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DIFFERENT EXPONENTIAL SPECTRA IN BANACH ALGEBRAS

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ABSTRACT. We compare the exponential spectrum of a Banach algebra element with the usual spectrum of the element.

1. Preliminaries. The exponential spectrum of a Banach algebra element was introduced by Robin Harte [**11**]. Since 1976, various authors have studied the relationship between the usual spectrum and the exponential spectrum in the setting of Banach algebras, see [**8, 9, 10, 13, 16**]. Although in general the exponential spectrum of an element in a Banach algebra is different from the usual spectrum [**11, 18**], it shares many properties with the usual spectrum. The present paper contributes to results which strengthen this viewpoint. In [**18**] it was shown how the exponential spectrum turns up naturally in a number of diverse applications, especially in the context of Toeplitz operators.

If A is a complex Banach algebra with unit 1, we denote the group of invertible elements by A^{-1} and the connected component of 1 in A^{-1} by $\operatorname{Exp} A$. Recall that $\operatorname{Exp} A$ is a normal open and closed subgroup of A^{-1} generated by elements e^a, $a \in A$. The *exponential spectrum* of $a \in A$ is the set $\varepsilon(a, A) := \{\lambda \in \mathbf{C} \mid \lambda - a \notin \operatorname{Exp} A\}$ and the spectrum of a is denoted in the usual way by $\sigma(a, A) := \{\lambda \in \mathbf{C} \mid \lambda - a \notin A^{-1}\}$. As observed in [**11**], $\varepsilon(a, A)$ is compact, and we have the inclusions

$$\partial\varepsilon(a, A) \subset \sigma(a, A) \subset \varepsilon(a, A) \subset \eta\sigma(a, A)$$

where ∂ denotes the boundary for subsets of the complex plane $\mathbf{C}$ and η the connected hull of compact sets in $\mathbf{C}$.

If $K \subset \mathbf{C}$, we use the symbol $\operatorname{acc} K$ to indicate the set of accumulation points of K and the symbol $\operatorname{iso} K$ for the set of isolated points of K.

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By an ideal in A we mean a two-sided ideal in A. An ideal J in A is called *inessential* [**2**, p. 106] whenever

$$b \in J \Longrightarrow \operatorname{acc} \sigma(b, A) \subset \{0\}$$

and J will be called *special* in A if, for every $b \in J$, $-1 \notin \sigma(b, A)$.

The radical of A will be denoted by $\operatorname{Rad} A$ and A is said to be semi-simple if $\operatorname{Rad} A = \{0\}$. An element $a \in A$ is quasinilpotent if $\sigma(a, A) = \{0\}$. The set of these elements will be denoted by $QN(A)$. Recall that if J is a closed ideal in A, then $b \in A$ is called *Riesz* relative to J if $b + J \in QN(A/J)$, see [**3**, Section R.1.].

An element $a \neq 0$ in a semi-simple Banach algebra is called *rank one* if there exists a linear functional τ_a such that $axa = \tau_a(x)a$ for all $x \in A$. For properties of these elements we refer to [**19**]. If A and B are Banach algebras, then a linear map $T : A \to B$ is a homomorphism if $T(ab) = TaTb$ for all $a, b \in A$ and $T1 = 1$.

The paper is organized as follows. In Section 3 we investigate how the exponential spectrum of $b \in B \subset A$ depends on the algebra if B is merely a subalgebra of A with the same unit as A and also in the case where B has a unit element different from the unit element in A. Section 4 deals with the exponential spectrum in quotient algebras. In Section 5 we study the behavior of the exponential spectrum under perturbation by radical elements, rank one elements and Riesz elements. In our final section it is observed that the exponential spectrum shares many analytic properties with the ordinary spectrum.

2. Homomorphisms and Exp. If A and B are Banach algebras and $T : B \to A$ is a bounded homomorphism, then it is easy to see that $T \operatorname{Exp} B \subset \operatorname{Exp} A$. We will show that this is true even if T is not bounded.

