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INVARIANT HYPERBOLIC TORI FOR HAMILTONIAN SYSTEMS WITH RÜSSMANN NONDEGENERACY CONDITIONS

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ABSTRACT. Following the procedure designed by Graff for proving the persistence theorem of invariant hyperbolic tori for Hamiltonian systems with some modifications, we obtain in this paper a KAM theorem for Hamiltonian systems with hyperbolic fixed point. Because the Hamiltonian of unperturbed systems satisfies the Rüssmann nondegeneracy condition, this generalizes the well-known result of Graff.

1. Introduction and main results. In the classical KAM theory, a stronger nondegeneracy condition is required, see [**1, 7, 8**]. To weaken that condition is, currently, an attractive topic, and there have been some profound works for Hamiltonian systems, for example, [**2–4, 11, 15, 17**]. However, in our opinion, the real weakening of the degeneracy condition should look like

> "image of the frequency map $y \to w(y)$ does not lie in a hyperplane of the frequency space."

This is precisely Rüssmann nondegeneracy condition [**13**]. Pöschel's work [**9, 10**] may imply Rüssmann's conjecture, because Rüssmann's condition actually restricts the frequency vectors of the unperturbed system to some "twisted manifold." Recently, Rüssmann's conjecture was proved in [**16**]. Almost certainly this can be done for other KAM-type theorems.

In the present paper, we shall consider such degeneracy problems. More precisely, we shall prove the following result about the existence of invariant hyperbolic tori for Hamiltonian systems with Rüssmann nondegeneracy.

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Consider the Hamiltonian systems

$$H(x,y,z) = h(y) + \langle z_-, \Omega z_+\rangle + R(x,y,z), \tag{1.1}$$

where $(x,y,z) \in \Sigma$, $\langle , \rangle$ stands for the usual inner product in C^l, and

$$\begin{aligned}\Sigma &= \{x : \mathrm{Re}\, x \in T^n, |\mathrm{Im}\, x| \le r\} \\ &\quad \times \{y : \mathrm{Re}\, y \in G, |\mathrm{Im}\, y| \le \rho\} \\ &\quad \times \{(z_+, z_-) : |z_\pm| \le \rho\} \\ &:= \Sigma_1 \times \Sigma_2 \times \Sigma_3.\end{aligned}$$

Here $T^n = R^n/2\pi Z^n$ denotes the usual n-dimensional torus, $G \subset R^n$ is a connected open bounded set, Ω is a matrix function of order l defined on $\Sigma_1 \times \Sigma_2$, $z = (z_{\pm 1}, \dots, z_{\pm l})$, and r and ρ are positive constants.

Theorem A. *Assume that*

(A1) h, Ω *and* R *are real analytic functions on* Σ;

(A2) $w(y) = (\partial h/\partial y)$ *satisfies the Rüssmann nondegeneracy condition on* G;

(A3) *on* $\Sigma_1 \times \Sigma_2$,

$$\mathrm{Re}\, \langle \nu, \Omega \nu\rangle \ge 2\mu|\nu|^2,$$

for every $\nu \in C^l$, *where* μ *is a given positive constant.*

Then there exist a nonempty Cantor set $G_{\delta_0} \subset G$ *and a constant* $M > 0$ *such that if*

$$||R|| < M,$$

for each $y_0 \in G_{\delta_0}$, *the reduced invariant hyperbolic torus*

$$\mathcal{J}_0 : y = y_0, \qquad z = 0$$

for the unperturbed system

$$h(y) + \langle z_-, \Omega z_+\rangle$$

persists under the perturbation R, *and*

$$|w^\infty(y_0) - w(y_0)| \le 2M^{1/2},$$

where $w^\infty(y_0)$ *denotes the frequency of the invariant hyperbolic torus drifting from* $\mathcal{J}_0$*, for the perturbed system* (1.1)*. Further,* $\operatorname{meas} G_{\delta_0}$ *uniformly converges to* $\operatorname{meas} G$*, as* $\delta_0 \to 0$*.*

Remark. By [**2**], Rüssmann's condition implies the standard nondegeneracy one. Hence, in (A2),

$$\operatorname{rank}\left(\frac{\partial w}{\partial y}\right) = n,$$

in the case $l = 0$, Theorem A is the classical KAM theorem; when $l > 0$, it is the well-known result of Graff [**6**], which is an important generalization of the KAM theorem. Zehnder [**18**] also proposed a further approach. Because in our result, the unperturbed system may possess some stronger degeneracy, Theorem A generalizes the above-mentioned results.

Our argument is similar to that in [**6**] and that in [**11**]. Nevertheless, because of the presence of the certain degeneracy in the unperturbed system, some modifications are necessary. For example, it will be seen below that, in our arguments, the chosen action $y = y_0$ remains unchanged in whole iteration processes.

2. Proof of Theorem A.

In this section, on the basis of the KAM iteration method related to [**6**], we give the proof of Theorem A. Throughout this paper, all norms of vectors, matrices and functions denote the maximum ones.

