n \sum_{m|n} 1 = d(n) \log^i n,$$

where $d$ is a divisor function. And, hence, noting (105) again,

$$\sum_{n\leq c} \frac{\Lambda_i(n)}{n} \leq \log^i c \sum_{n\leq c} \frac{d(n)}{n} \ll \log^{i+2} c = \log^{(i/2)+1} x, \tag{177}$$

for $D < i \leq k$ and the $\ll$ is $i$-uniform.

Noting (153),

$$\begin{aligned} \binom{k}{i} &= \frac{k(k-1)\cdots(k-i+1)}{i!} \leq \frac{1}{i!}k^i \\ &\leq \frac{1}{i!} \log^{(i/2)-(\varepsilon/2)i} x \quad \text{for } 1 \leq i \leq k. \end{aligned} \tag{178}$$

Using this and (176),

$$\binom{k}{i}\sum_{n\le c}\frac{\Lambda_i(n)}{n} \ll \frac{1}{i!}\log^{i-(\varepsilon/2)i} x \le \frac{1}{i!}\log^{i-(\varepsilon/2)} x,$$

for $1 \le i \le D$, and the $\ll$ depends on $D$.

Using (178) again and (177),

$$\begin{aligned}\binom{k}{i}\sum_{n\le c}\frac{\Lambda_i(n)}{n} &\ll \frac{1}{i!}(\log x)^{i+1-(\varepsilon/2)i}\\ &\le \frac{1}{i!}(\log x)^{i+1-(\varepsilon/2)D}\\ &= \frac{1}{i!}\log^{i-(\varepsilon/2)} x,\end{aligned}$$

for $D < i \le k$ and the $\ll$ is absolute.

Substituting the last two formulae in (175) and noting that $D$ depends on $\varepsilon$, we have

$$I \ll x\sum_{i=1}^{k}\frac{1}{i!}\log^{k-(\varepsilon/2)} xC(x)A\left(\frac{x}{2}\right) \ll x\log^{k-(\varepsilon/2)} xC(x)A\left(\frac{x}{2}\right),$$

and the $\ll$ depends on $\varepsilon$. This proves the lemma.

**Theorem.**

$$R(x) \ll x\exp(-\log^{(1/2)-\varepsilon} x),$$

*and the $\ll$ depends on $\varepsilon$.*

*Proof.* Substituting Lemma 32 and Lemma 33 in (110) and then substituting (110) and Lemma 34 in (108), we obtain

$$\begin{aligned}|R(x)|\log^k x \le& \left\{\frac{39}{40}+O\left(\frac{1}{\log\log x}\right)\right\}x\log^k xC(x)A\left(\frac{x}{2}\right)\\ &+O\left\{\log^{k-(1/2)} x|R(x)|+x(2k\log\log x)^k\right\},\end{aligned} \tag{179}$$

and the $O$'s depend on $\varepsilon$.

By (153), when $x$ is suitably large,

$$\begin{aligned}(2k\log\log x)^k &\le (2\sqrt{\log x}\log\log x)^k\\ &\le (e^{-1}\log x)^k = e^{-k}\log^k x\\ &= \exp(-[\log^{(1-\varepsilon)/2} x])\log^k x\\ &\ll \exp(-\log^{(1/2)-\varepsilon} x)\log^{k-1} x\\ &= C(x)\log^{k-1} x,\end{aligned}$$

the last step following from (149), and the $\ll$ depending on $\varepsilon$. Substituting in (179) we get

$$\begin{aligned}|R(x)|\log^k x \le \left\{\frac{39}{40} + O\left(\frac{1}{\log\log x}\right)\right\} x\log^k x C(x) A\left(\frac{x}{2}\right)\\ + O\{\log^{k-(1/2)} x|R(x)| + xC(x)\log^{k-1} x\},\end{aligned}$$

and the $O$'s depend on $\varepsilon$. By substituting (152) and dividing both sides of this inequality by $x\log^k xC(x)$, we get

$$\begin{aligned}|Q(x)| \le \left\{\frac{39}{40} + O\left(\frac{1}{\log\log x}\right)\right\} A\left(\frac{x}{2}\right)\\ + O\left\{\log^{-(1/2)} x|Q(x)| + \frac{1}{\log x}\right\},\end{aligned}$$

and the $O$'s depend on $\varepsilon$. From this we see that there exists a constant $\varepsilon_2$ depending only on $\varepsilon$ such that, when $x \ge \varepsilon_2$,

$$|Q(x)| < \frac{79}{80} A\left(\frac{x}{2}\right) + \frac{1}{160}|Q(x)| + \frac{1}{160}A(1),$$

(for the last term noticing that $A(1) > 0$ from the definition of $A$). From the definition of $A$ we have $A(1) \le A(x/2)$ for $x \ge 2$. Substituting this in the above inequality and reducing, we get

$$|Q(x)| < A\left(\frac{x}{2}\right) \quad \text{for } x \ge \varepsilon_2,$$

that is, by (151),

$$|Q(x)| < \sup_{1\le t\le (x/2)} |Q(t)| \quad \text{for } x \ge \varepsilon_2.$$

Consequently,

$$Q(x) \ll 1,$$

and the $\ll$ depends on $\varepsilon$. Substituting this in (152) and noticing (149), we complete the proof of the theorem.

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