Lemma 2.1. *Let $T : B \to A$ be a homomorphism with B a Banach algebra and A a commutative Banach algebra. Suppose $b \in B$ and U is a neighborhood of $\sigma(b, B)$. If $f : U \to \mathbf{C}$ is an analytic function, then $Tf(b) - f(Tb) \in \operatorname{Rad} A$.*

Proof. If ϕ is a multiplicative linear functional on A, then ϕT is a multiplicative linear functional on B and so both ϕ and ϕT are continuous. If Γ is a closed rectifiable curve in U winding $+1$ round $\sigma(b,B)$, then

$$\begin{aligned}(\phi T)f(b) &= \phi T\left[\frac{1}{2\pi i}\int_\Gamma f(z)(z-b)^{-1}\,dz\right]\\ &= \phi\left[\frac{1}{2\pi i}\int_\Gamma f(z)(z-Tb)^{-1}\,dz\right].\end{aligned}$$

Hence $\phi[Tf(b)-f(Tb)]=0$ for every multiplicative linear functional ϕ and so $Tf(b)-f(Tb)\in\operatorname{Rad}A$. □

In order to prove our next result we need the following fact. If A is a Banach algebra, then $\operatorname{Exp}A+\operatorname{Rad}A\subset\operatorname{Exp}A$. Indeed, if $a\in\operatorname{Exp}A$ and $r\in\operatorname{Rad}A$, then $a+r=a(1+a^{-1}r)$. Since the spectrum of $1+a^{-1}r$ consists of the point 1 only, it follows from [**20**, Theorem 10.30] that $1+a^{-1}r\in\operatorname{Exp}A$. Since $\operatorname{Exp}A$ is closed under multiplication, $a+r\in\operatorname{Exp}A$.

Lemma 2.2. *Let $T:B\to A$ be a homomorphism with B a Banach algebra and A a commutative Banach algebra. Then $T\operatorname{Exp}B\subset\operatorname{Exp}A$.*

Proof. If $e^b\in\operatorname{Exp}B$, then by Lemma 2.1 $Te^b=e^{Tb}+r$ with $r\in\operatorname{Rad}A$. By the remarks preceding the lemma, $Te^b\in\operatorname{Exp}A$. □

Theorem 2.1. *Let $T:B\to A$ be a homomorphism with B and A Banach algebras. Then $T\operatorname{Exp}B\subset\operatorname{Exp}A$.*

Proof. Let $e^b\in\exp B$, and let C be a maximal commutative subalgebra of B containing b. Let D be a maximal commutative subalgebra of A containing TC. By the previous lemma, $Te^b\in\operatorname{Exp}D\subset\operatorname{Exp}A$ because D is a closed subalgebra of A. The theorem follows from the fact that T is a homomorphism and $\operatorname{Exp}A$ is closed under multiplication. □

Corollary 2.1. *Let B and A be Banach algebras such that $1\in B\subset A$. Then $\operatorname{Exp}B\subset\operatorname{Exp}A$.*

3. Subalgebras. In [**10**] it was investigated how the exponential spectrum depends on the algebra in the case that A and B are Banach algebras with B a closed subalgebra of A and such that A and B have the same unit element. We are going to investigate how the exponential spectrum depends on the algebra if B is merely a subalgebra of A and A and B have the same unit element.

Proposition 3.1. *Let A and B be Banach algebras such that $1 \in B \subset A$. Then $\varepsilon(x, A) \subset \varepsilon(x, B)$ for every $x \in B$.*

Proof. This follows from Corollary 2.1. □

We give an example to show that the inclusion in Proposition 3.1 may be strict.