Outline of the proof. We utilize the rapidly convergent iteration to prove Theorem A. To this end, choose rapidly convergent sequences as follows:

$$\begin{gathered} s_0 = \delta_0^{28(n+1)}, \qquad M_0 = s_0^{(18/7)}, \qquad r_0 = \rho, \\ \delta_{i+1} = \delta_i^{8/7}, \qquad M_{i+1} = M_i^{8/7}, \qquad s_i = M_i^{(7/18)}, \\ r_{i+1} = r_i - 6\delta_i, \quad i = 0, 1, \dots , \end{gathered}$$

where δ_0 is a positive constant satisfying conditions (A)–(M) listed below.

From (A2) in Theorem A and Lemma 9, there exists $r \in N$ such that for all $y \in G$, the collection of vectors

$$\frac{\partial^{|\alpha|} w(y)}{\partial y^\alpha}, \quad \alpha \in Z_+^n, \quad 0 \le |\alpha| = \alpha_1 + \cdots + \alpha_n \le r,$$

generates R^n. Let $w' : G \to R^n$ be a real analytic function. Using Lemma 8 there is $C_4 > 0$, satisfying that as

$$\|w' - w\| \le C_4, \tag{2.1.1}$$

for a given $\tau > nr - 1$, Lebesgue measure of the set

$$G(\gamma, w') = \{y : |\langle k, w' \rangle| \ge \gamma |k|^{-\tau},\ 0 \ne k \in Z^n, y \in G\}$$

uniformly converges to $\operatorname{meas} G$, as $\gamma \to 0$.

Set

$$O_{i+1} = \{y : |\langle k, w^{i+1}(y) \rangle| \ge \delta_i |k|^{-\tau},\ 0 \ne k \in Z^n, y \in G\},$$
$$i = 0, 1, \dots ,$$

where $w^{i+1}(y)$ is given below. Define

$$G_{\delta_0} = \bigcap_{i=0}^{\infty} O_{i+1}.$$

Fix $y_0 \in G_{\delta_0}$, and set

$$\begin{aligned} D_i = &\{x : \operatorname{Re} x \in T^n, |\operatorname{Im} x| \le r_i\} \\ &\times \{y : |y - y_0| \le 4s_i, y \in G\} \\ &\times \{(z_+, z_-) : |z_\pm| \le 6s_i\}, \quad i = 0, 1, \dots . \end{aligned}$$

Rewrite (1.1) in the following form:

$$H(x, y, z) = h^1(y) + \langle z_-, \Omega^1(x, y) z_+ \rangle + \overline{H}(x, y, z), \tag{2.1.2}$$

where

$$h^1(y) = h(y) + [R]_x(y,0), \tag{2.1.3}$$

$$[R]_x(y,0) = \frac{1}{(2\pi)^n}\int_{T^n} R(x,y,0)\,dx, \tag{2.1.4}$$

$$\Omega^1(x,y) = \Omega(x,y) + R_{z_+z_-}(x,y,0), \tag{2.1.5}$$

(2.1.6)

$$\overline{H}(x,y,z) = R(x,y,z) - [R]_x(y,0) - \left\langle z_-, R_{z_+z_-}(x,y,0)z_+\right\rangle,$$

$$w^1(y) = \frac{\partial h^1(y)}{\partial y}.$$

First we need to construct a canonical transformation T_1 by a suitable generating function $S^1(x, y^1, z_+, z_-^1)$:

$$T_1\colon (x^1, y^1, z_+^1, z_-^1) \in D_1 \to D \ni (x, y, z_+, z_-),$$

$$T_1\colon \begin{aligned} x^1 &= x + S^1_{y^1},\\ y &= y^1 + S^1_x,\\ z_+^1 &= z_+ + S^1_{z_-^1},\\ z_- &= z_-^1 + S^1_{z_+}, \end{aligned} \tag{2.1.7}$$

such that

$$\begin{aligned} H^1(x^1,y^1,z^1) &= H\circ T_1(x^1,y^1,z^1)\\ &= h^1(y^1) + \left\langle z_-^1, \Omega^1(x^1,y^1)z_+^1\right\rangle + R^1(x^1,y^1,z^1), \end{aligned} \tag{2.1.8}$$

and that on D_0,

$$||h^1 - h|| \le M_0, \qquad ||\Omega^1 - \Omega|| \le M_0^{(7/36)}, \tag{2.1.9}$$

and that on D_1,

$$||R^1|| \le M_1, \ \left\|\frac{\partial(x,y,z)}{\partial(x^1,y^1,z^1)} - I\right\| \le M_0^{(7/36)}, \tag{2.1.10}$$

where I stands for the identity matrix.

Generally, if we have that on D_i,

$$H^i(x^i, y^i, z^i) = h^i(y^i) + \langle z_-^i, \Omega^i(x^i, y^i) z_+^i \rangle + R^i(x^i, y^i, z^i), \tag{2.1.11}$$

satisfying

$$\operatorname{Re} \langle \nu, \Omega^i(x^i, y^i)\nu \rangle \geq \mu |\nu|^2, \qquad ||R^i|| \leq M_i, \tag{2.1.12}$$

then by a suitable generating function $S^{i+1}(x^i, y^{i+1}, z_+^i, z_-^{i+1})$ we can construct a canonical transformation

$$T^{i+1} \colon (x^{i+1}, y^{i+1}, z^{i+1}) \in D_{i+1} \to D_i \ni (x^i, y^i, z^i),$$