Example 3.1. It follows from [**1**, Example 3.7] that there exists a Hilbert space H, a compact operator T on H and a subalgebra B of the algebra $A := BL(H)$ of bounded linear operators on H such that $T \in B$, $\sigma(T, A)$ is countable but $\sigma(T, B) \supset \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}$. Hence,

$$\varepsilon(T, A) = \sigma(T, A) \subsetneqq \sigma(T, B) \subset \varepsilon(T, B).$$

Proposition 3.2. *Let A and B be Banach algebras such that $1 \in B \subset A$. If $x \in B$ and if D is a nonvoid clopen subset of $\varepsilon(x, B)$, then $D \cap \varepsilon(x, A) \neq \varnothing$.*

Proof. If D is a nonvoid clopen subset of $\varepsilon(x, B)$, then $D \cap \sigma(x, B)$ is a clopen subset of $\sigma(x, B)$. Also, it is nonvoid because $D \cap \partial\varepsilon(x, B) \neq \varnothing$ and $\partial\varepsilon(x, B) \subset \sigma(x, B)$. By [**7**, Corollary 2.3], $D \cap \sigma(x, B) \cap \sigma(x, A) \neq \varnothing$ and so $D \cap \varepsilon(x, A) \neq \varnothing$. □

Corollary 3.1. (i) *The set* $\operatorname{iso} \varepsilon(x, B) \subset \varepsilon(x, A)$, *and hence* $\operatorname{iso}(\varepsilon(x, B) \backslash \varepsilon(x, A)) = \varnothing$.

(ii) *If $\varepsilon(x, B) \backslash \varepsilon(x, A) \neq \varnothing$, then it is uncountable.*

(iii) *if $\varepsilon(x, B)$ is totally disconnected, then $\varepsilon(x, B) = \varepsilon(x, A)$.*

The following results deal with the situation when A and B do not have the same unit element. Note that if p is an idempotent in A with $0 \neq p \neq 1$ and if $B := pAp$, then B is a closed subalgebra of A with identity p. If $x \in B$, then the exponential function of x in B is denoted by e_B^x and it coincides with $p + \sum_{n=1}^{\infty}(x^n/n!)$ while the exponential function of x in A is denoted (as usual) by e^x and it coincides with $1 + \sum_{n=1}^{\infty}(x^n/n!)$.

Proposition 3.3. *Let A be a Banach algebra with p an idempotent in A such that $0 \neq p \neq 1$. If $B := pAp$, then $B \cap A^{-1} = \varnothing$, and so we have $0 \in \varepsilon(x, A)$ for every $x \in B$.*

Proof. If $x \in B \cap A^{-1}$, then x^{-1} exists in A. Since $x(1-p) = 0$, it follows that $0 = x^{-1}x(1-p) = 1 - p$ and so $p = 1$ which is a contradiction. Hence, $0 \in \sigma(x, A) \subset \varepsilon(x, A)$. $\square$

Proposition 3.4. *Let A be a Banach algebra with $0 \neq p \neq 1$ an idempotent in A that commutes with every element in A, and let $B := pA$. Then*

(i) $p\mathrm{Exp}\, A = \mathrm{Exp}\, B \subset \overline{\mathrm{Exp}\, A}$,

(ii) $\varepsilon(x, B) \subset \varepsilon(x, A)$ *for every* $x \in B$.

Proof. (i) If $x \in p\mathrm{Exp}\, A$, then for some $c \in A^k$ and using the fact that p is an idempotent which commutes with every element of A, we have

$$x = pe^{c_1} \cdots e^{c_k} = p^k e^{c_1} \cdots e^{c_k} = e_B^{pc_1} \cdots e_B^{pc_k} \in \mathrm{Exp}\, B.$$

The inclusion $\mathrm{Exp}\, B \subset p\mathrm{Exp}\, A$ follows similarly. By [**13**, Theorem 6], $p\mathrm{Exp}\, A \subset \overline{\mathrm{Exp}\, A}$.

(ii) If $x \in B$ and if $\lambda \notin \varepsilon(x, A)$, then $\lambda - x \in \mathrm{Exp}\, A$. By (i)

$$\lambda p - x = p(\lambda - x) \in p\mathrm{Exp}\, A = \mathrm{Exp}\, B,$$

i.e., $\lambda \notin \varepsilon(x, B)$. $\square$

If A is a Banach algebra and J is a proper ideal of A endowed with an identity p with $0 \neq p \neq 1$, then $\sigma(x+\alpha, A) = \sigma(x+\alpha p, J) \cup \{\alpha\}$ for any

$x \in J$ and any $\alpha \in \mathbf{C}$, [**5**, Proposition 5]. For the exponential spectrum we have the following. Although in general $\varepsilon(p(x), A) \subset p\varepsilon(x, A)$ where $p(z)$ is a complex polynomial ([**11**, 3.2]), it follows that $\varepsilon(x + \alpha, A) = \varepsilon(x, A) + \alpha$ for every $\alpha \in \mathbf{C}$.

Proposition 3.5. *Let A be a Banach algebra, and let J be a proper ideal of A endowed with an identity $0 \neq p \neq 1$ such that p commutes with every element of A. Then J is closed and $\varepsilon(x + \alpha p, J) \cup \{\alpha\} \subset \varepsilon(x + \alpha, A)$ for any $x \in J$ and any $\alpha \in \mathbf{C}$.*

Proof. Since $J = pJp = pAp$ we have that J is closed in A. If $x \in J$ and $\alpha \in \mathbf{C}$, then $\varepsilon(x+\alpha, A) = \varepsilon(x, A)+\alpha$ and $\varepsilon(x+\alpha p, J) = \varepsilon(x, J)+\alpha$. This together with Proposition 3.4(ii) and Proposition 3.3 proves our result. $\square$

Proposition 3.6. *Let A be a Banach algebra, and let J be an inessential ideal of A endowed with an identity $0 \neq p \neq 1$. Then J is closed and $\varepsilon(x + \alpha, A) = \varepsilon(x + \alpha p, J) \cup \{\alpha\}$ for any $x \in J$ and any $\alpha \in \mathbf{C}$.*

Proof. It follows in the same way as in the proof of the previous result that J is closed in A. Since J is inessential, $\sigma(x, A)$ is either finite or a sequence converging to zero for every $x \in J$ and so $\sigma(x, A) = \varepsilon(x, A)$ for every $x \in J$. This together with [**5**, Proposition 5] and the remarks preceding Proposition 3.5 proves the assertion of the proposition. $\square$

4. Quotient algebras. In this section we investigate how the exponential spectrum behaves in quotient algebras. The key matter in this section is the fact that elements in inessential ideals have a simple spectrum.

It is well known [**5**, Proposition 1] that if J_1 and J_2 are closed ideals in a Banach algebra A, then for all $a \in A$

$$\sigma(a + J_1, A/J_1) \cup \sigma(a + J_2, A/J_2) = \sigma(a + J_1 \cap J_2, A/J_1 \cap J_2).$$

For the exponential spectrum it follows from $J_1 \cap J_2 \subset J_i$, $i = 1, 2$, and [**2**, Theorem 3.3.8] that

$$a + J_1 \cap J_2 \in \operatorname{Exp}(A/J_1 \cap J_2) \Longrightarrow a + J_1 \in \operatorname{Exp}(A/J_1)$$

and

$$a + J_2 \in \operatorname{Exp}(A/J_2)$$

and so

$$\varepsilon(a + J_1, A/J_1) \cup \varepsilon(a + J_2, A/J_2) \subset \varepsilon(a + J_1 \cap J_2, A/(J_1 \cap J_2)).$$

Our next example shows that this inclusion can be strict.

Example 4.1. Let $\Gamma := \{z \in \mathbf{C} \mid |z| = 1\}$ and A be the Banach algebra $C(\Gamma)$ of complex valued continuous functions with the sup norm. Let $f \in A$ with $f(z) = z$, $z \in \Gamma$. If $F_0 = \{z \in \Gamma \mid \operatorname{Im} z \geq 0\}$ and $F_1 = \{z \in \Gamma \mid \operatorname{Im} z \leq 0\}$, then $J_0 := \{g \in A \mid g(F_0) = 0\}$ and $J_1 := \{g \in A \mid g(F_1) = 0\}$ are both closed ideals in A. Furthermore, $\varepsilon(f + J_0, A/J_0) = F_0$ and $\varepsilon(f + J_1, A/J_1) = F_1$ and so $\varepsilon(f + J_0, A/J_0) \cup \varepsilon(f + J_1, A/J_1) = \Gamma \subsetneqq \mathbf{D} = \{z \in \mathbf{C} \mid |z| \leq 1\} = \varepsilon(f + J_0 \cap J_1, A/J_0 \cap J_1)$. $\square$