of the form

$$T^{i+1} \colon \begin{aligned} x^{i+1} &= x^i + S^{i+1}_{y^{i+1}}, \\ y^i &= y^{i+1} + S^{i+1}_{x^i}, \\ z_+^{i+1} &= z_+^i + S^{i+1}_{z_-^{i+1}}, \\ z_-^i &= z_-^{i+1} + S^{i+1}_{z_+^i}, \end{aligned} \tag{2.1.13}$$

such that

$$\begin{aligned} H^{i+1}(x^{i+1}, y^{i+1}, z^{i+1}) &= H^i \circ T^{i+1}(x^{i+1}, y^{i+1}, z^{i+1}) \\ &= h^{i+1}(y^{i+1}) + \langle z_-^{i+1}, \Omega^{i+1}(x^{i+1}, y^{i+1}) z_+^{i+1} \rangle \\ &\quad + R^{i+1}(x^{i+1}, y^{i+1}, z^{i+1}), \end{aligned} \tag{2.1.14}$$

and that on D_i,

$$||h^{i+1} - h^i|| \leq M_i, \qquad ||\Omega^{i+1} - \Omega^i|| \leq M_i^{(7/36)}, \tag{2.1.15}$$

and that on D_i,

$$||R^{i+1}|| \leq M_{i+1}, \tag{2.1.16}$$

(2.1.17)

$$||T^{i+1} - id|| \leq M_i^{(7/18)}, \qquad \left\| \frac{\partial(x^i, y^i, z^i)}{\partial(x^{i+1}, y^{i+1}, z^{i+1})} - I \right\| \leq M_i^{(7/36)}.$$

Since

$$\text{(A)} \qquad \delta_0 < 2^{-7},$$

we have $\sum_{j=0}^{\infty} \delta_i \le 2\delta_0$. From

$$\text{(B)} \qquad \delta_0 \le \frac{1}{24}\rho_0$$

it follows that $r_i \ge \rho_0/2$, $i = 1, 2, \dots$. Define

$$D_\infty = \{|\operatorname{Im} x| \le \rho_0/2\} \times \{y = y_0\} \times \{z_\pm = 0\},$$
$$\mathcal{U}_i = T_1 \circ T_2 \circ \cdots \circ T_i, \qquad \mathcal{U}'_i = T'_1 \cdot T'_2 \cdot \cdots \cdot T'_i.$$

Then $\mathcal{U}_i \colon D_i \to D_0$. Hence if T_i, T'_i satisfy (2.1.17), then for sufficiently small δ_0,

$$\mathcal{U}_\infty = \lim_{i\to\infty} \mathcal{U}_i, \qquad \mathcal{U}'_\infty = \lim_{i\to\infty} \mathcal{U}'_i$$

hold uniformly on D_∞, and $\mathcal{U}_\infty \colon D_\infty \to D_0$. Put $\operatorname{Im}\xi = 0$ on D_∞; then, according to Lemma 7, $\mathcal{U}_\infty \colon T^n \to D_0$ is a continuous embedding, and on T^n

$$\xi' = w^\infty(y_0),$$

where $w^\infty(y_0) = w(y_0) + \sum_{i=1}^{\infty} w^i(y_0)$; refer to [**6**, Section 2-d] for details. By (A), $w^\infty(y_0)$ exists, and

$$|w^\infty(y_0) - w(y_0)| \le 2M^{1/2}.$$

The discussion of the measure of invariant tori. Obviously,

$$G \setminus G_{\delta_0} \subset \bigcup_{i=0}^{\infty} (G \setminus O_{i+1}).$$

From (A2), Lemma 9 and Lemma 8, there exists the function $l(\delta) > 0$ such that

$$\text{(2.2.1)} \qquad \operatorname{meas}(G \setminus O_{i+1}) < l(\delta_i).$$

and as $\delta \to 0$, $l(\delta) \to 0$. It is easy to prove that, for some positive constant κ and sufficiently small δ,

$$l(\delta) < \delta^\kappa.$$

Hence, as

$$\delta_0 < \min\left\{2^{-7/\kappa}, \left(\frac{C_4}{2}\right)^{(1/(36n+35))}\right\}, \tag{C}$$

using Lemma 8, Lemma 9 and (2.2.1), we have

$$\operatorname{meas}(G \setminus G_{\delta_0}) \le \sum_{i=0}^{\infty} \operatorname{meas}(G \setminus O_{i+1}) \le \sum_{i=0}^{\infty} c\delta_i^\kappa \le 2c\delta_0^\kappa,$$

which prove the convergence of that measure in Theorem A.