If we combine the remarks preceding the example it follows that

$$\begin{aligned}\sigma(a + J_1 \cap J_2, A/(J_1 \cap J_2)) &\subset \varepsilon(a + J_1, A/J_1) \cup \varepsilon(a + J_2, A/J_2)\\ &\subset \varepsilon(a + J_1 \cap J_2, A/(J_1 \cap J_2)).\end{aligned}$$

To prove our main results in this section, we need the following lemma.

Lemma 4.1. *If J is an inessential special ideal in a Banach algebra A, then* $\operatorname{Exp} A + J \subset \operatorname{Exp} A$.

Proof. If $b \in \operatorname{Exp} A$ and $j \in J$, then $b + j = b(1 + b^{-1}j)$. Since J is inessential, the spectrum of $1 + b^{-1}j$ is either finite or a sequence converging to 1. Because J is special, $0 \notin \sigma(1 + b^{-1}j)$ and so, by [**20**, Theorem 10.30], $1 + b^{-1}j \in \operatorname{Exp} A$. Since $\operatorname{Exp} A$ is closed under multiplication, $b + j \in \operatorname{Exp} A$. $\square$

Our next result improves [**9**, Proposition 2.1].

Theorem 4.1. *Let J_1 and J_2 both be closed ideals in a Banach algebra A with J_1 inessential and special. Then, for all $a \in A$,*

$$a + J_1 \cap J_2 \in \operatorname{Exp}(A/J_1 \cap J_2) \Longleftrightarrow a + J_1 \in \operatorname{Exp}(A/J_1)$$

and

$$a + J_2 \in \operatorname{Exp}(A/J_2).$$

Proof. For the nontrivial implication suppose $a + J_1 \in \operatorname{Exp}(A/J_1)$ and $a + J_2 \in \operatorname{Exp}(A/J_2)$. By [**5**, Proposition 1], we already know that $a + J_1 \cap J_2 \in (A/J_1 \cap J_2)^{-1}$. In view of [**2**, Theorem 3.3.8], there exist $b_1 \in \operatorname{Exp} A$ and $j_1, j_2 \in J_1$ such that $b_1 a = 1 + j_1$ and $ab_1 = 1 + j_2$. Also there exist $b_2 \in \operatorname{Exp} A$ and $k_1, k_2 \in J_2$ such that $b_2 a = 1 + k_1$ and $ab_2 = 1 + k_2$. Then $(b_1 - j_1 b_2)a = 1 - j_1 k_1$ and $a(b_1 - b_2 j_2) = 1 - k_2 j_2$. Since $j_1 k_1,\ k_2 j_2 \in J_1 \cap J_2$ and $b_1 - j_1 b_2,\ b_1 - b_2 j_2 \in \operatorname{Exp} A$ (Lemma 4.1) it follows that $a + J_1 \cap J_2 \in \operatorname{Exp}(A/J_1 \cap J_2)$. $\square$

Corollary 4.1. *Let J_1 and J_2 both be closed ideals in a Banach algebra A with J_1 inessential and special. For all $a \in A$,*

$$\varepsilon(a + J_1 \cap J_2, A/(J_1 \cap J_2)) = \varepsilon(a + J_1, A/J_1) \cup \varepsilon(a + J_2, A/J_2).$$

Let A and B be Banach algebras such that $1 \in B \subset A$. If J is an ideal in both A and B, we will denote the closure of J in A by J_A and the closure of J in B by J_B. It is well known [**7**, Theorem 3.4] that if J is an inessential ideal in both A and B, then for every $b \in B$,

$$\sigma(b, B) = \sigma(b, A) \cup \sigma(b + J_B, B/J_B).$$

To derive an analogous result for the exponential spectrum, note that if J is an ideal in B and $b \in B$, then in view of Proposition 3.1 and the fact that the canonical homomorphism from B onto B/J_B is bounded

$$\varepsilon(b, A) \cup \varepsilon(b + J_B, B/J_B) \subset \varepsilon(b, B).$$

Our next example shows that the above inclusion can be strict.

Example 4.2. Consider the Banach algebras A and B in Example 3.1 with $1 \in B \subset A$, $T \in B$ with $\varepsilon(T, A)$ countable and $\varepsilon(T, B)$ uncountable. Since B has a finer norm than A, see [**1**, Example

3.7], $\operatorname{Exp} B \subset \operatorname{Exp} A$. Let C be the subalgebra of B generated by T. Since C is a closed subalgebra of B, $\operatorname{Exp} C \subset \operatorname{Exp} B \subset \operatorname{Exp} A$. If $J := TC$, then J is an ideal in C and the spectrum of T in C is uncountable because the spectrum of T in B is uncountable. Then $\varepsilon(T, A) \cup \varepsilon(T + J_C, C/J_C) = \varepsilon(T, A) \cup \{0\}$ is countable while $\varepsilon(T, C)$ is uncountable. Hence $\varepsilon(T, C) \underset{\neq}{\supseteq} \varepsilon(T, A) \cup \varepsilon(T + J_C, C/J_C)$.

Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose J is an ideal both in A and in B. If $x \in B$, $\varepsilon(x, B) = \varepsilon(x, A) \cup \varepsilon(x + J_B, B/J_B)$ and $\varepsilon(x + J_A, A/J_A) = \varepsilon(x + J_B, B/J_B)$, then $\varepsilon(x, B) = \varepsilon(x, A)$.

Theorem 4.2. *Let A and B be Banach algebras such that $1 \in B \subset A$. If J_B is special in B, then, for every $x \in B$,*

$$x \in \operatorname{Exp} B \Longleftrightarrow x \in \operatorname{Exp} A \quad \textit{and} \quad x + J_B \in \operatorname{Exp}(B/J_B).$$

Proof. The forward implication follows from Corollary 2.1 and the fact that the canonical homomorphism from B onto B/J_B is bounded. If $x \in \operatorname{Exp} A$ and $x + J_B \in \operatorname{Exp}(B/J_B)$, it follows from [**7**, Theorem 3.4] that $x \in B^{-1}$. By [**2**, Theorem 3.3.8] there exist elements $b \in \operatorname{Exp} B$ and $j \in J_B$ such that $x = b + j$. Since J_B is inessential [**2**, Corollary 5.7.6] and special in B, it follows from Lemma 4.1 that $x \in \operatorname{Exp} B$. $\square$

Corollary 4.2. *Let A and B be Banach algebras such that $1 \in B \subset A$. If J is an inessential ideal in A and B and if J_B is special in B, then for every $x \in B$,*

$$\varepsilon(x, B) = \varepsilon(x, A) \cup \varepsilon(x + J_B, B/J_B).$$

Corollary 4.3. *Let A and B be Banach algebras such that $1 \in B \subset A$. Suppose J is an inessential ideal in A and B such that J_B is special in B. If $x \in B$ and if $J_B \subset J_A$, then*

$$\varepsilon(x, B) \backslash \varepsilon(x + J_B, B/J_B) \subset \varepsilon(x, A) \backslash \varepsilon(x + J_A, A/J_A).$$

5. Perturbation results. In this section we study the behavior of the exponential spectrum under perturbation by radical elements, rank one elements and Riesz elements.

It is well known [**14**, Theorem 2.5] that if A is a Banach algebra, then $\operatorname{Rad} A = \{x \in A \mid A^{-1} + x \subset A^{-1}\}$. If $a \in A$ and $b \in \operatorname{Rad} A$, then

$$a \in \operatorname{Exp} A \Longleftrightarrow a + b \in \operatorname{Exp} A.$$

To prove the forward implication, note that $a + b = a(1 + a^{-1}b)$. Since the spectrum of $1 + a^{-1}b$ consists of the point 1 only, it follows from [**20**, Theorem 10.30] that $1 + a^{-1}b \in \operatorname{Exp} A$. Since $\operatorname{Exp} A$ is closed under multiplication $a + b \in \operatorname{Exp} A$. To prove the converse implication, note that $a = a + b - b$. Hence, $\varepsilon(a + b, A) = \varepsilon(a, A)$ for every $a \in A$ and $b \in \operatorname{Rad} A$.