Inductive iterations. To prove the theorem, we consider one cycle of the iteration scheme. To this end, assuming (2.1.14)–(2.1.17) hold for k, we need to prove that they also hold for $k+1$. For simplicity, we omit "k" and rewrite "$k+1$" as "+." Then, by (2.1.14),

$$H(x,y,z) = h(y) + \langle z_-, \Omega(x,y)z_+\rangle + R(x,y,z). \tag{2.3.1}$$

Rewrite it in the following form:

$$H(x,y,z) = h^+(y) + \left\langle z_-, \Omega^+(x,y)z_+\right\rangle + \overline{H}(x,y,z), \tag{2.3.2}$$

where

$$\begin{aligned} h^+(y) &= h(y) + [R]_x(y,0), \\ [R]_x(y,0) &= \frac{1}{(2\pi)^n}\int_{T^n} R(x,y,0)\,dx, \end{aligned} \tag{2.3.3}$$

$$\begin{aligned} w^+(y) &= w(y) + \frac{\partial}{\partial y}[R]_x(y,0), \\ \Omega^+(x,y) &= \Omega(x,y) + R_{z_+z_-}(x,y,0), \end{aligned} \tag{2.3.4}$$

$$\overline{H}(x,y,z) = R(x,y,z) - [R]_x(y,0) - \left\langle z_-, R_{z_+z_-}(x,y,0)z_+\right\rangle . \tag{2.3.5}$$

We consider the generating function $S^+(x, y^+, z_+, z_-^+)$:

$$\begin{aligned} S^+(x,y^+,z_+,z_-^+) &= A(x,y^+) + B(x,y^+)z_+ + C(x,y^+)z_-^+ \\ &+ \frac{1}{2}\left\langle z_+, D(x,y^+)z_+\right\rangle + \frac{1}{2}\left\langle z_-^+, E(x,y^+)z_-^+\right\rangle , \end{aligned} \tag{2.3.6}$$

where A, B, C, D, E are determined by the following equations:

$$\partial A + R(x,y^+,0) - [R]_x(y^+,0) = 0, \tag{2.3.7}$$

$$\partial B + B\Omega(x,y^+) + R_{z_+}(x,y^+,0) = 0, \tag{2.3.8}$$

$$\partial C - C{\Omega^+}^T(x,y^+) + R_{z_-^+}(x,y^+,0) = 0, \tag{2.3.9}$$

$$\partial D + D\Omega(x,y^+) + \Omega^T(x,y^+)D + R_{z_+z_+}(x,y^+,0) = 0, \tag{2.3.10}$$

$$\partial E - E{\Omega^+}^T(x,y^+) - \Omega^+(x,y^+)E + R_{z_-^+z_-^+}(x,y^+,0) = 0, \tag{2.3.11}$$

where "∂" denotes the operator $\partial = \sum_{k=1}^{n} w_k^+(y_0)(\partial/\partial x_k)$, and "$T$" stands for the transpose. By Lemmas 2 and 3, these equations have unique solutions.

Introduce a canonical transformation

$$T^+\colon \begin{aligned} x^+ &= x + S^+_{y^+}, \\ y &= y^+ + S^+_x, \\ z_+^+ &= z_+ + S^+_{z_-^+}, \\ z_- &= z_-^+ + S^+_{z_+}. \end{aligned} \tag{2.3.12}$$

Under this transformation, $H(x,y,z)$ becomes

$$\begin{aligned} H^+(x^+,y^+,z^+) &= H\circ T^+(x^+,y^+,z^+) \\ &= h^+(y^+) + \left\langle z_-^+, \Omega^+(x^+,y^+)z_+^+\right\rangle \\ &\quad + \left\langle h_y^+(y^+) - w^+(y_0), S_x^+\right\rangle \\ &\quad + [h^+(y) - h^+(y^+) - \left\langle h_y^+(y^+), S_x^+\right\rangle] \end{aligned}$$

$$
\begin{aligned}
&\quad - \langle z_-^+, (\Omega^+(x + S_{y^+}^+, y^+) - \Omega^+(x, y^+))(z_+ + S_{z_-^+}^+)\rangle \\
&\quad + \left\langle z_-^+ + S_{z^+}^+, (\Omega(x, y^+ + S_x^+) - \Omega(x, y^+))z_+ \right\rangle \\
&\quad + [R^*(x, y^+ + S_x^+, z_+, z_-^+ + S_{z_-^+}^+) - R^*(x, y^+, z_+, z_-^+)] \qquad (2.3.13) \\
&\quad + [R^*(x, y^+, z_+, z_-^+) - \sum_{i=0}^{2} R^{*(i)}(x, y^+, z_+, z_-^+)] \\
&= h^+(y^+) + \left\langle z_-^+, \Omega^+(x^+, y^+) z_+^+ \right\rangle \\
&\quad + P_1 + P_2 - P_3 + P_4 + P_5 + P_6 \\
&= h^+(y^+) + \left\langle z_-^+, \Omega^+(x^+, y^+) z_+^+ \right\rangle \\
&\quad + R^+(x^+, y^+.z^+),
\end{aligned}
$$

where

$$R^*(x, y, z) = R(x, y, z) - [R]_x(y, 0),$$

and $R^{*(i)}$ denotes the sum of the ith order terms in Taylor's expansion of R^*.

If on D, $||R|| \leq M$, then

$$
\begin{aligned}
||[R]_x(\cdot, 0)|| &\leq M, \\
||[R]_{z_+ z_-}(\cdot, 0)|| &\leq \frac{M}{s^2} < M^{(7/36)},
\end{aligned}
$$

and hence on D,

$$||h^+ - h|| \leq M, \qquad ||\Omega^+ - \Omega|| \leq M^{(7/36)}. \qquad (2.3.14)$$

By the Cauchy estimate, on $\{|\operatorname{Im} x| \leq r\} \times \{|y^+ - y_0| \leq 4s\}$,

$$|R_{z_+}(x, y^+, 0)|, \ |R_{z_-}(x, y^+, 0)| \leq \frac{M}{s}, \qquad (2.3.15)$$

(2.3.16)

$$|R_{z_+ z_+}(x, y^+, 0)|, \ |R_{z_- z_-}(x, y^+, 0)| \leq \frac{M}{s^2}.$$

Using Lemma 1 yields that, on D,

$$
\begin{aligned}
|\Omega^+(x, y^+)| &\leq \Theta_1, \\
\operatorname{Re} \left\langle \nu, \Omega^+(x, y^+)\nu \right\rangle &\geq \mu|\nu|^2, \quad \nu \in C^l.
\end{aligned}
$$

By Lemmas 4, 2 and 3, on $\{|\operatorname{Im} x| \le r - 2\delta\} \times \{|y^+ - y_0| \le 2s\}$,

$$||A|| \le C_1 M \delta^{-(2n+2)}, \tag{2.3.17}$$
$$||B||,\ ||C|| \le C_2 M s^{-1}, \tag{2.3.18}$$
$$||D||,\ ||E|| \le C_2 M s^{-2}, \tag{2.3.19}$$

where $C_1 = 4^n((n+1)/e)^{n+1}$, $C_2 = 2l^{1/2}\mu^{-1}$.

Similarly, on $\{|\operatorname{Im} x| \le r - 3\delta\} \times \{|y^+ - y_0| \le s\} \times \{|z_\pm| \le 6s\}$,

$$||S_x^+|| \le C_3 M \delta^{-(2n+2)}, \tag{2.3.20}$$
$$||S_{y^+}^+||,\ ||S_{z_+}^+||,\ ||S_{z_-^+}^+|| \le C_3 M s^{-1} \delta^{-(2n+2)}, \tag{2.3.21}$$
$$||S_{z_-^+ z_-^+}^+||,\ ||S_{z_+ z_+}^+|| \le C_3 M s^{-2} \delta^{-(2n+2)}, \tag{2.3.22}$$
$$||S_{z_+ z_-^+}|| = 0, \tag{2.3.23}$$

where $C_3 = 5 \max\{C_1, C_2\}$.

Now let us check that T^+ maps D_+ into

$$D_*\colon \{|\operatorname{Im} x| \le r - 5\delta\} \times \{|y - y_0| \le s\} \times \{|z_\pm| \le s\}.$$

Indeed, if

$$\delta_0 \le C_3^{-1/(10(n+1))} \le C_3^{-1/(4n+41)}, \tag{D}$$

then by (2.3.21),

$$|x^+ - x| \le ||S_{y^+}^+|| \le C_3 M s^{-1} \delta^{-2(n+1)} < \delta.$$

From the choice of r and r^+ it follows that if $|\operatorname{Im} x^+| \le r_+$, then $|\operatorname{Im} x| \le r - 5\delta$. We derive from (2.3.20) that

$$|y^+ - y| \le ||S_x^+|| \le C_3 M \delta^{-(2n+3)} < s_+,$$

provided

$$\delta_0 \le C_3^{-1/(10(n+1))} \le C_3^{-1/(38n+37)}. \tag{E}$$

Therefore,

$$|y-y_0| \le |y-y_+| + |y_+-y_0| \le s_+ + 4s_+ < s.$$

If

(F) $$\delta_0 \le \min\{C_3^{-1/(10(n+1))},\ 7^{-1/(4n+4)}\},$$

then by (2.3.22),

$$\begin{aligned} |z_+| &\le |z_+^+| + \|S^+_{z_-^+}\| \\ &\le 6s_+ + C_3 M s^{-1}\delta^{-(2n+2)} \le 7s_+ < s, \\ |z_-| &\le |z_-^+| + \|S^+_{z_+}\| \\ &\le 6s_+ + C_3 M s^{-1}\delta^{-(2n+2)} \le 7s_+ < s. \end{aligned} \tag{2.3.24}$$

To summarize, we have

$$T^+ \colon D_+ \to D_* \subset D.$$

In the following, we estimate R^+. By Lemma 1 and (2.3.20), we have

$$\begin{aligned} \|P_1\| &\le n\|h_y^+(y^+) - h_y^+(y_0)\|\,\|S_x^+\| \\ &\le n^2\|h_{yy}^+\|\,\|y^+-y_0\|\,\|S_x^+\| \\ &\le 4n^2\Theta_0\cdot s_+\cdot c_3 M\delta^{-(2n+3)} \\ &\le 4n^2\Theta_0 C_3\delta^{17n+16}M^{8/7} \le \frac{1}{6}M^{8/7}, \end{aligned} \tag{2.3.25}$$

provided

(G) $$\delta_0 \le (24n^2\Theta_0 C_3)^{-1/(17n+16)}).$$

Similarly, as

(H) $$\delta_0 \le (6n^2\Theta_0\cdot C_3^2)^{-1/(56n+54)},$$

we have

$$\begin{aligned} \|P_2\| &\le \|h_y^+(y) - h^+(y^+) - \langle h_y^+(y^+), S_x\rangle\| \\ &\le n^2\|h_{yy}^+\|\|S_x^+\|^2 \\ &\le n^2\Theta_0\cdot C_3^2 M^2\delta^{-(4n+6)} \\ &\le n^2\Theta_0\cdot C_3^2\delta^{56n+54}M^{8/7} \le \frac{1}{6}M^{8/7}. \end{aligned} \tag{2.3.26}$$