This observation together with [**2**, Theorem 3.3.8] shows that the canonical homomorphism $A \mapsto A/\operatorname{Rad} A$ is exponential preserving. In order to prove our first main result in this section, we need the following lemma.

Lemma 5.1. *Let $A \neq \mathbf{C}$ be a semi-simple Banach algebra, and let b be a rank one element in A. If $a \in \operatorname{Exp} A$, then $a + b \notin \operatorname{Exp} A \Leftrightarrow \tau_b(a^{-1}) = -1$.*

Proof. If $a \in \operatorname{Exp} A$, then $a + b = a(1 + a^{-1}b)$. Since $\operatorname{Exp} A$ is closed under multiplication, by [**19**, Lemmas 2.7, 2.8] $a + b \notin \operatorname{Exp} A \Leftrightarrow -1 \in \varepsilon(a^{-1}b, A) = \sigma(a^{-1}b, A) = \{0, \tau_b(a^{-1})\} \Leftrightarrow \tau_b(a^{-1}) = -1$. $\square$

Theorem 5.1. *Let A be a semi-simple Banach algebra and $a \in A$. If $b \in A$ is rank one, then $\operatorname{acc} \varepsilon(a + b, A) \subset \eta\varepsilon(a, A)$.*

Proof. By Lemma 5.1, $\varepsilon(a + b, A)\backslash\varepsilon(a, A) = \{\lambda \in \mathbf{C}\backslash\varepsilon(a, A) \mid \tau_{-b}((\lambda - a)^{-1}) = -1\}$. Since the function $\tau_{-b}(\lambda - a)^{-1} + 1$ is analytic and the set $\varepsilon(a + b, A)$ compact, it follows from [**6**, Theorem 3.7] that the set $\varepsilon(a + b, A)\backslash\eta\varepsilon(a, A)$ consists of isolated points of $\varepsilon(a + b, A)$ and so $\operatorname{acc} \varepsilon(a + b, A) \subset \eta\varepsilon(a, A)$. $\square$

The proof of Theorem 5.1 is a modification of the proof of Theorem 2.4 in [**15**] which is the analogue of Theorem 5.1 for the Browder spectrum and the ordinary spectrum. See also [**12**, Theorem 5]. Our next example shows that the inclusion in Theorem 5.1 may be strict.

Example 5.1. Let A be the Banach algebra $BL(l^2(\mathbf{Z}))$ of bounded linear operators on $l^2(\mathbf{Z})$. If e_n is the nth vector of the canonical basis of $l^2(\mathbf{Z})$, define elements a and b in A as follows:

$$ae_n = \begin{cases} 0 & \text{if } n = -1, \\ e_{n+1} & \text{if } n \neq -1, \end{cases}$$
$$be_n = \begin{cases} e_0 & \text{if } n = -1, \\ 0 & \text{if } n \neq -1. \end{cases}$$

Then b is a rank one element in A, $\varepsilon(a+b, A) = \{z \in \mathbf{C} \mid |z| = 1\}$ and $\varepsilon(a, A) = \{z \in \mathbf{C} \mid |z| \leq 1\}$. Hence, $\operatorname{acc} \varepsilon(a+b, A) \subsetneq \varepsilon(a, A)$.