By Lemma 1, (2.3.21) and (2.3.22), we get

(2.3.27)

$$\begin{aligned}||P_3|| &\le n^2||z_-^+||\,||\Omega^+(x+S_{y^+}^+,y^+)-\Omega^+(x,y^+)||\,||z_+ + S_{z_-^+}^+|| \\ &\le n^2\cdot 6s_+\cdot n\left\|\frac{\partial\Omega^+}{\partial x}\right\|||S_{y^+}^+||(6s+C_3Ms^{-1}\delta^{-(2n+2)}) \\ &\le 6n^3s_+\Theta_1\cdot\delta^{-1}\cdot C_3M\delta^{-(2n+2)} \\ &\quad\times s^{-1}(6s+C_3M\delta^{-(2n+2)}s^{-1}) \\ &\le 6(6+C_3)n^3\Theta_1\cdot C_3\delta^{17n+16}M^{8/7}\le\frac{1}{6}M^{8/7};\end{aligned}$$

(2.3.28)

$$\begin{aligned}||P_4|| &\le n^2||z_-^+ + S_{z_+}^+||\,||\Omega(x,y^+ + S_x^+)-\Omega^+(x,y^+)||\,||z_+|| \\ &\le n^3(6s_+ + C_3M\delta^{-(2n+2)}s^{-1})\Theta_1 s^{-1}\cdot C_3M\delta^{-(2n+3)}\cdot 6s \\ &\le 6(6+C_3)n^3\Theta_1C_3\delta^{17n+16}M^{8/7}\le\frac{1}{6}M^{8/7},\end{aligned}$$

provided

$$\text{(I)} \qquad \delta_0 \le (36(6+C_3)n^3\Theta_1C_3)^{-1/(17n+16)},$$

where the constant Θ_1 is given in Lemma 1. Applying the mean value theorem, Cauchy's estimate, (2.3.20) and (2.3.21), we have

$$\text{(2.3.29)} \qquad \begin{aligned}||P_5|| &\le n(||R_y^*||\,||S_x^+|| + ||R_{z_-}^*||\,||S_{z_+}^+||) \\ &\le 2nM\cdot s^{-1}\cdot C_3M\delta^{-(2n+3)} \\ &\quad + 2nMs^{-1}C_3M\delta^{-(2n+2)}s^{-1} \\ &\le 2nC_3(\delta^{31n+30}+\delta^{3n+3})M^{8/7}\le\frac{1}{6}M^{8/7}.\end{aligned}$$

Here we have used the inequality

$$\text{(J)} \qquad \delta_0 \le (24nC_3)^{-1/(3n+3)}.$$

Using Taylor's expansion and (2.3.24) yields

$$\text{(2.3.30)} \qquad \begin{aligned}||P_6|| &\le n^3\cdot 3!M\cdot s^{-3}|z|^3 \le 6n^3Ms^{-3}7^3s_+^3 \\ &\le 6\cdot 7^3n^3M^{(1/42)}M^{8/7}\le\frac{1}{6}M^{8/7},\end{aligned}$$

provided

(K) $$\delta_0 \le (6^2 \cdot 7^3 n^3)^{-1/(n+1)}.$$

Hence from (2.3.25)–(2.3.30) we derive that, on D_+,

(2.3.31) $$||R^+|| \le M^{8/7} = M_+.$$

Finally, we prove (2.1.17) for $k+1$. By (2.3.20) and (2.3.21),

(2.3.32) $$||T^+ - id|| \le M^{(7/36)}.$$

Taking δ_0,

(L) $$\delta_0 \le (2C_3)^{-1/(14n+14)};$$

hence,

$$C_3 M \delta^{(2n+2)} s^{-2} \le \frac{1}{2}.$$

From (2.3.20) and (2.3.21) it follows that on the domain

$$\{|\mathrm{Im}\, x| \le r - 3\delta\} \times \{|y^+ - y_0| \le s\} \times \{|z_\pm| \le 6s\},$$
$$\begin{pmatrix} x^+ \\ z_+^+ \end{pmatrix} = \begin{pmatrix} x \\ z_+ \end{pmatrix} + \begin{pmatrix} S_{y^+}^+ \\ S_{z_-}^+ \end{pmatrix} (x, y^+, z_+, z_-^+)$$

and

$$\left\| \begin{pmatrix} S_{y^+}^+ \\ S_{z_-}^+ \end{pmatrix} \right\| \le C_3 M \delta^{-(2n+2)} s^{-1} \le \frac{1}{2} s.$$

By Lemmas 5 and 6, on

$$\{|\mathrm{Im}\, x^+| \le r - 5\delta\} \times \{|y^+ - y_0| \le s\} \times \{|z_\pm^+| \le 6s\},$$

we have

$$\begin{pmatrix} x \\ z_+ \end{pmatrix} = \begin{pmatrix} x^+ \\ z_+^+ \end{pmatrix} + \begin{pmatrix} \phi \\ \varphi \end{pmatrix} (x^+, y^+, z^+)$$

and ϕ and φ are real analytic; furthermore,

$$\left\| \begin{pmatrix} \phi \\ \varphi \end{pmatrix} \right\| \le 2C_3 M s^{-1} \delta^{-(2n+2)},$$
$$\left\| \frac{\partial(\phi, \varphi)}{\partial(x^+, z_+^+)} \right\| \le 4C_3 M s^{-1} \delta^{-(2n+2)}.$$

Then, on D_+,

$$\left\| \frac{\partial(x,y,z)}{\partial(x^+,y^+,z^+)} - I \right\| \le M^{(7/36)}; \tag{2.3.33}$$

refer to [**6**, Section 3-f] for details. Equations(2.3.14), (2.3.31)–(2.3.33) imply that (2.1.14) and (2.1.15) hold for $k+1$. Thus we complete the proof of the theorem.

3. Technique lemmas. In this section we shall list some lemmas which have been used in the proof of Theorem A.

Lemma 1. *For each $i \in N$, set*

$$\hat{D}_i = \{|Im\, x| \le r_i - \delta_i\} \times \{|y - y_0| \le 3s_i\} \times \{|z_\pm^\pm| \le 5s_i\}.$$

Assume (A1)–(A3) *hold. Then there exist* $\Theta_0, \Theta_1 > 0$ *such that on* $\hat{D}_i$*,*

(1) $||h_{yy}^{i+1}|| \le \Theta_0$;

(2) $||\Omega^{i+1}|| \le \Theta_1$;

(3) Re $\langle \nu, \Omega^{i+1}(x,y)\nu \rangle \ge \mu|\nu|^2$.

Proof. Let $C_5 = ||h_{yy}||_{G'} + 1$, where $G' = \{|\mathrm{Im}\, y| \le \rho, \mathrm{Re}\, y \in G\}$. From the iteration processes, it is known that

$$h_{yy}^{i+1}(y) = h_{yy}(y) + \sum_{j=1}^{i} ([R^j]_x(y,0))_{yy}.$$

Hence, if

$$\delta_0 < \min\left\{2^{-(7/(16n+16))}, \left(\frac{\mu}{4}\right)^{(1/(16n+16))}\right\}, \tag{M}$$

then

$$\begin{aligned}||h_{yy}^{i+1}|| &\le ||h_{yy}||_{G'} + \sum_{j=1}^{i} |(([R^j]_x(y,0))_{yy}| \\ &\le C_5 + \sum_{j=1}^{i} 2M_j s_j^{-2} \\ &\le C_5 + \sum_{j=1}^{i} 2\delta_j^{16n+16} \\ &\le C_5 + 4\delta_0^{16n+16} \\ &\le C_5 + \mu = \Theta_0,\end{aligned}$$

which proves (1).

Since on D_i,

$$\Omega^{i+1}(x,y) = \Omega(x,y) + \sum_{j=1}^{i} R^j_{z_+z_-}(x,y,0),$$

using (M) on $\hat{D}_i$,

$$\begin{aligned}||\Omega^{i+1}|| &\le ||\Omega||_{\Sigma_1\times\Sigma_2} + \sum_{j=1}^{i} |R^j_{z_+z_-}(x,y,0)| \\ &\le ||\Omega||_{\Sigma_1\times\Sigma_2} + \sum_{j=1}^{i} 2M_j s_j^{-2} \\ &\le ||\Omega||_{\Sigma_1\times\Sigma_2} + \mu = \Theta_1.\end{aligned}$$

Hence on $\hat{D}_i$,

$$\begin{aligned}\text{Re}\,\left\langle \nu, \Omega^{i+1}(x,y)\nu \right\rangle &\ge \text{Re}\,\langle \nu, \Omega(x,y)\nu\rangle - \mu|\nu|^2 \\ &\ge 2\mu|\nu|^2 - \mu|\nu|^2 = \mu|\nu|^2.\end{aligned}$$

This completes the proof. □

Lemma 2 [**6**]. *Consider the equation*

$$V_x w + V(x)\Phi(x) + A(x)V(x) = F(x),$$

where $w = (w_1, \dots, w_n)$, $x = (x_1, \dots, x_n)$, $\Phi(x)$ *and* $A(x)$ *are real analytic on* $\Sigma_r\colon \{|Im\, x| \leq r\}$. *Assume*

$$\text{Re}\ \langle \nu, \Phi(x)\nu\rangle \geq \mu|\nu|^2,$$

$$\text{Re}\ \langle \nu, A(x)\nu\rangle \geq \mu|\nu|^2,$$

for all $\nu \in C^l$. *Then for each real analytic matrix* $F(x)$, *there exists a unique real analytic matrix* $V(x)$ *such that*

$$|V(x)| \leq 2l^{1/2}\mu^{-1}|F(x)|.$$

Lemma 3 [**6**]. *Consider the equation*

$$V_x(x,z)w + A(x,z)V(x,z) = f(x,z),$$

where $w = (w_1, \dots, w_n)$, $x = (x_1, \dots, x_n)$, $z = (z_1, \dots, z_l)$, $A(x,z)$ *is a real analytic matrix on the domain*