Theorem 5.2. *Let J be a closed inessential ideal in a Banach algebra A and $a \in A$. If $b \in A$ is Riesz relative to J and $ab = ba$, then $\operatorname{acc} \varepsilon(a+b, A) \subset \eta\varepsilon(a, A)$.*

Proof. It follows from [**17**, Theorem 5.3] and [**2**, Theorem 5.7.4(iii)] that $\operatorname{acc} \sigma(a+b, A) \subset \eta\sigma(a+b+J, A/J)$. Since $b + J \in QN(A/J)$ and $b+J$ and $a+J$ commute in A/J, $\sigma(a+b+J, A/J) = \sigma(a+J, A/J)$. If we combine these remarks, then

$$\begin{aligned} \operatorname{acc} \varepsilon(a+b, A) &\subset \eta \operatorname{acc} \varepsilon(a+b, A) \\ &= \eta \operatorname{acc} \sigma(a+b, A) \\ &\subset \eta\sigma(a+b+J, A/J) \\ &= \eta\sigma(a+J, A/J) \\ &\subset \eta\sigma(a, A) \\ &= \eta\varepsilon(a, A). \qquad \square \end{aligned}$$

Note that the commutativity condition in the above theorem cannot be omitted. It follows from [**8**, Example 1] that there exists a Banach algebra A and elements a and b in A with $ab \neq ba$ and b Riesz in A relative to some closed ideal in A such that

$$\operatorname{acc}\varepsilon(a+b, A) = \sqrt{2}\mathbf{D} \subsetneqq \mathbf{D} = \eta\varepsilon(a, A)$$

with $\mathbf{D} = \{\lambda \mid |\lambda| \leq 1\}$. For another perturbation result involving the exponential spectrum, subspaces and quotient spaces, we refer to [**16**, Theorem 3.9].

6. Analytic properties of the exponential spectrum. In this final section we will indicate briefly that the exponential spectrum shares many analytic properties with the ordinary spectrum. For compact subsets K_1 and K_2 of $\mathbf{C}$, the *Hausdorff distance* between K_1 and K_2 is defined by

$$\Delta(K_1, K_2) = \max\{\sup_{z \in K_2} \operatorname{dist}(z, K_1), \sup_{z \in K_1} \operatorname{dist}(z, K_2)\}.$$

Let $r > 0$ and K be a compact subset of $\mathbf{C}$. If $K + r$ denotes $\{z \mid \operatorname{dist}(z, K) \leq r\}$, then $K_1 \subset K_2 + \Delta(K_1, K_2)$. We shall say the function $x \mapsto \varepsilon(x, A)$ is continuous at $a \in A$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x - a\| < \delta$ implies $\Delta(\varepsilon(x, A), \varepsilon(a, A)) < \varepsilon$. As usual, we say $x \mapsto \varepsilon(x, A)$ is *continuous* on E if it is continuous at every point of E. If, for given $\varepsilon > 0$,the number δ is independent of $a \in E$, then we say that $x \mapsto \varepsilon(x, A)$ is *uniformly continuous* on E. The examples in [**2**, pp. 48, 49] show that in general the exponential spectrum function is not continuous and if it is continuous it need not be uniformly continuous.

Results concerning analytic properties of the spectrum function in a Banach algebra appear in [**2**, Chapter III]. We formulate some analogous results for the exponential spectrum. Since the proofs are easy modifications of the corresponding proofs for the spectrum, we omit the proofs.

Theorem 6.1. *Let A be a Banach algebra and suppose $x, y \in A$ commute. Then $\varepsilon(y, A) \subset \varepsilon(x, A) + r(x - y)$ and consequently $\Delta(\varepsilon(x, A), \varepsilon(y, A)) \leq r(x - y) \leq \|x - y\|$. Furthermore, if A is commutative, then the exponential spectrum function is uniformly continuous on A.*

Theorem 6.2. *Let A be a Banach algebra. The exponential spectrum function $x \mapsto \varepsilon(x, A)$ is upper semi-continuous on A, i.e., for every open set U containing $\varepsilon(x, A)$ there is a $\delta > 0$ such that $\|x - y\| < \delta$ implies $\varepsilon(y, A) \subset U$.*

Theorem 6.3. *Let A be a Banach algebra and $x \in A$. Suppose U and V are two disjoint open sets such that $\varepsilon(x, A) \subset U \cup V$ and $\varepsilon(x, A) \cap U \neq \varnothing$. Then there exists $r > 0$ such that $\|x - y\| < r$ implies $\varepsilon(y, A) \cap U \neq \varnothing$.*

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