$$\Sigma_{r,R}\colon \{|Im\, x| \leq r\} \times \{|y| \leq R\}.$$

Assume on $\Sigma_{r,R}$,

$$\text{Re}\ \langle \nu, A(x,z)\nu\rangle \geq \mu|\nu|^2$$

for all $\nu \in C^l$. *Then for each real analytic function* $f(x,z)$ *defined on* $\Sigma_{r,R}$, *the equation admits a unique solution* $V(x,z)$ *such that*

$$|V(x,z)| \leq 2l^{1/2}\mu^{-1}|f(x,z)|.$$

Lemma 4 [**1**]. *Consider the scale equation*

$$\partial U(x) + f(x) = 0,$$

where $\partial = \sum_{k=1}^{n} w_k(\partial/\partial x_k)$. *Assume*

(1) f *is a real analytic and* 2π*-periodic function on* Σ_r,

(2) $||f||_{\Sigma_r} \leq M$,

(3) $[f] = (1/(2\pi)^n) \int_{T^n} f(x)\, dx = 0$,

(4) $|\langle k, w\rangle| \geq K|k|^{-\tau}$, *for all* $0 \neq k \in Z^n$, *where* $K > 0$, $\tau > n$ *are constants.*

Then on $\Sigma_{r-2\delta}$, $0 < 2\delta < r < 1$, *the equation admits a unique solution* $U(x)$ *such that* $[U] = 0$, *and*

$$|U(x)| \leq Mc\delta^{-(2n+1)},$$

where $c = 4^n K^{-1}((n+1)e^{-1})^{n+1}$.

Lemma 5 [**5**]. *Assume on the* k*-dimensional ball* $|y| \leq 6r$,

$$x = y + \phi(y),$$

where $\phi(x)$ *is real analytic and* $||\phi|| \leq r/2$. *Then there exists a unique real analytic function* f *defined on* $|x| \leq r$ *such that*

$$y = x + f(x),$$

and on $|x| \leq r$,

$$||f|| \leq ||\phi||, \qquad ||f_x|| \leq ||\phi||/r.$$

Lemma 6 [**5**]. *Assume on* Σ_r, $\phi(y)$ *is real analytic. Define*

$$x = y + \phi(y).$$

Then for given $\delta \in (0, r/2)$ *with* $||\phi|| \leq \delta/2$, *there exists a unique real analytic function* $f(x)$ *on* $\Sigma_{r-2\delta}$ *such that*

$$y = x + f(x)$$

and on $\Sigma_{r-2\delta}$,

$$||f|| \leq 2||\phi||, \qquad ||f_x|| \leq 4\delta^{-1}||\phi||.$$

Lemma 7 [**6**]. *Let $V_0(x)$ be a smooth vector field on D_0. Define the flow*:

$$\phi_0^t(x)\colon \frac{d}{dt}\phi_0^t(x) = V_0(\phi_0^t(x)),$$
$$\phi_0^0(x) = x.$$

Assume there exists an invertible transformation $T_i\colon D_i \to D_{i-1}$ with $|\Pi_{i=1}^{\infty} T_i'| < \infty$, where T_i' denotes the Jacobian of T_i. The transformation

$$U_i = T_1 \circ \cdots \circ T_i\colon D_i \to D_0$$

naturally induce flows

$$\phi_i^t = U_i^{-1} \circ \phi_0^t \circ U_i$$

with corresponding vector fields V_i on D_i:

$$V_i(x) = \frac{d}{dt}(\phi_i^t(x))\Big|_{t=0}.$$

Assume

(1) *V_i converges to V_∞ as $i \to \infty$ and $||V_i - V_\infty|| \le cd_{i+1}$ on D_∞, where c is independent of i, and $d_i = \operatorname{dist}(D_i, \partial D_{i-1})$;*

(2) *the segment $x = x_0 + vt$, $0 \le t \le 1$, belongs to D_∞; and on this segment, $V_\infty = v$;*

(3) *$\|(\partial V_i/\partial x)\| \le B$ on D_i, where B is independent of i;*

(4) *$U_\infty = \lim_{i\to\infty} U_i$ exists and is continuous.*

Then for $0 \le t \le (1/(B + C))$,

$$\phi_0^t(U_\infty(x_0)) = U_\infty(x_0 + vt) \subset D_0.$$

Lemma 8 [**16**]. *Let $S \subset R^n$ be a connected bounded closed region. Assume that $w : S \to R^n$ is C^r and satisfies that for any $y \in S$ the collection of vectors*

$$\frac{\partial^{|\alpha|} w(y)}{\partial y^\alpha}, \quad \alpha \in Z_+^n, \quad 0 \le |\alpha| = \alpha_1 + \cdots + \alpha_n \le r \tag{3.1}$$

generates a linear space R^n. Then, for a given $\tau > nr - 1$, Lebesgue measure of the set

$$S(\gamma, w') = \{y: |\langle k, w'(y)\rangle| \geq \gamma |k|^{-\tau}, 0 \neq k \in Z^n\}$$

uniformly converges to meas S *with respect to all C^r-functions $w' : S \to R^n$, which belong to some C^r-neighborhood of w, as $\gamma \to 0$.*

Lemma 9 [**12, 14**]. *If real analytic function $w : S \to R^n$ satisfies the Rüssmann's nondegeneracy condition on S, then there exists $r \in N$ such that, for all $y \in S$, the collection of vectors* (3.1) *generates R^n.*

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